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TRIQUOTIENT AND INDUCTIVELY PERFECT MAPS

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We give some conditions under which triquotient maps are inductively perfect.

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1. Introduction

All spaces in this paper are assumed to be regular and all maps are continuous. We recall that a surjective map $f: X \to Y$ of topological spaces is said to be *inductively perfect* if there is a closed set $F \subset X$ such that $f(F) = Y$ and $f|_F$ is perfect.

Triquotient maps were introduced by Michael in [7], and generalize the concepts of open and inductively perfect maps. The question of when open maps are inductively perfect has been considered in [1, 5, 7, 14]. The question of when compact covering and s-covering maps are inductively perfect have been considered in [7, 8, 12, 13, 16] (see also [4, Corollary to Theorem 1]).

Definition [7]. A map $f: X \to Y$ of topological spaces is said to be *triquotient* if for each $y \in Y$ there is a family η_y of open subsets of X such that $X \in \eta_y$ and

(a) if $U \in \eta_y$ and γ is an open cover of $U \cap f^{-1}(y)$ by open subsets of X then there exists a finite number of elements $U_i \in \gamma$, $1 \leq i \leq k$, such that $\bigcup_{i=1}^{k} U_i \in \eta_y$,

(b) if $U \in \eta_y$, then $y \in \operatorname{Int} f(U)$,

(c) if $U \in \eta_y$, then there is a neighbourhood $O(y)$ such that $U \in \eta_\xi$ for every $\xi \in O(y)$.

The above definition of a triquotient map is slightly different from that of Michael [7], but obviously is equivalent to it (put $U^* = \{y \in Y$: there is $O \in \eta_y$ such that $O \subset U\}$ and $\eta_y = \{U \subset X: y \in U^*\}$).

Our main results are:

Theorem 1. *Let $f: X \to Y$ be a triquotient map onto a paracompact space Y. If there exists a perfect extension $f^*: X^* \to Y$ of f such that X is a G_δ-set in X^*, then f is inductively perfect.*

Theorem 2. *If $f: X \to Y$ is a triquotient map of a metric space (X, d) onto a metric space Y such that every fiber $f^{-1}(y)$ is a complete metric space with respect to d and, for every open $U \subset X$, $f(U)$ is a G_δ-set in Y, then f is inductively perfect.*

The proof of Theorem 1 is, to some extent, similar to the proof of a theorem of Pasynkov [14, Theorem 8]. The idea of the proof of Lemma 1 for the case of subsets of the real line is due to Novikov [3, Section 14, Lemma 1]. Theorem 2 is based on Lemma 1 and Theorem 1. Corollaries to these results give further partial answers to the question of when triquotient maps are inductively perfect.

It should be remarked that Theorem 1 yields Theorem 1.6 of [7] but not the more general Theorem 6.6 of [7], and that Theorem 2 strengthens Corollary 1.2 in [10].

