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Set-valued stable maps

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Abstract

Various classes of maps (stable, transquotient, set-valued triquotient, harmonious, point-harmonious) are studied. It is proved that compositions of finitely many closed and open maps preserve consonance. © 2000 Published by Elsevier Science B.V. All rights reserved.

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0. Stable and transquotient maps

All maps in this section are continuous and onto and all the spaces are regular. Let $\mathcal{O}_X$ be the family of all open subsets of X. A *system associated* with a map $f : X \to Y$ is a family $(\eta_y)_{y \in Y}$, $\eta_y \subset \mathcal{O}_X$, satisfying $U \cap f^{-1}(y) \neq \emptyset$ for all $U \in \eta_y$. The system is said to be *transmittable* if the following condition holds:

(0.1) For every $U \in \eta_y$ there is a neighborhood $O(y)$ of y such that $U \in \eta_{y'}$ for every $y' \in O(y)$.

It is clear that if $U \in \eta_y$ then $y \in \operatorname{Int} f(U)$. Notice, that we can define for one map many transmittable systems, for example, we obtain a trivial transmittable system, if $\eta_y = \{X\}$ for every $y \in Y$.

Definition 0.1. The map $f : X \to Y$ is stable if it admits a transmittable system satisfying the following condition:

(0.2) If $U \in \eta_y$ and V is open in X such that $V \supset U \cap f^{-1}(y)$, then $V \in \eta_y$.

Definition 0.2. A map $f : X \to Y$ is transquotient (respectively, triquotient) if it admits a transmittable system satisfying the following condition:

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(0.3) If $U \in \eta_y$, $\gamma \subset \mathcal{O}_X$ and $\bigcup \gamma \supset f^{-1}(y) \cap U$, then there exist $U_{\alpha_1}, \ldots, U_{\alpha_n} \in \gamma$ and an open subset $O(y)$ such that $\bigcup_{i=1}^n U_{\alpha_i} \in \eta_{y'}$ for every $y' \in O(y) \setminus \{y\}$ (respectively, for every $y' \in O(y)$).

It is clear that every triquotient map is transquotient. Example 2 of [12] shows that the converse is false even for Polish spaces.

We collect in this section a few general properties of stable maps which were established in [12] (see also the results was proved recently by Michael [7]).

Remark 0.3. Let η_y be a transquotient family for f. Define a new family for f:

$$\tilde{\eta}_y = \{V \in \mathcal{O}_X\colon \exists U \in \eta_y, V \supset U \cap f^{-1}(y)\}.$$

Note that the system $\{\tilde{\eta}_y\}_{y \in Y}$ is transmittable since for $V \in \tilde{\eta}_y$ we can consider $\gamma = \{V\}$ in (0.3) and then, obviously, $V \in \tilde{\eta}_{y'}$ for every $y' \in O(y)$. (Since every cover of $f^{-1}(y) \cap V$ is a cover of $f^{-1}(y) \cap U$ then by (0.3) $\tilde{\eta}_y$ is a transquotient family for f.)

If V is an open subset of X and there is $U \in \tilde{\eta}_y$ such that $V \supset f^{-1}(y) \cap U$ (in particular, if $V \supset f^{-1}(y)$), then, obviously, V is also an element of $\tilde{\eta}_y$. Hence the transquotient map f is stable. □

Lemma 0.4. *A map between separable metric spaces is stable if and only if it is transquotient.*

Proof. According to Remark 0.3 we only need to show that every stable map $f: X \to Y$ is transquotient. Suppose the family η_y satisfies Definition 0.1 and does not satisfy Definition 0.2. This means that (0.3) fails for some $U \in \eta_y$ and family $\gamma = \{U_i\}_{i \in \omega}$. It is obvious that there is an open decreasing base $\{O_n(y)\}_{n \in \omega}$ at y such that for every $\gamma_n = \bigcup_{i=0}^n U_i$ there is a $y'_n \in O_n(y) \setminus O_{n+1}(y)$, for which $\gamma_n \notin \eta_{y'_n}$ and, hence,

$$M_n = (U \cap f^{-1}(y'_n)) \setminus \gamma_n \neq \emptyset$$

(we assume that $O_0(y)$ is such a neighborhood of y that $U \in \eta_{y'}$ for each $y' \in O_0(y)$).

