Topology and its Applications 159 (2012) 2004–2008

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

An alternative approach to the decomposition of functions

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A R T I C L E   I N F O

Dedicated to the Centennial of the Egorov–Luzin Theorem

MSC:
primary 26A21, 03E15, 28A05, 54C10
secondary 26A15, 54C08

Keywords:
Borel-measurable
Open-Borel
Countably continuous
Borel class
Open function
Closed function

A B S T R A C T

Let a function $f:X\to Y$ with compact fibers takes any open set into a union of an open and a closed set. Then there exist subsets $T_n\subset X$ $(n=1,2,\ldots)$ such that every restriction $f|T_n$ is open or closed function onto Y_n and Y_n $(n=0,1,\ldots)$ cover Y.
This result is related to the problem of preservation of Borel classes by open-Borel functions and to Luzin's question about decomposition of Borel-measurable functions into countably many continuous ones.

1. Introduction

During his research, the author has repeatedly encountered situations where function $f:X\to Y$ between arbitrary subspaces X and Y of the Cantor set $\mathbf{C}$ possess the following two properties at the same time:

(1) f has compact fibers and takes clopen sets $U\subset X$ to F_σ and G_δ-sets $B\subset Y$;
(2) there exist $T_n\subset X$ such that every restriction $f|T_n$ is a closed or an open function onto $Y_n=f(T_n)$ and $Y=\bigcup_{n=1}^{\infty}Y_n$.

This brought up a hypothesis that (1) often implies (2). It seems natural to verify this hypothesis for the case of sets $B=F\cup O$, where O is an open and F is a closed set. We will point out in Theorems 3.1 and 4.1 affirmative answers.

This can be applied to the problem of defining a natural common class $\mathfrak{A}$ for open and closed functions, such that every function $f\in\mathfrak{A}$ with compact fibers preserves Borel class in $\mathbf{C}$ [6, Problem 3.6]. Corollary 4.2 gives us a natural answer.

Our approach gives more. We can apply it to investigate the question of Luzin whether every Borel-measurable function with arbitrary domain has a countable decomposition into continuous ones [3,2].

This question was answered negatively by P.S. Novikov (see [3]) and was subsequently generalized in [1,3,4,11].

Corollary 5.1 gives a positive answer to Luzin's question for $\mathbf{G}\cup\mathbf{F}$-measurable functions.

2. Related materials and basic definitions

A clopen set is a set that is both open and closed. As usual, $\mathbf{C}$ denotes the Cantor set.

Given an arbitrary (not necessarily continuous) function $f:X\to Y$, we say that a function f is

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.04.032

– *Borel-measurable* if the preimage $f^{-1}(W)$ of every open set $W \subset Y$ is a Borel set[1];
– *open-Borel* if the image $f(U)$ of every open set $U \subset X$ is a Borel set.

In particular, f is

– $\mathbf{G} \cup \mathbf{F}$-*measurable* if the preimage $f^{-1}(W)$ of every open set $W \subset Y$ is the union of an open set and a closed set;
– *open* (*closed*) if the image of every open (closed) set is open (closed) set;
– *open* (*closed*)-$\mathbf{G} \cup \mathbf{F}$ if the image $f(W)$ of every open (closed) set $W \subset X$ is the union of an open set and a closed set;
– *half-open* if the image $f(W)$ of every clopen set $W \subset X$ is the union of an open and a closed set.

We will denote by $S_1(y)$ a sequence with its limit point ($y_i \neq y_j$ for $i \neq j$):

$$S_1(y) = \{y\} \cup \{y_i : y_i \to y\}$$

It is easy to check that a function f is closed iff it is closed at each point $y \in Y$; i.e., for every $S_1(y)$, every sequence $x_i \in f^{-1}(y_i)$ ($y_i \neq y_j$ for $i \neq j$) has a limit point in $f^{-1}(y)$.

Indeed, suppose, contrary to our claim, that f is closed and, for some $S_1(y)$, there is no limit point in $f^{-1}(y)$ for some $x_i \in f^{-1}(y_i)$. Now we have two possibilities:

1) there is a limit point $x \in X \setminus f^{-1}(y)$ for x_i;
2) there is no limit points in X for x_i.

In case 1) the image $f(T)$ of the closed set $T = \{x\} \cup cl_X\{x_i\}$ is not closed in Y. In case 2) the image $f(T)$ of the closed set $T = cl_X\{x_i\}$ is not closed in Y. This contradicts our assumption.