2. Proofs and corollaries

Proof of Theorem 1. Let $X = \bigcap_{i\in\omega} O_i$ where O_i is open in X^*, $i \in \omega$. Let us choose for each $y \in Y$ some $U_0^y \in \eta_y$ and for each point $x \in U_0^y$ choose a neighbourhood $U(x) \subset U_0^y$ such that $\mathrm{cl}_{X^*} U(x) \subset O_0$. There are finitely many $U(x_i)$, $i = 1, 2, \ldots, k$, such that $U_1^y = \bigcup_{i=1}^k U(x_i) \in \eta_y$. Obviously, $\mathrm{cl}_{X^*} U_1^y \subset O_0$. Now $y \in \mathrm{Int} f(U_1^y)$ and there is a neighbourhood $O(y)$ of y such that $U_1^y \in \eta_\xi$ for each $\xi \in O(y)$. Let us consider the open cover $\{(\mathrm{Int} f(U_1^y)) \cap O(y)\}_{y\in Y}$ of Y and let $\gamma = \{U_\alpha\}_{\alpha\in A}$ be a locally finite open refinement. For every $\alpha \in A$ we can choose $\xi(\alpha) \in Y$ such that $U_\alpha \subset O(\xi(\alpha)) \cap \mathrm{Int} f(U_1^{\xi(\alpha)})$. Let us consider the family δ of sets $W_\alpha^{\xi(\alpha)} = U_1^{\xi(\alpha)} \cap f^{*-1}(U_\alpha)$, and put $X_0 = \mathrm{cl}_{X^*}(\bigcup_{\alpha\in A} W_\alpha^{\xi(\alpha)})$. Then $X_0 = \bigcup_{\alpha\in A} \mathrm{cl}_{X^*} W_\alpha^{\xi(\alpha)} \subset O_0$ and $f_0 = f^*|_{X_0}$ is perfect. Since for each $y \in Y$ there is some $W_\alpha^{\xi(\alpha)} \in \eta_y$, we may consider O_1, X_0, f_0 instead of O_0, X^*, f^* and repeat the construction and so on. Thus we have $O_i \supset X_i \supset X_{i+1}$, $i \in \omega$, where X_i is closed in X^* and $f_i: X_i \to Y$ is perfect. Let $F = \bigcap_{i\in\omega} X_i$. It is obvious that $F \subset X$ and F is closed in X^*, hence $f|_F = f^*|_F$ is perfect and $f(F) = Y$. □

Corollary 1 [7, Theorem 1.6]. *A triquotient map $f: X \to Y$ of a Čech-complete space X onto a paracompact space Y is inductively perfect.*

Proof. Let us extend f to $\tilde{f}: \beta X \to \beta Y$ and put $X^* = \tilde{f}^{-1}(Y)$, $f^* = \tilde{f}|_{X^*}$. □

Remark 1. Let $f: X \to Y$ be a triquotient map onto a countable, hence paracompact, space Y, and let $f^*: X^* \to Y$ be a perfect extension of f such that all the fibers $f^{-1}(y)$, $y \in Y$, are G_δ-sets in $f^{*-1}(y)$. Then $X = f^{-1}(Y)$ is also a G_δ-set in X^* and, by Theorem 1, f is inductively perfect.

Corollary 2. *A triquotient map $f: X \to Y$ of a subspace X of a perfectly normal Čech-complete space Z onto a countable space Y with Čech-complete fibers is inductively perfect.*

Proof. Let $X^* = (\beta Z) \times Y$, where Z is Čech-complete and perfectly normal and $X \subset Z$, and let $f^*: X \to Y$ be the projection. Then f^* is perfect. We may identify X with the graph of f and assume $X \subset X \times Y \subset (\beta Z) \times Y$. By Remark 1 it is sufficient to show that every fiber $f^{-1}(y)$ is a G_δ-set in βZ. Now $\beta Z \backslash Z = \bigcup_{i\in\omega} F_i$, where F_i is compact, and $Z \backslash \mathrm{cl}_Z f^{-1}(y) = \bigcup_{i\in\omega} T_i$, where T_i is closed in Z, $i \in \omega$. Since $f^{-1}(y)$ is Čech-complete, $\mathrm{cl}_{\beta Z}(f^{-1}(y)) \backslash f^{-1}(y) = \bigcup_{i\in\omega} B_i$, where B_i is compact, $i \in \omega$. It is easily seen that $\beta Z \backslash f^{-1}(y) = \bigcup_{i\in\omega} (F_i \cup \mathrm{cl}_{\beta Z}(T_i) \cup B_i)$, and so $f^{-1}(y)$ is a G_δ-set in βZ. □

Remark 2. Taking the completion of X, we see that every metrizable space X satisfies the condition of Corollary 2, thus Corollary 2 generalizes Theorem 1.4 from [7].