Let $M = \bigcup\{M_n\colon n \in \omega\}$. Now let us show that $f^{-1}(y) \cap U \cap \mathrm{cl}_X M \neq \emptyset$. Suppose the contrary, then $U' = U \setminus \mathrm{cl}_X M \supset f^{-1}(y) \cap U$ and $U' \in \eta_y$. By the transmittable property, $U' \in \eta_{y'_n}$ for some n. By our construction, $\gamma_n \supset \gamma_n \cap U \cap f^{-1}(y'_n) = U' \cap f^{-1}(y'_n)$, and hence $\gamma_n \in \eta_{y'_n}$. That is impossible.

If $x \in f^{-1}(y) \cap U \cap \mathrm{cl}_X M$, then x belongs to some γ_n. It is clear that $x \in T = f^{-1}(y) \cap U \cap \mathrm{cl}_X(\bigcup\{M_k\colon k > n\})$. Taking into consideration that $M_k \subset U \setminus \gamma_n$ for $k > n$, and hence $T \cap \gamma_n = \emptyset$, we get a contradiction with $x \in \gamma_n$. □

Recall that a map $f: X \to Y$ is said to be *harmonious* if one can assign to every compact set $K \subset Y$ a family η_K of open subsets of X satisfying to following conditions:

(0.4) If $U \in \eta_K$, then there exists a compact set $B \subset U$ such that $f(B) = K$ and for every open set $V \supset B$ we have $V \in \eta_K$.

(0.5) If $U \in \eta_K$, then there exists an open set $O(K)$ such that $U \in \eta_{K'}$ for every compact set $K' \subset O(K)$.

We obtain the definition of a *point-harmonious* map by replacing K and K' in the above definition by one-point compact sets $\{y\}$ and $\{y'\}$. It is easy to see that every point-harmonious map is triquotient. Every stable map $f: X \to Y$ with compact fibres is point-harmonious [12].

1. Some remarks on consonant spaces

Consonant spaces were introduced by Dolecki, Greco and Lechnicki in [3] (see also [8]).

Definition 1.1. A family $\mathcal{Q} = \{Q_\alpha\}_{\alpha \in A}$ of (arbitrary) subsets of X is said to be compact if for every $Q_\alpha \in \mathcal{Q}$ and every open cover γ of Q_α there exits a finite subfamily $\gamma_1 \subset \gamma$ and $Q_\beta \in \mathcal{Q}$ such that $\bigcup \gamma_1 \supset Q_\beta$.

Definition 1.2. A space X is consonant if for every compact family $\mathcal{Q}$ in X there exists a family $\mathcal{K}(\mathcal{Q}) = \{K_\beta\}_{\beta \in B}$ of compact subsets of X such that the following conditions are satisfied:

(1.1) For every $Q_\alpha \in \mathcal{Q}$ and every open set $U \supset Q_\alpha$ there exists $K_{\beta(\alpha)} \in \mathcal{K}(\mathcal{Q})$ such that $\mathcal{K}_{\beta(\alpha)} \subset U$.

(1.2) For every $K_\beta \in \mathcal{K}(\mathcal{Q})$ and for every open set $O \supset K_\beta$ there is $Q_\alpha \in \mathcal{Q}$ such that $O \supset Q_\alpha$.

The following theorem generalizes some results from [2,8,14] proved for open, perfect and triquotient maps.

Theorem 1.3. *A transquotient image of a consonant space is consonant.*

Proof. Let $f: X \to Y$ be a transquotient map and η_y a corresponding transquotient family. Denote

$$\mathcal{O}(\mathcal{Q}) = \bigl\{O \in \mathcal{O}_X\colon \exists Q_\alpha \in \mathcal{Q},\ Q_\alpha \subset O\bigr\}.$$

It is easy to see that $\mathcal{O}(\mathcal{Q})$ is compact if and only if for every $V \in \mathcal{O}(\mathcal{Q})$ and every open cover γ of V there is a finite subfamily $\gamma_1 \subset \gamma$ such that $\bigcup \gamma_1 \in \mathcal{O}(\mathcal{Q})$ [3].

Let $\mathcal{Q}_Y$ be a compact family in Y. Define an open family $\mathcal{Q}_X$ of X as follows:

(1.3) $\mathcal{Q}_X = \{V \in \mathcal{O}_X\colon \exists U \in \mathcal{O}(\mathcal{Q}_Y)$ such that $\forall y \in U$ there is $O(y) \subset U$ such that $f^{-1}(y) \cap V \neq \emptyset$ and $V \in \eta_{y'}$ for every $y' \in O(y) \setminus \{y\}\}$.