Conversely, if, for every $S_1(y)$, some sequence of points $x_i \in f^{-1}(y_i)$ has a limit point in $f^{-1}(y)$ and there is a closed $T \subset X$ for which $f(T)$ is not closed in Y, then there is $S_1(y)$ such that $y \notin f(T)$ and $y_i \in f(T)$. Hence, the sequence of points $x_i \in f^{-1}(y_i) \cap T$ has no limit point in $f^{-1}(y)$.

Analogously, it is easy to check that a function f is open iff it is open at each point $x \in X$; i.e., for every $S_1(y)$ where $y = f(x)$ and for every neighborhood $O(x)$ of x, the intersection $f^{-1}(y_i) \cap O(x) = \emptyset$ for only finitely many points y_i.

3. Half-open functions in the Cantor set C

The following theorem is, in a way, a generalization of Lemma 1 from [8].

Theorem 3.1. *Let $f : X \to Y$ be a half-open function with compact fibers, $X, Y \subset \mathbf{C}$. Then there exist $T_n \subset X$ ($n = 1, 2, \ldots$) such that every restriction $f|T_n$ is a closed function onto a subset $f(T_n) = Y_n$ that is closed in Y, and the restriction $f|f^{-1}(Y_0)$, where $Y_0 = Y \setminus \bigcup_{n=1}^{\infty} Y_n$, is an open function onto Y_0.*

Proof. Theorem 3.1 is an obvious corollary to the following Theorem 3.2. Indeed, by Theorem 3.2, f is an open function at every point of the set $D_0 = X \setminus \bigcup_{n=1}^{\infty} T_n$. Hence, if $f^{-1}(L) \subset D_0$ for some $L \subset Y$, then the restriction $f|f^{-1}(L)$ is an open function onto L. It follows from definitions of Y_0 and D_0 that $f^{-1}(Y_0) \subset D_0$. □

Theorem 3.2. *Under conditions of Theorem* 3.1, *there exist $T_n \subset X$ ($n = 1, 2, \ldots$) such that every restriction $f|T_n$ is a closed function onto a subset $f(T_n) = Y_n$ that is closed in Y and f is an open function at every point*[2] *of the set $D_0 = X \setminus \bigcup_{n=1}^{\infty} T_n$.*

Proof. We start with the following notion:

Definition 3.1. Let $f : X \to Y$ be a function, $Z \subset X$. We say that f is $cl_Y f(Z)$-*closed extendable* if, for every sequence $y_i \to y$, where $y_i \in f(Z)$ and $y \in cl_Y f(Z)$, every sequence $x_i \in f^{-1}(y_i) \cap Z$ has a limit point $x \in f^{-1}(y)$.

Lemma 3.1. *Let $f : X \to Y$ be a $cl_Y f(Z)$-closed extendable function of metric spaces X, Y. We define a set T_y for every $y \in cl_Y f(Z)$ by setting*

$$T_y = \left\{x \in f^{-1}(y)\colon \exists y_i \in f(Z) \text{ for which } y_i \to y \text{ and } \exists x_i \in Z \cap f^{-1}(y_i) \text{ for which } x_i \to x\right\}.$$

Then $g = f|\bigcup_{y \in Y} T_y$ is a closed function onto $cl_Y f(Z)$.

Remark. Definition 3.1 does not involve the condition $y_i \neq y_j$ for $i \neq j$; hence, every T_y is nonempty for all $y \in f(Z)$ and g in Lemma 3.1 is “onto”.

[1] Borel measurable functions are often called Borel mappings.
[2] A nuance: $f|D_0$ is not necessarily an open function.

Proof of Lemma 3.1. Suppose the opposite: g is not closed, and hence there is an $S_1(y) \subset cl_Y f(Z)$ such that, for some $x_i \in g^{-1}(y_i)$, there is no limit point in $g^{-1}(y)$.

By the definition of T_{y_i}, for every $x_i \in g^{-1}(y_i)$ there exist $y_{i_j} \in f(Z)$ such that $y_{i_j} \to y_i$ and there exist $x_{i_j} \in Z \cap f^{-1}(y_{i_j})$ such that $x_{i_j} \to x_i$.

It is clear, that $\{y_{i_j}\}$ contains a subsequence $y_{i_{j(i)}} \to y$. Moreover, we can choose $j(i)$ such that for the corresponding points $x_{i_{j(i)}} \in Z \cap f^{-1}(y_{i_{j(i)}})$ the following condition holds:

$$\operatorname{dist}(x_{i_{j(i)}}, x_i) < 1/i. \tag{\#}$$

Since $y_{i_{j(i)}} \in f(Z)$, $y_{i_{j(i)}} \to y$, and f is $cl_Y f(Z)$-closed extendable, by Definition 3.1 the sequence $x_{i_{j(i)}}$ has a limit point $x \in f^{-1}(y)$. By definition of T_y we have $x \in T_y$. According to (#), $x_i \to x$, and hence $x \in g^{-1}(y)$. Thus, we obtain a contradiction. □

In order to prove Theorem 3.2 we need the following notion and Lemma 3.2.

