Topology and its Applications 159 (2012) 3229–3233

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Countably open and closed functions

Alexey Ostrovsky

Helmut-Kaeutner Str. 25, 81739 Munich, Germany

ARTICLE INFO

Article history:
Received 18 August 2010
Received in revised form 25 June 2012
Accepted 25 June 2012

MSC:
54C10
26A21
54E40
54H05

Keywords:
Open function
Closed function

ABSTRACT

It is obvious that every open function $f : X \to Y$ has for every $y \in Y$ a cover ε_y of the subspace $f^{-1}(y)$ by singletons such that

(∗) every open neighborhood of every $x \in \varepsilon_y$ also contains $x' \in \varepsilon_{y'}$ for every point y' from some neighborhood of y.

If x (and x') in (∗) can be two-point set with the same image, we obtain a simple generalization of the notion of open function. In this case we prove that there exist $X_i \subset X$ $(i = 1, 2, \ldots)$ such that each restriction $f|X_i$ is an open function onto $f(X_i)$ and the sets $f(X_i)$ cover Y.

1. Introduction

Our purpose is to introduce the notion of a 2-open (2-closed) function[1] $f : X \to Y$ and to prove that for every such function there is a subset $X^* \subset X$ for which $f(X^*) = Y$ and the restriction $f|X^*$ is countably open (countably closed).

A function $f : X \to Y$ between topological spaces is called *countably open* (*resp. countably closed*) if X has a countable disjoint cover $\mathcal{C}$ such that for every set $C \in \mathcal{C}$ the restriction $f|C : C \to f(C)$ is an open (closed) function onto $f(C)$.

The following idea of generalization of closed functions is based on a property of so-called y-universal sets [2]: their intersections with the preimages of points $y \in Y$ are closed sets.

Definition 1.1. A function $f : X \to Y$ between topological spaces is *n-open* (resp. *n-closed*) if for every $y \in Y$ there is a cover ε_y of the subspace $f^{-1}(y)$ by sets of cardinality $\leqslant n$ (resp. by $\leqslant n$ closed sets) such that

- for each point $y \in Y$, a set $B \in \varepsilon_y$ and a neighborhood $O_B \subset X$ of B there is a neighborhood $O_y \subset Y$ of y such that for each $y' \in O_y$ there is a set $B' \in \varepsilon_{y'}$ with $B' \subset O_B$.

It can be shown that a function is 1-open (resp. 1-closed) if and only if it is open (resp. closed) in the standard sense: the image $f(W)$ of every open (resp. closed) set $W \subset X$ is an open (resp. closed) set.

E-mail address: Alexey.Ostrovskiy@UniBw.de.

[1] All functions in this paper are assumed to be surjective.

0166-8641/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.topol.2012.06.011

2. Some properties of *n*-open functions in metrizable spaces

It follows immediately from the proof of [2, Theorem 1] that for every continuous n-open function $f: X \to f(X)$, defined on a Polish space X, there exists a subset $X_0 \subset X$ such that the restriction $f|X_0$ is a closed function with compact fibers onto Y.

This statement remains valid if we replace "X is Polish" by "f is an n-to-one function" [1].

Remark 2.1. 1. Definition 1.1 forces ε_y to satisfy all the conditions for ε_y in [4, Definition 1′].

2. Obviously, item • in Definition 1.1 is equivalent to the following:

- for each point $y \in Y$, a set $B \in \varepsilon_y$ and a neighborhood $O_B \subset X$ of B there is a neighborhood $O_y \subset Y$ of y such that for each $y' \in O_y \setminus \{y\}$ there is a set $B' \in \varepsilon_{y'}$ with $B' \subset O_B$. □

Given an n-open function $f: X \to Y$ between metrizable separable spaces, we will produce below two modifications $\tilde{\varepsilon}_y$ and $\hat{\varepsilon}_y$ of each family ε_y analogously to ones in [4].