It is clear that $f(V) \supset U$. The family $\mathcal{Q}_X$ is compact. Indeed, let $\{V_i\colon i \in A\}$ be an open cover of $V \in \mathcal{Q}_X$. By definition of $\mathcal{Q}_X$ and Remark 0.3 we can suppose that

$$\bigcup_{i \in A} V_i \in \mathcal{Q}_X.$$

Let us chose according to (1.3) $U \in \mathcal{O}(\mathcal{Q}_Y)$ and for every $y \in U$ chose, according to definition of transquotient map and Remark 0.3, finitely many $V_{i_n}(y)$ and $O(y) \subset U$ such that $\bigcup_{i_n} V_{i_n}(y) \in \eta_{y'}$ for every $y' \in O(y) \setminus \{y\}$ and $f^{-1}(y) \cap \bigcup_{i_n} V_{i_n}(y) \neq \emptyset$. Since $\mathcal{Q}_Y$

is a compact family and $\{O(y)\colon\ y \in Y\}$ is a cover of $U \in \mathcal{O}(\mathcal{Q}_Y)$, there are finitely many $O(y_j)$ such that $\bigcup_j O(y_j) \in \mathcal{O}(\mathcal{Q}_Y)$. Then by (1.3)

$$\bigcup_j \bigcup_{i_n} V_{i_n}(y_j) \in \mathcal{Q}_X.$$

We shall verify the conditions (1.1) and (1.2) of the definition of consonant spaces.

(i) Let $U \in \mathcal{O}(\mathcal{Q}_Y)$. By Remark 0.3 and (1.3) $f^{-1}(U) \in \mathcal{Q}_X$. The space X is consonant, hence, there is a compact set $K \subset f^{-1}(U)$, $K \in \mathcal{K}_X(\mathcal{Q})$ such that $V \in \mathcal{Q}_X$ for every open set $V \supset K$. Denote $K_1 = f(K)$. It is clear that K_1 is a compact subset in U. Denote $\mathcal{K}(\mathcal{Q}_Y)$ the family of all obtained compact sets K_1, then by our construction the condition (1.1) is satisfied.

(ii) For every open set $O \supset K_1$, where $K_1 \in \mathcal{K}(\mathcal{Q}_Y)$, we have $O \in \mathcal{O}(\mathcal{Q}_Y)$.

Indeed, for such an open set O we have $f^{-1}(O) \supset K$, $K \in \mathcal{K}(\mathcal{Q}_X)$ and hence $f^{-1}(O) \in \mathcal{Q}_X$. By definition of $\mathcal{Q}_X$ there is $U \in \mathcal{O}(\mathcal{Q}_Y)$ such that $U \subset O$ and hence $O \in \mathcal{O}(\mathcal{Q}_Y)$. □

Theorem 1.4. *If $f : X \to Y$ is a composition of finitely many closed and open maps and X, Y are separable metric spaces* (*the in-between spaces need not be metric*) *and X is consonant, then Y is consonant.*

Proof. It is easy to see that open maps and closed maps $f : X \to Y$ are stable. Indeed, define for an open map:

$$\eta_y = \{U \in \mathcal{O}_X\colon\ y \in f(U)\}$$

and for a closed map:

$$\eta_y = \{U \in \mathcal{O}_X\colon\ U \supset f^{-1}(y)\}.$$

(If f is closed, then for every $y \in Y$ and every open set $U \supset f^{-1}(y)$ there is an open set $O(y)$ such that $f^{-1}(O(y)) \subset U$.) Since a composition of stable maps is stable [12] (see, also Proposition 2.6), we have by Lemma 0.4 that f is transquotient. If we combine this with Theorem 1.3, we get that Y is consonant. □

Example 1 in [12] shows that the stable image of a consonant space X (even if X is a complete metrizable space) need not be consonant.

Denote by $\mathbb{C}$ and $\mathbb{Q}$, the Cantor set and the space of rational numbers, respectively.