For every $n = 1, 2, \ldots$, denote $Z_n = \{x \in X\colon\ \exists S_1(f(x)) \subset Y$ such that for all $y_i \in S_1(f(x)),\ \operatorname{dist}(x, f^{-1}(y_i)) > 1/n\}$.

Hence,

$$f(Z_n) = \bigl\{y \in Y\colon\ \text{there exist } S_1(y) \text{ and } x \in f^{-1}(y) \text{ such that } \operatorname{dist}\bigl(x, f^{-1}(y_i)\bigr) > 1/n\bigr\}. \tag{$*$}$$

Lemma 3.2. *f is a $cl_Y f(Z_n)$-closed extendable function.*

Proof. Indeed, suppose the contrary: there exist $y_k \to y$, $y_k \in f(Z_n)$, $y \in cl_Y f(Z_n)$, and $x_k \in f^{-1}(y_k) \cap Z_n$ that have no limit points in $f^{-1}(y)$. Since every $f^{-1}(y)$ is compact, we can suppose that $y_k \neq y_j$ for $k \neq j$:

According to condition ($*$) we can take $S_1(y_k) = \{y_k\} \cup \{y_{k_i} : y_{k_i} \to y_k\}$ such that $\operatorname{dist}(x_k, f^{-1}(y_{k_i})) > 1/n$.

Let $O_{1/n}(x_k) = \{x \in X\colon\ \operatorname{dist}(x_k, x) < 1/n\}$ be the $1/n$-ball centered at x_k.

Since X is a subspace of the Cantor set $\mathbf{C}$, the points x_k have a limit point $z \in \mathbf{C}$.

It is clear that there is a number N such that, for every $k > N$, we have:

(i) $O_{1/2n}(z) \subset O_{1/n}(x_k)$.

By the above $z \notin f^{-1}(y)$ and $f^{-1}(y)$ is compact, hence:

(ii) there is a clopen in X neighborhood $O(z) \subset O_{1/2n}(z)$ of the point z, that does not intersect $f^{-1}(y)$ and hence $y \notin f(O(z))$.

By assumption, f is half-open, $f(O(z)) = F \cup O$, where F is a closed and O is an open subset in Y.

Since z is limit point for x_k and $f(x_k) = y_k$ we conclude that $\{y_k\} \subset f(O(z))$ and since $y_k \to y \notin f(O(z))$, the closed set F contains at most finitely many points y_k, and we can suppose that $\{y_k\} \subset O \setminus F$.

The items (i) and (ii) shows that for every $k > N$

$$f\bigl(O(z)\bigr) \subset f\bigl(O_{1/n}(x_k)\bigr),$$

and hence

$$f\bigl(O(z)\bigr) \subset \bigcap_{k>N} f\bigl(O_{1/n}(x_k)\bigr).$$

By the above, $\operatorname{dist}(x_k, f^{-1}(y_{k_i})) > 1/n$, hence, $y_{k_i} \notin f(O(z)) = F \cup O$. This contradicts the condition $y_{k_i} \to y_k$: the open set $O \setminus F$ contains y_k but not y_{k_i}. □

We now turn to the proof of Theorem 3.2. By lemma above f is a $cl_Y f(Z_n)$-closed extendable function, hence we conclude from the Lemma 3.1 for $Z = Z_n$, $T_n = \bigcup_{y \in Y} T_y$ and $Y_n = cl_Y f(Z_n)$ that $f|T_n$ is a closed function onto a subset Y_n that is closed in Y and finally that f is an open function at every point of the set D_0. □

4. Open-$\mathbf{G} \cup \mathbf{F}$ functions in separable metric spaces

Let X and Y be separable metric spaces; hence, there exists a metric compact space $K \supset X, Y$.

First, one can observe that the proof of Lemma 3.1 remains valid if we replace $\mathbf{C}$ by K and "half-open" by "open-$\mathbf{G} \cup \mathbf{F}$" and the following result can be proved in the same way as Theorem 3.1 if we consider a metric compact K instead of $\mathbf{C}$.