Let us consider for every $y \in Y$ the following family $\tilde{\varepsilon}_y \supset \varepsilon_y$ of all nonempty subsets $B \subset f^{-1}(y)$ satisfying the conditions of above Definition 1.1:

$\tilde{\varepsilon}_y = \{B \subset f^{-1}(y)$: $|B| \leqslant n$ and for every open $U \supset B$ there is a neighborhood $O(y)$ of y such that for every $y' \in O(y) \setminus \{y\}$ there is $B' \in \varepsilon_{y'}$ for which $B' \subset U\}$.

If every B is a finite set, for every $B_0 \in \tilde{\varepsilon}_y$ there is a minimal subset $B_m \subset B_0$, such that

- $B_m \in \tilde{\varepsilon}_y$;
- for every $B' \in \tilde{\varepsilon}_y$, if $B' \subset B_m$ then $B' = B_m$.

Denote by $\hat{\varepsilon}_y$ the family of all minimal subsets $B_m \in \tilde{\varepsilon}_y$:

$$\hat{\varepsilon}_y = \{B \in \tilde{\varepsilon}_y\colon \forall B' \in \tilde{\varepsilon}_y \ (B' \subset B \Rightarrow B' = B)\}.$$

Elements of the family $\hat{\varepsilon}_y$ are minimal elements of the family $\tilde{\varepsilon}_y$ and are called *poles*.

The following lemma is an analogue of [4, Lemma 4(ii)].

Lemma 2.1. *For every k-pole $D = \{x_1, \dots, x_k\} \subset f^{-1}(y)$, there exist in X exactly k arbitrarily small, disjoint balls $O(x_1), \dots, O(x_k)$ centered at points $x_1, \dots, x_k$ such that if $B \in \tilde{\varepsilon}_y$ and $B \subset \bigcup_{i=1}^{k} O(x_i)$ then B intersects all $O(x_i)$.*

Proof. Suppose the opposite. Without loss of generality we can suppose that for $k-1$ points $\{x_1, \dots, x_{k-1}\}$ every set $\bigcup_{i=1}^{k-1} O_{1/p}(x_i) \cap f^{-1}(y)$, where $O_{1/p}(x_i)$ is a $1/p$-ball centered at x_i, contains an elements $B_p \in \tilde{\varepsilon}_y$ for every $p = 1, 2, \dots$.

Since $f(B_p) = y$, for the neighborhood $U = \bigcup_{i=1}^{k-1} O_{1/p}(x_i)$ of B_p there is an open ball $O(y)$ centered at y such that for every $y' \in O(y) \setminus \{y\}$ there is $B' \in \varepsilon_{y'}$ for which $B' \subset U$.

Hence, by definition of $\tilde{\varepsilon}_y$, the set of points $\{x_1, \dots, x_{k-1}\}$ is an element of $\tilde{\varepsilon}_y$ and simultaneously a subset of $D = \{x_1, \dots, x_k\}$ that contradicts the assumption that $D \in \hat{\varepsilon}_y$ is a minimal element. □

The following Corollary 2.1 will be used in the proof of Theorem 3.1.

Corollary 2.1. *Let $f: X \to Y$ be a 2-open function between metrizable separable spaces. Then for every two-pole $D = \{x_1, x_2\} \subset f^{-1}(y)$, there exist two arbitrarily small, disjoint neighborhoods $O(x_1)$, $O(x_2)$ of points x_1, x_2 such that the set $O(x_1) \cup O(x_2) \cap f^{-1}(y)$ contains only two-poles and they intersect with both $O(x_1)$ and $O(x_2)$.* □

3. 2-open functions

Given a 2-open function $g: X \to Y$ between metrizable separable spaces, denote by X^* the union of all poles (namely, two-poles and one-poles) of g and denote by $f = g|X^*$ the restriction of g to X^*. It is easy to check, that f is a 2-open function relative to families $\tilde{\varepsilon}_y$.

A subset A of a topological space X is *locally closed* if it can be written as the intersection $A = U \cap F$ of open and closed sets $U, F \subset X$, respectively. It is clear that a set is locally closed if and only if it is open in its closure.