Proposition 1.5 (Axiom of projective determinancy). *Every projective* (*in particular, analytic*) *consonant space $X \subset \mathbb{C}$ is a Polish space.*

Indeed, it was proved in [10, Lemmas 1 and 2] that for every projective set X we have:

(1.4) $\mathbb{Q}$ is a closed subspace of $X \times \mathbb{C}$; or

(1.5) X is a closed subspace of $\mathbb{C} \times (\mathbb{C} \setminus \mathbb{Q})$.

According to [3], the product of a consonant space and a compact space is consonant and consonance is hereditary with respect to closed subspaces. Bouziad proved that the space of rational numbers $\mathbb{Q}$ is not consonant [2], then (1.4) is impossible, hence X is a Polish space as G_δ in $\mathbb{C} \times \mathbb{C}$.

Remark 1.6. Let $X, Y \subset \mathbb{C}$ and X be a consonant space. I do not know if $f : X \to Y$ is inductively perfect when f is a point-harmonious map (even if X is an analytic).

According to Hausdorff, a space Y is an F_{II}-space if there are no closed subsets in Y of the first category in itself. By a theorem of Hurewicz every metric space of the first category contains a closed subspace homeomorphic to $\mathbb{Q}$ (and a separable metric space is of the first category if and only if it contains a dense G_δ-subset homeomorphic to $\mathbb{Q}$ [10, Lemma 7]). Hence, every metric consonant space is an F_{II}-space [2].

Theorem 2 of [13] yields that the map f in Remark 1.6 is inductively perfect, if every $f^{-1}(y)$ is compact, X is an F_{II}-space and for every open set $U \subset X$, $f(U)$ is a Borel (or co-analytic) set in Y. It is clear that f is also perfect if $X = Z \times Y$, f is a projection onto the second coordinate and Z is a compact set. Notice, that Aarts and Lutzer posed the following question [1]: "if Z is a compact set and Y is an F_{II}-space, must $Z \times Y$ be an F_{II}-space?" In the case of separable metric spaces the answer is affirmative even if Z is an analytic F_{II}-space [9].

2. Set-valued stable maps

Let 2^X denote the space of non-empty closed subsets of X. Recall that a set-valued map $G : Y \to 2^X$ is lower semi-continuous (l.s.c.) if for every open set $U \subset X$ the set $\{y\colon G(y) \cap U \neq \emptyset\}$ is open in Y and G is upper semi-continuous (u.s.c.) if for every open set $U \subset X$ the set $\{y\colon G(y) \subset U\}$ is open in Y.

A *system associated* with a set-valued map $G : Y \to 2^X$ is a family $(\eta_y)_{y \in Y}$ where each η_y is a family of open subsets of X satisfying $U \cap G(y) \neq \emptyset$ for all $U \in \eta_y$. The system is said to be transmittable if the following condition holds:

(2.1) If $U \in \eta_y$, then there exits a neighborhood $O(y)$ of y such that $U \in \eta_{y'}$ for every $y' \in O(y)$.

Definition 2.1. The map $G : Y \to 2^X$ is set-valued stable if it admits a transmittable system satisfying the following condition:

(2.2) If $U \in \eta_y$ and V is an open set in X such that $V \supset U \cap G(y)$ then $V \in \eta_y$.

Similar to open and closed maps we have:

Proposition 2.2. *L.s.c. and u.s.c. maps are set-valued stable.*

For an l.s.c. map define

$$\eta_y = \{U \in \mathcal{O}_X\colon U \cap G(y) \neq \emptyset\}$$

and for a u.s.c. map define

$$\eta_y = \{U \in \mathcal{O}_X\colon U \supset G(y)\}.$$

Definition 2.3. A map $G : Y \to 2^X$ is set-valued triquotient if it admits a transmittable system satisfying the following condition:

(2.3) If $U \in \eta_y$ and $\gamma = \{U_\alpha\}_{\alpha \in A}$ is an open cover of $G(y) \cap U$ then there exist $U_{\alpha_1}, \dots, U_{\alpha_n} \in \gamma$ and an open set $O(y)$ such that $\bigcup\{U_{\alpha_i}\colon i = 1, \dots, n\} \in \eta_{y'}$ for every $y' \in O(y)$.

It is easy to see that if a map $f : X \to Y$ is stable (respectively, triquotient) then the set-valued map $G : Y \to 2^X$, where $G(y) = f^{-1}(y)$ is set-valued stable (respectively, set-valued triquotient).