Theorem 4.1. *Let $f : X \to Y$ be an open-$\mathbf{G} \cup \mathbf{F}$ function with compact fibers of separable metric spaces X, Y. Then there exist subsets $T_n \subset X$ $(n = 1, 2, \ldots)$ such that every restriction $f|T_n$ is a closed function onto a subset $f(T_n) = Y_n$ that is closed in Y, and $f|f^{-1}(Y_0)$, where $Y_0 = Y \setminus \bigcup_{n=1}^{\infty} Y_n$ is an open function onto Y_0.*

Corollary 4.1. *Let $f : X \to Y$ be a continuous function with compact fibers of separable metric spaces X, Y. Let $Z \subset X$ be a subspace such that $f(Z) = Y$ and the restriction $f|Z$ be an open-$\mathbf{G} \cup \mathbf{F}$ function onto Y.*

Then,

(a) *there exist closed $T_n \subset X$ $(n = 1, 2, \ldots)$ such that every restriction $f|T_n$ is a closed function onto a closed subset $Y_n \subset Y$;*
(b) *there is a G_δ-set $T_0 \subset X$ such that the restriction $f|T_0$ is a closed function onto a G_δ-set $Y_0 = Y \setminus \bigcup_{n=1}^{\infty} Y_n$.*

Proof. Indeed, it is easily seen that if $f : X \to Y$ is a continuous function and there is a subset $Z \subset X$ such that the restriction $f|Z$ is closed and $f(Z) = Y$, then Z is closed in X.

It is clear that Y_0 in Theorem 4.1 is a G_δ-set in Y.

According to Novikov's Lemma [5, Lemma 1], the set

$$L = \bigcup_{y \in Y} cl_X\bigl(f^{-1}(y) \cap Z\bigr)$$

is a G_δ-set in X.

It is easy to check that $f|L$ is a half-open function with compact fibers onto Y.

By [5, Theorem 2], there is a subset T_0, closed in $f^{-1}(Y_0)$, such that $f|T_0$ is a closed function onto Y_0. □

By analogy with to [7, Theorem 7], using the classical Taimanov–Saint Raymond theorem on the preservation of Borel classes by closed functions, we can easily deduce from Corollary 4.1 (for $\alpha > 1$) the following:

Corollary 4.2. *Let $f : X \to Y$ be a continuous function with compact fibers between subspaces X, Y of the Cantor set $\mathbf{C}$. Let $Z \subset X$, $f(Z) = Y$, and the restriction $f|Z$ be a half-open function onto Y. If X is a Borel set of additive or multiplicative class α in $\mathbf{C}$, then Y is a Borel set of the same class in $\mathbf{C}$.*

For $\alpha = 1$ this corollary follows from [8, Theorem 1].

5. G ∪ F-measurable functions in separable metric spaces and the Cantor set C

By Egorov's theorem and Luzin's theorem in the classical measure theory, for every Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}$, there exist X_n such that $X = \bigcup_n X_n$, $\mu(A_0) = 0$, and each $f|X_n$ is continuous.

In this connection, Luzin posed the question of whether, for arbitrary $X, Y \subset \mathbf{R}$ and B-measurable function $f : X \to Y$, there exist X_n such that $X = \bigcup_n X_n$ and each $f|X_n$ is continuous.

The following theorem is an analogue of Theorem 3.1. Its proof is somewhat simpler than the proof of Theorem 3.1.

Theorem 5.1. *Let $f : X \to Y$ be a $\mathbf{G} \cup \mathbf{F}$-measurable function of separable metric space X, Y. Then X can be covered by countably many subsets $T_n \subset X$, $n = 0, 1, 2, \ldots$, such that every restriction $f|T_n$ is continuous.*[3]

Proof. We will suppose without loss of generality that Y is a subset of the Cantor set $\mathbf{C}$.[4]

For every $n = 1, 2, \ldots$, denote

$$X_n = \bigl\{x \in X: \text{ there are } x_i \in X \text{ such that } x_i \to x \text{ and } \operatorname{dist}\bigl(f(x), f(x_i)\bigr) > 1/n\bigr\}. \qquad (**)$$

In order to prove Theorem 5.1 we need the following lemma:

Lemma 5.1. *Every restriction $f|X_n$ is a continuous function* $(n > 0)$.

Proof. Suppose the contrary; then there exist x and $x_j \to x$, where $x, x_j \in X_n$, such that $f(x_j) \not\to f(x)$.

Let $y^* \in \mathbf{C}$ be an accumulation point for the points $f(x_j)$; then we can suppose that $f(x_j) \to y^* \in \mathbf{C}$ and, for some $d > 0$, we have $\operatorname{dist}(y^*, f(x)) > d$.

Take a clopen (in $\mathbf{C}$) neighborhood $O_\delta(y^*)$ of y^*, where $\delta = \min\{d/2, 1/2n\}$. Then, for a subset $D = O_\delta(y^*) \cap Y$ that is clopen in Y, we can suppose that $f(x_j) \in D$ and $f(x) \notin D$.