Theorem 3.1. *For a 2-open function $f: X^* \to Y$ between metrizable separable spaces, there is a countable family $\mathcal{L}$ of subsets of the union $X^* = \bigcup\{\hat{\varepsilon}_y\colon y \in Y\}$ of all poles of f such that:*

(1) *for each set* $L \in \mathcal{L}$ *the restriction* $f|L : L \to f(L)$ *is an open function onto its image* $f(L)$ *which is a locally closed set in* Y;
(2) *the set* $Y_0 = Y \setminus \bigcup_{L\in\mathcal{L}} f(L)$ *is of type* G_δ *in* Y *and for the set* $X_0 = X^* \cap f^{-1}(Y_0)$ *the restriction* $f|X_0 : X_0 \to Y_0$ *is an open surjective function.*

Proof. For any disjoint open subsets U_1, U_2 of X, consider the set $F = F(U_1, U_2) \subset Y$ of all points $y \in Y$ such that

- there is a set $B \in \tilde{\varepsilon}_y$ with $B \subset U_1 \cup U_2$;
- each set $B \in \tilde{\varepsilon}_y$ with $B \subset U_1 \cup U_2$ meets both sets U_1 and U_2.

For every $i \in \{1, 2\}$ consider the set

$$L_i = L_i(U_1, U_2) = U_i \cap \bigcup_{y\in F(U_1,U_2)} \bigcup\{B \in \tilde{\varepsilon}_y\colon\ B \subset U_1 \cup U_2\}.$$

In order to prove Theorem 3.1 we need the following Lemmas 3.1 and 3.2.

Lemma 3.1. *The restrictions* $f|L_1 : L_1 \to F$ *and* $f|L_2 : L_2 \to F$ *are open.*

Proof. Take any point $x_1 \in L_1$, and any neighborhood $V_1 \subset U_1$ of x_1 in X. By the definition of the set L_1 there is a set $B \in \tilde{\varepsilon}_y$, $y \in F$, with $x_1 \in B \subset U_1 \cup U_2$. By the definition of the set F, the set B intersects both sets U_1 and U_2. Since $x_1 \in B$ and $|B| = 2$, we conclude that $B \subset V_1 \cup U_2$. Since $B \in \tilde{\varepsilon}_y$, the point y has a neighborhood $O_y \subset Y$ such that for each $y' \in O_y$ there is a set $B' \in \varepsilon_{y'}$ with $B' \subset V_1 \cap U_2$.

We claim that $O_y \cap F \subset f(L_1 \cap V_1)$. Indeed, for every $y' \in O_y \cap F$ there is a set $B' \in \varepsilon_{y'}$ with $B' \subset V_1 \cup U_2 \subset U_1 \cup U_2$. Since $y' \in F$ and $B' \in \varepsilon_{y'} \subset \tilde{\varepsilon}_{y'}$, the set B' meets both sets U_1 and U_2 by the definition of the set F. Choose any point $x' \in B' \cap U_1 = B' \cap V_1$ and observe that $x' \in L_1 \cap V_1$ and $y' = f(x') \in f(L_1 \cap V_1)$. □

Lemma 3.2. *For any disjoint open subsets* $U_1, U_2 \subset Y$ *the set* $F = F(U_1, U_2)$ *is locally closed in* Y.

Proof. Let us take for every $y \in F$ a neighborhood $O_y \subset Y$ such that for each $y' \in O_y$ there is $B' \in \varepsilon_{y'}$ with $B' \subset U_1 \cap U_2$.

Denote

$$O_F(U_1, U_2) = \bigcup\{O_y\colon\ y \in F\}.$$

Then F is a closed subset of the open set $O_F(U_1, U_2)$.

Indeed, suppose the opposite, that there exist $y' \in O_y$ and $y_i \in F$ $(i = 1, 2, \ldots)$ for which $y_i \to y'$ and $y' \notin F$, hence $\exists B' \in \varepsilon_{y'}$ for which the condition $B' \subset f^{-1}(y') \cap (U_1 \cup U_2)$ implies that $B' \cap U_1 = \emptyset$ or $B' \cap U_2 = \emptyset$.