Theorem 2.4. *Let $\Gamma \subset Y \times X$ be the graph of a set-valued map $G : Y \to 2^X$, π_Y be the projection of $Y \times X$ onto the first coordinate and $g = \pi_Y|\Gamma$ be the restriction of π_Y to Γ. Then $g : \Gamma \to Y$ is triquotient if and only if $G : Y \to 2^X$ is set-valued triquotient.*

Proof. (1) Suppose $G : Y \to 2^X$ is a set-valued triquotient map and let us show that g is triquotient. Define $\eta_y(g) = \{U'\colon U'$ is open in Γ and $U' \supset (O(y) \times U) \cap \Gamma$ where $O(y)$ is a neighborhood of y and $U \in \eta_y(G)\}$.

First note that the transmittable property for g is satisfied. In fact, suppose that $U' \in \eta_y(g)$, hence $U' \supset (O(y) \times U) \cap \Gamma$ where $U \in \eta_y(G)$. By definition of set-valued triquotient maps there exists a neighborhood $O'(y)$ of y such that $U \in \eta_{y'}(G)$ for every $y' \in O'(y)$. Denote $O''(y) = O'(y) \cap O(y)$, then $U' \supset (O''(y) \times U) \cap \Gamma$ and hence $U' \in \eta_{y''}(g)$ for every $y'' \in O''(y)$ and the transmittable property for g is satisfied.

If γ is an open (in Γ) cover of $g^{-1}(y) \cap U'$ where $U' \in \eta_y(g)$, then one may choose for every $x \in g^{-1}(y) \cap U'$ some $L_\alpha \in \gamma$, and $R_\alpha = (O_\alpha \times U_\alpha) \cap \Gamma$, $x \in R_\alpha$ (where O_α and U_α are open sets in Y and X) such that $R_\alpha \subset L_\alpha$. By definition of set-valued triquotient maps there exists $U_{\alpha_1}, \dots, U_{\alpha_n}$ and an open set $O(y)$ such that

$$U'' = \bigcup_{i=1}^n U_{\alpha_i} \in \eta_y(G)$$

for every $y' \in O(y)$. It is clear that

$$\bigcup_{i=1}^n L_{\alpha_i} \supset \left(\left(\bigcap_{i=1}^n O_{\alpha_i}\right) \times U''\right) \cap \Gamma \in \eta_y(g)$$

for every $y' \in O(y) \cap \bigcap_{i=1}^n O_{\alpha_i}$.

(2) Suppose $g : \Gamma \to Y$ is triquotient and let us show that $G : Y \to 2^X$ is a set-valued triquotient. Define $\eta_y(G) = \{U \supset \pi_X(U')\colon U'$ is open in $X \times Y$ and $U' \cap \Gamma \in \eta_y(g)\}$.

First note that the transmittable property for G is satisfied. In fact, suppose that $U \in \eta_y(G)$. Let us choose for every $x \in g^{-1}(y) \cap U'$ some $R_\alpha = O_\alpha \times U_\alpha$, $x \in R_\alpha$ (where O_α and U_α are open in Y and X) such that $R_\alpha \subset U'$. By definition of triquotient maps there exists $R_{\alpha_1}, \dots, R_{\alpha_n}$ and an open set $O(y)$ such that for $U'' = \bigcup_{i=1}^n R_{\alpha_i}$ we have

$U'' \cap \Gamma \in \eta_{y'}(g)$ for every $y' \in O(y)$. It is clear that $U \supset \pi_X(U') \supset \pi_X(U'')$ for every $y' \in O(y)$, and hence $U \in \eta_{y'}(g)$ for every $y' \in O(y)$.

Finally if $\gamma = \{U_\alpha\}_{\alpha\in A}$ is an open cover of $G(y) \cap U$, where $U \in \eta_y(G)$, then $\{(U_\alpha \times Y) \cap \Gamma\}_{\alpha\in A}$ is an open cover of $U' \cap \Gamma \cap g^{-1}(y)$. By assumption g is triquotient, and hence there is $U'' = \bigcup_{i=1}^n (U_\alpha \times Y)$ such that $U'' \cap \Gamma \in \eta_y(g)$. Now easily follows that $\bigcup_{i=1}^n U_\alpha \supset \pi_X(U'')$ and $\bigcup_{i=1}^n U_\alpha \in \eta_y(G)$. □

The following theorem was obtained by Michael in [6]:

Theorem 2.5. *Let X be a complete metric space, Y a metric space and $G : Y \to 2^X$ an l.s.c. map. Then there exists $F : Y \to 2^X$ such that*

(1) *$F(y) \subset G(y)$ and $F(y)$ is compact for all $y \in Y$.*

(2) *F is a u.s.c. map.*

Theorem 2.6. *Let X be a complete metric space, Y be a metric space and $G : Y \to 2^X$ be a set-valued triquotient map such that $G(S)$ is a closed subspace of X for every subset S of Y of the form $S = \{y_n: n \in \omega\}$ with $y_0 = \lim y_n$. Then there exits $F : Y \to 2^X$ such that*

(2.4) *$F(y) \subset G(y)$ and $F(y)$ is compact for all $y \in Y$.*

(2.5) *F is a u.s.c. map.*

Proof. Let $\Gamma \subset Y \times X$ be the graph of a set-valued map $G : Y \to 2^X$, π_Y be the projection of $Y \times X$ onto the first coordinate and $g = \pi_Y|\Gamma$ be the restriction of π_Y to Γ. By a Theorem 2.4 g is triquotient. It is easy to prove that every $g^{-1}(S)$ is complete in the metric $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

The following statement follows from the proof of Lemma 2 [12]:

If $g : \Gamma \to Y$ is a triquotient map and $g^{-1}(S)$ is a complete subspace for every subset S of Y of the form $S = \{y_n: n \in \omega\}$ with $y_0 = \lim y_n$, then there exists a closed subset $X' \subset \Gamma$ such that the restriction $g_0 = g|X'$ is a perfect map onto Y. Now we show that the map $F : Y \to 2^X$ where $F(y) = \pi_X(g_0^{-1}(y))$ is u.s.c. First note that if $U \subset X$ is open, then $\{y: F(y) \subset U\} = \{y: g_0^{-1}(y) \subset U \times Y)$. The map g_0 is closed und the reader can easily prove that $\{y: F(y) \subset U\}$ is open, and hence F is u.s.c. □

Given two set-valued maps $\Phi : Z \to 2^Y$ and $F : Y \to 2^X$, we define the composition $G = F \circ \Phi : Z \to 2^X$ as follows:

$$G(z) = \bigcup \bigl\{F(y): y \in \Phi(z)\bigr\}.$$

Proposition 2.7. *A composition of set-valued stable maps is set-valued stable.*

In fact, let $\eta_z(G)$ be given by

$$\eta_z(G) = \bigl\{U: \exists T \in \eta_z(\Phi),\ \forall y \in T \cap \Phi(z),\ U \in \eta_y(F)\bigr\}.$$

First let us prove that $\eta_z(G)$ is a transmittable system. Let us assume that $U \in \eta_z(G)$ and $y \in T \cap \Phi(z)$. By assumption F is set-valued stable and there is an open set $O(y) \subset T$

such that $U \in \eta_{y'}(F)$ for each $y' \in O(y)$. Let $T' = \bigcup\{O(y)\colon\ y \in T \cap \Phi(z)\}$. Obviously $T' \supset T \cap \Phi(z)$, and hence $T' \in \eta_z(\Phi)$. The map Φ is set-valued stable and we may choose an open set $O(z)$ such that $T' \in \eta_{z'}(\Phi)$ for every $z' \in O(z)$ and then $U \in \eta_{z'}(G)$.

Finally suppose $V \supset U \cap G(z)$ for an open set V where $U \in \eta_z(G)$. Since F is set-valued stable, $V \in \eta_y(F)$ for every $y \in T \cap \Phi(z)$, and hence $V \in \eta_z(G)$.

3. Set-valued harmonious maps

Denote by $\mathcal{K}(X)$ the family of all compact subsets of X.

Definition 3.1. A set-valued map $G : Y \to 2^X$ is said set-valued harmonious if one can assign to every $K \in \mathcal{K}(Y)$ a family η_K of open subsets of X such that:

(3.1) If $U \in \eta_K$ then there is a non-empty $B \in \mathcal{K}(X)$ such that $G^{-1}(B) \supset K$ and for every open set $V \supset B$ we have $V \in \eta_K$.

(3.2) If $U \in \eta_K$ then there is an open set $O(K)$ such that $U \in \eta_{K'}$ for every compact set $K' \subset O(K)$.