Since D is clopen in Y, $f^{-1}(D)$ is a union of a closed set F and an open set O.

According to $(**)$, every x_j is a limit point for some sequence $\{x_{j_i}\}$ such that $\operatorname{dist}(f(x_j), f(x_{j_i})) > 1/n$; hence, $f(x_{j_i}) \notin D$.

Since $x_j \in f^{-1}(D)$ and x_j is a limit point for $x_{j_i} \notin f^{-1}(D)$, we have $x_j \notin O$. Hence $\{x_j\}$ is a subset of the closed set $F \setminus O$. This contradicts the fact that $x_j \to x$; hence, $x \in F$ and $f(x) \notin f(D) \supset f(F)$. □

[3] It is easy to see that we can assume T_n to be disjoint, taking $X_n = T_n \setminus \bigcup_{k<n} T_k$.

[4] If X and Y are Borel subsets of $\mathbf{C}$, our assertion follows from [1]. But the proof of Theorem 5.1 gives a little more: we can only suppose that preimages of elements of some *clopen* base of Y are union of open and closed sets.

I do not know whether the hypothesis at the beginning that (1) implies (2) is still true for Borel (or analytic) $X, Y \subset \mathbf{C}$ even if f is continuous.

The task is now to prove Theorem 5.1 for $n=0$. According to $(**)$, for every $x \in X \setminus \bigcup_n T_n$, where $T_n = X_n$, the condition $x_i \to x$ implies $\operatorname{dist}(f(x), f(x_i)) = 0$, and hence f is continuous at every point of $T_0 = X \setminus \bigcup_n X_n$. This finishes the proof of Theorem 5.1. □

Note, that an analysis similar to that applied in the proof of Theorem 3.1 shows that X_n can be supposed to be closed subsets of X $(n = 1, 2, \ldots)$ and X_0 can be supposed to be a G_δ-set in X.

Corollary 5.1. *Let $f : X \to Y$ be a $\mathbf{G} \cup \mathbf{F}$-measurable function of the separable metric spaces X, Y. Then X can be divided into countably many subsets H_n such that f is continuous on each of these sets.*

Proof. Indeed, let us define $H_1 = T_1$ and $H_n = T_{n+1} \setminus T_n$, $n > 0$. Obviously, f is continuous on H_n $(n = 1, 2, \ldots)$ and on $H_0 = X \setminus \bigcup_k X_k$. It is clear that $H_i \cap H_k = \emptyset$ if $i \neq k$ and $\bigcup_{i=0}^{\infty} H_i = X$. □

Corollary 5.2. *Let $f : X \to Y$ be a one-to-one function of the separable metric spaces X, Y such that f and f^{-1} are $\mathbf{G} \cup \mathbf{F}$-measurable. Then X is a countable homeomorphism; i.e., X can be divided into countably many sets X_n such that all restrictions $f|X_n$ are homeomorphisms.*

Proof. Indeed, let us take the sets $H_n^X \subset X$ and $H_k^Y \subset Y$ such that each of $f|H_n^X$ and $f^{-1}|H_k^Y$ is a one-to-one continuous function.

Then every restriction $f|(X_n)$, where $X_n = H_n^X \cap f^{-1}(H_k^Y)$, is a homeomorphism. □

6. Closed-$\mathbf{G} \cup \mathbf{F}$ functions in metric spaces

Note that, while proving Theorem 3.1, we have also proved the following Theorem 6.1 for arbitrary metric spaces; moreover its proof is simpler than that of Theorem 3.1: we do not need to consider a limit point $z \in \mathbf{C}$ for the points x_k.

Theorem 6.1. *Let $f : X \to Y$ be a closed-$\mathbf{G} \cup \mathbf{F}$ function of the metric spaces X, Y.*[5] *Then there are subsets $T_n \subset X$ $(n = 1, 2, \ldots)$ such that every restriction $f|T_n$ is a closed function onto a subset Y_n, that is closed in Y, and $f|f^{-1}(Y_0)$, where $Y_0 = Y \setminus \bigcup_{n=1}^{\infty} Y_n$, is an open function onto Y_0.*

For more details (about open-$\mathbf{G} \cap \mathbf{F}$ functions and $\mathbf{G} \cap \mathbf{F}$-measurable functions) we refer the reader to [9] and [10].

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[5] Note that we did not really have to use in Theorems 3.1 and 6.1 the metrizability of Y: it is sufficient to suppose that Y is a regular space with the first axiom of countability.