Lemma 1. *Let $h: Z \to Y$ be a map of a metric space Z with a σ-discrete open base $\bigcup_{n\in\omega} \{H_{n,\alpha}: \alpha \in A\}$ onto a topological space Y. If $X \subset Z$ and $h(X) = Y$, then $\bigcup_{y\in Y} T_y = Z \backslash \bigcup_{n\in\omega} \bigcup_{\alpha\in A} T_{n,\alpha}$, where $T_y = \mathrm{cl}_Z(h^{-1}(y) \cap X)$ and $T_{n,\alpha} = H_{n,\alpha} \backslash h^{-1}h(H_{n,\alpha} \cap X)$.*

Proof. (1) Let $x \in \bigcup_{y\in Y} T_y$, then $x \in T_y$ for some $y \in Y$. It is clear that $h(x) = y$ and $x \in Z$. If $x \in T_{n,\alpha}$ then $x \in H_{n,\alpha}$ and $x \notin h^{-1}h(H_{n,\alpha} \cap X)$. Hence $H_{n,\alpha} \cap X \cap h^{-1}(y) = \emptyset$ and $x \notin T_y$.

(2) Let $x \in Z$ and $x \notin T_{n,\alpha} (n \in \omega,\ \alpha \in A)$. This implies that if $x \in H_{n,\alpha}$, then $x \in h^{-1}h(H_{n,\alpha} \cap X)$. Hence $x \in \mathrm{cl}_Z(h^{-1}(h(x)) \cap X)$, so $x \in T_{h(x)}$. □

Proof of Theorem 2. Let $\tilde{X}$ and $\tilde{Y}$ be the completions of X and Y, respectively, and let $\tilde{f}: X_0 \to \tilde{f}(X_0)$ be an extension of f, where X_0 is a G_δ-set in $\tilde{X}$. Let $g: \beta X_0 \to \beta\tilde{f}(X_0)$ be the extension of $\tilde{f}$ and let $X^* = g^{-1}(Y)$. It is clear that $g^* = g|_{X^*}: X^* \to Y$ is perfect. Since X_0 is a G_δ-set in βX_0, the set $Z = X_0 \cap X^*$ is a G_δ-set in X^*. We shall show that X is a G_δ-set in Z and hence in X^*. Let $h = g|_Z$. Since the fibers $f^{-1}(y)$ are complete with respect to d, they are closed in Z and, by Lemma 1, $\bigcup_{y\in Y} T_y = X = Z \backslash \bigcup_{n\in\omega} \bigcup_{\alpha\in A} T_{n,\alpha}$. Since for each $n \in \omega$ the set $\bigcup_{\alpha\in A} T_{n,\alpha}$ is an F_σ-set in Z, X is a G_δ-set in Z. □

We conclude with several rather specialized remarks.

Recall that a subset $Y \subset \mathbb{R}$ is called a σ-set provided every F_σ subset of Y is a G_δ-set in Y [6, Section 40, VI]. The continuum hypothesis implies the existence of uncountable σ-sets. It is known [6, Section 40, VI, Theorem 1] that any Borel subset of a σ-set is simultaneously an F_σ and a G_δ. Thus Theorem 2 remains valid for a σ-set Y provided the map f satisfies the following property:

(*) For each open set $U \subset X$, $f(U)$ is a Borel subset of Y.

We note that a continuous map $f: X \to Y$ satisfies (*) if X, $Y \subset \mathbb{R}$, X is a Borel set and $f^{-1}(y)$ is compact for each $y \in Y$ [3, Section 14, Theorem 3.5].

Under the assumption of Martin's Axiom, for each $Y \subset \mathbb{R}$ with $|Y| < 2^{\omega}$ each subset of Y is G_{δ}-set [15, Chapter 6, Theorem 12]. This and Theorem 2 imply:

Corollary 3 [10]. *Let X be a separable metric space, and $Y \subset \mathbb{R}$ with $|Y| < 2^{\omega}$. If $f: X \to Y$ is a triquotient map such that for every $y \in Y$ the fiber $f^{-1}(y)$ is complete in the metric induced by X, then f is inductively perfect.*

It should be noted that under the hypothesis of Corollary 3, the class of triquotient maps coincides with the class of s-covering and compact covering maps.

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