If, for example, $B' \cap U_1 = \emptyset$, then $B' \subset U_2$ and by definition of a 2-open function, there is $y_i \in Int(f(U_2))$ such that $f^{-1}(y_i) \cap U_2$ contains some elements of ε_{y_i} contradicting the condition that $y_i \in F$. □

Fix a countable base $\mathcal{B}$ of the topology and consider the countable family $\mathcal{L} = \{L_1(U_1, U_2), L_2(U_1, U_2)\colon\ U_1, U_2 \in \mathcal{B}, U_1 \cap U_2 = \emptyset\}$ of subsets of X, which satisfies the condition (1) of Theorem 3.1 according to Lemmas 3.1 and 3.2. Lemma 3.2 implies that the set $Y_0 = Y \setminus \bigcup_{L\in\mathcal{L}}$ is of type G_δ in Y. The definition of the sets $F(U_1, U_2)$ guarantees that the set $X_0 = X^* \cap f^{-1}(Y_0)$ is a union of one-poles of the function f, which implies that the restriction $f|X_0 : X_0 \to Y_0$ is an open function. This completes the proof of Theorem 3.1. □

The following Corollary 3.1 is immediate from Theorem 3.1.

Corollary 3.1. *Let* $g : X \to Y$ *be a* 2*-open function. Then there exists a subset* $Z \subset X$ *such that the restriction* $g|Z$ *is a countably open function onto* Y.

4. 2-closed functions

The following equivalent definition of n-closed functions shows that the proof of Lemma 4.1 below can also be obtained by modifying only slightly the method of [4, Lemma 5].

We recall that a function $f : X \to Y$ is n-closed if for every $y \in Y$ there is a family ε_y of nonempty closed subsets of X such that

- $f^{-1}(y) = \bigcup \varepsilon_y$;
- $|\varepsilon_y| \leqslant n$;
- for every open in X subset $O_B \supset B \in \varepsilon_y$ there exists a neighborhood $O(y)$ of y such that for every $y' \in O(y)$ there exists $B' \in \varepsilon_{y'}$ with $B' \subset O_B$.

Theorem 4.1. *Let $f: X \to Y$ be a 2-closed function between normal spaces. Denote $M_y = \bigcap \varepsilon_y$ and $M = \bigcup_{y \in Y} M_y$. Then the restrictions $f|M$ and $f|f^{-1}(Y \setminus f(M))$ are closed functions.*

Proof. The proof of the theorem is based essentially on the following lemma.

Lemma 4.1. (a) *For every $y \in Y \setminus f(M)$ there is its neighborhood $O(y) \subset Y \setminus f(M)$ such that $f|f^{-1}(O(y))$ is a closed function onto $O(y)$;*

(b) *$f|M$ is a closed function onto a closed in Y subset $f(M)$.*

Proof. (a) For every $y \in Y \setminus f(M)$ let us consider pairwise disjoint neighborhoods $O(K)$, $K \in \varepsilon_y$, and define

$$O_K(y) = \{y' \in Y:\ \exists K' \in \varepsilon_{y'},\ K' \subset O(K)\},$$
$$O(y) = \bigcap_{K \in \varepsilon_y} O_K(y).$$

Obviously, $O(y) \subset Y \setminus f(M)$ and that $f|f^{-1}(O(y))$ is a 1-closed functions (with the cover of $f^{-1}(y)$ by $\bigcup \varepsilon_y$) onto $O(y)$.

The proof of (b) in fact reproduces [6, Lemma 3]:

If M_y is nonempty and $V \supset M_y$ is an open set, then for every $K \in \varepsilon_y$ the set $B_K = K \setminus V$ is closed and

$$\bigcap_{K \in \varepsilon_y} B_K = \emptyset.$$

Since X is a normal space and $|\varepsilon_y| < \aleph_0$, there exist the open sets $W_K \supset B_K$ such that

$$\bigcap_{K \in \varepsilon_y} W_K = \emptyset.$$

Obviously, every $W_K \cup V \supset K$ is open and according to the definition of 2-closed functions, there exists an open set $U_K(y)$ such that, for every $y' \in U_K(y)$, there is $K' \in \varepsilon_{y'}$ such that $W_K \cup V \supset K'$.