Definition 3.2. A map $G : Y \to 2^X$ is a set-valued compact-covering if for every $K \in \mathcal{K}(Y)$ there is $B \in \mathcal{K}(X)$ such that $G^{-1}(B) \supset K$.

Theorem 3.3. *Every set-valued compact-covering map between separable metric spaces is set-valued harmonious.*

Proof. Let $K \in \mathcal{K}(Y)$. Denote $\{O_j\}_{j\in\omega}$ a decreasing open base at K. For $U \in \mathcal{O}_X$ put:

$U \in \eta_K$ if and only if there exists O_{j_0} such that for every compact set $K_1 \subset O_{j_0}$ there exists $B^1 \in \mathcal{K}(X)$ such that $B^1 \subset U$ and $G^{-1}(B^1) \supset K$.

Since G is a set-valued compact-covering, $X \in \eta_K$ and hence every family η_K is non-empty.

Let δ be a countable base of X containing all finite unions of its elements, i.e.:

$$U_1 \cup U_2 \cup \cdots \cup U_k \in \delta \quad \text{if } U_i \in \delta \text{ for } i = 1, \ldots, k.$$

It is clear that for every compact set $B_\alpha \in \mathcal{K}(X)$ and every open set $V \supset B_\alpha$ there exists a $W_i \in \delta$ such that

(3.3) $V \supset W_i \supset B_\alpha$ $(\alpha \in A,\ i \in N)$.

Suppose that the condition (3.1) of Definition 3.1 does not hold, i.e.:

there exists $U \in \eta_K$ such that for every $B_\alpha \in \mathcal{K}(X)$ with $G^{-1}(B) \supset K$, there exists an open set $V \supset B_\alpha$ such that $V \notin \eta_K$.

If $V \notin \eta_K$ then by our definition of η_K for every $W \subset V$ we have $W \notin \eta_K$. Hence, according to (3.1), for every $B_\alpha \in \mathcal{K}(X)$, take some $W_{i(\alpha)} \in \delta$ for which $W_{i(\alpha)} \supset B_\alpha$

and $W_{i(\alpha)} \notin \eta_K$. But if $W_{i(\alpha)} \notin \eta_K$ then by definition of η_K there exists a compact set $K_{i(\alpha)} \in \mathcal{K}(Y)$ in $O_{i(\alpha)}$ such that

(3.4) if $G^{-1}(T) \supset K_{i(\alpha)}$ and $T \in \mathcal{K}(X)$ then $T \not\subset W_{i(\alpha)}$.

Denote

(3.5) $K_1 = K \cup \bigcup_{\alpha \in A} K_{i(\alpha)}$.

It is clear that $O_{i(\alpha)}$ is a decreasing sequence and $K_1 \in \mathcal{K}(Y)$. By assumption $U \in \eta_K$ and a compact set $T \in \mathcal{K}(X)$ exists for which $T \subset U$ and $G^{-1}(T) \supset K_1$. It is clear that for some α_0 we have $W_{i(\alpha_0)} \supset B_{\alpha_0} = T$. It is easy to see that $G^{-1}(T) \supset K_{i(\alpha_0)}$ is impossible by (3.4). The condition (3.2) is satisfied by definition of η_K. □

A continuous map $f : X \to Y$ is compact-covering if the set-valued map $G : Y \to 2^X$, where $G(y) = f^{-1}(y)$ is set-valued compact-covering map.

Corollary 3.4. *A map between separable metric spaces is harmonious if and only if it is compact-covering.*

A new characteristic of harmonious maps, via games, was given by Debs and Saint Raymond [4, Remark 4.8].

Instead of $\mathcal{K}(Y)$ we can consider the family of all countable, compact sets in Y, define set-valued point-harmonious, set-valued s-covering maps and s-covering maps. When we obtain from the proof of Theorem 3.3 that every set-valued s-covering map between separable metric spaces is set-valued point-harmonious (about s-covering maps see also the work of Just and Wicke [5]).

Pillot proved the interesting results: each triquotient map from a consonant space is harmonious [14, Corollaire 11.4] and each triquotient map with consonant fibers is point-harmonious [14, Proposition 11.2]. The following result about set-valued point-harmonious maps is an immediate corollary from his proof: if $G : Y \to 2^X$ is a set-valued triquotient map and every $G(y)$ is consonant, then G is a set-valued point-harmonious map.

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