Since $|\varepsilon_y| < \aleph_0$, the set $U(y) = \bigcap_{K \in \varepsilon_y} U_K(y)$ is open and we have for every point $y' \in U(y) \cap f(M)$

$$M_{y'} = \bigcap \varepsilon_{y'} \subset \bigcap_{K \in \varepsilon_y} (W_K \cup V) = \left(\bigcap_{K \in \varepsilon_y} W_K\right) \cup V = V.$$

Let us denote $g = f|M: M \to f(M)$.

In summary it can be said, therefore, that if $y \in f(M)$ and $V \supset g^{-1}(y)$ is an open set, then for every point $y' \in U(y) \cap f(M)$ we have $V \supset g^{-1}(y')$. Hence, g is a closed function.

It follows from item (a) of Lemma 4.1 that $Y \setminus f(M)$ is open in Y; hence, $f(M)$ is closed in Y. □

Let us return to the proof of Theorem 4.1.

According to item (a) of Lemma 4.1 f is closed at every point $y \in Y \setminus f(M)$, hence the restriction $f|f^{-1}(Y \setminus f(M))$ is a closed function onto $Y \setminus f(M)$. According to item (b) of Lemma 4.1 $f|M$ is a closed function onto $f(M)$. □

A countably closed function $f: X \to Y$ with compact fibers is called *countably perfect*.

Corollary 4.1. *Let $f: X \to Y$ be a continuous,* 2*-closed function between metrizable separable spaces. Then there is $X^* \subset X$ such that the restriction $f|X^*$ is a countably perfect function onto Y.*

Proof. Indeed, by Theorem 4.1 the restriction of f to $f^{-1}(Y \setminus f(M)) \cup M$ is a countably closed function onto Y.

Taimanov proved [8, Lemma 2] that if $f: X \to Y$ is a continuous and closed function, then there exists a countable set $Y_0 \subset Y$ such that for every point $y \in Y \setminus Y_0$ its fiber $f^{-1}(y)$ is compact.

Hence, there exists a countable set $Y_0 \subset Y$ such that for $X_1^* = (f^{-1}(Y \setminus f(M)) \cup M) \setminus f^{-1}(Y_0)$ the restriction $f|(X_1^*)$ is a countably perfect function onto $Y \setminus Y_0$.

Obviously, for the set $X_2^* = \{x_i \in f^{-1}(y_i):\ y_i \in Y_0\}$ the restriction $f|X_2^*$ is a countably perfect, one-to-one function onto Y_0. Finally, define $X^* = X_1^* \cup X_2^*$. □

If $n < \aleph_0$, then the conclusions of Theorem 3.1 and Corollary 4.1 remain valid for n-closed and n-open functions.

For n-open functions, this follows from the consideration of poles.

For n-closed functions the proof in fact reproduces [6, Lemma 3].

If $n = \aleph_0$ and each fiber $f^{-1}(y)$ is compact, then the question of whether the conclusion of Corollary 4.1 is true remains an open question [5, p. 230].

Finally note that we have not really used in Theorems 3.1 and 4.1 the continuity of functions. A set-valued reformulation of Theorem 3.1 can be made using [3, Theorem 2.4] and its analogue [7].

References

[1] G. Debs, J. Saint Raymond, Compact covering properties of finite-to-one mappings, Topology Appl. 81 (1997) 55–84.
[2] A. Ostrovsky, Images of absolute G_δ-sets and sets close to them by k-covering mappings, PhD dissertation, MGU, Moscow, 1975, pp. 1–45.
[3] A. Ostrovsky, Set-valued stable maps, Topology Appl. 104 (2000) 227–236.
[4] A. Ostrovsky, s-covering maps with complete fibers, Topology Appl. 102 (2000) 1–11.
[5] A. Ostrovsky, Maps of Borel sets, Proc. Steklov Inst. Math. 252 (2006) 225–247.
[6] A. Ostrovsky, Finite-to-one continuous s-covering mappings, Fund. Math. 194 (2007) 89–93.
[7] S. Nedev, J. Pelant, V. Valov, A non-separable Christensen's theorem and set tri-quotient maps, Topology Appl. 156 (2009) 1234–1240.
[8] A.D. Taimanov, On closed mappings. II, Mat. Sb. 52 (1960) 579–588 (in Russian).