ELSEVIER

Topology and its Applications 102 (2000) 1–11

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

s-covering maps with complete fibers

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Received 4 September 1996; received in revised form 24 November 1997, 7 June 1998

Abstract

The purpose of this note is to study s-covering maps $f: X \to Y$ with compact or more generally with complete (in the given metric on X) fibers $f^{-1}(y)$. The question of whether f is inductively perfect has been recently negatively solved by Debs and Saint Raymond. In the first part of this paper we show that the answer is "yes" if $f^{-1}(S)$ is complete in the given metric on X for every convergent including its limit sequence $S \subset Y$. In the second part point-harmonious maps are used in considering the properties of f. We prove that if X is a Borel set of the multiplicative class α then Y is a Borel set of the multiplicative class $3+\alpha$. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Point-harmonious maps; s_α-covering maps; Inductively perfect maps; Borel sets

AMS classification: 54C20; 54H05; 54E40; 54C10

All spaces in this paper are separable and metrizable, all the maps are continuous. A map $f: X \to Y$ is s-covering or countable-compact-covering (respectively compact-covering) if every countable and compact (respectively compact) set $S \subset Y$ is the image of some compact $B \subset X$.

Since the inverse image under a perfect map of a compact set is always compact, every inductively perfect map is always s-covering.

We recall that a surjective map $f: X \to Y$ is said to be inductively perfect if there exists (necessarily closed) $X' \subset X$ such that $f(X') = Y$ and $f|X'$ is perfect.

A map $f: X \to Y$ is called uniformly complete if every fiber $f^{-1}(y)$ is a complete metric space for the given metric on X. It is clear that if X is a Polish (= separable, complete metrizable) space or every fiber $f^{-1}(y)$ is compact, then $f: X \to Y$ is uniformly complete.

A section for a map $f: X \to Y$ is any subset $S \subset X$ which intersects every non-empty $f^{-1}(y)$ in exactly one point.

PII: S0166-8641(98)00140-0

Let $f : X \to Y$ be a compact-covering map. Then f is inductively perfect in the following cases:

(k1) Y is an F_σ [10];

(k2) X is of some "low" Borel class or X and Y are coanalytic + axiom of Σ^1_1-determinancy [2]. [1]

The question of when an s-covering map $f : X \to Y$ is inductively perfect has been positively solved in the following cases:

(s1) f is uniformly complete and for every open set $U \subset X$, $f(U)$ is a G_δ in Y [8];

(s2) there is a perfect extension $f^* : X^* \to Y$ of f and X is a G_δ-set in X^* [8];

(s3) Y is a countable space [10,4];

(s4) f is an s_2-covering two-to-one map [9] or f is an s-covering n-to-one map [3].

In Section 1 we give some new conditions under which an s-covering map f is inductively perfect.

It is well known that perfect maps and, hence, inductively perfect maps preserve the property of being a Borel set (in some compactification) of the class α [15]. Corollary 4 in Section 3 implies that:

Let X, Y be a subset of a Polish space P. Let $f : X \to Y$ be an s-covering map with compact fibers $f^{-1}(y)$. If X is a Borel set of the multiplicative class α then Y is also a Borel set of the multiplicative class $3 + \alpha$.

1. Uniformly complete s-covering maps

Our starting point is the following theorem which gives further partial answers to the question of when an s-covering map is inductively perfect.

Observe that conditions (a)–(c) of the proof of Theorem 1 were considered first in [6] and they define the class of triquotient maps (see [8]).

Theorem 1. *If $f : X \to Y$ is an s-covering map such that for every convergent sequence*

$$S = \{y\} \cup \bigcup \{y_i : y_i \to y\} \quad \text{in } Y,$$

$f^{-1}(S)$ is complete (for the given metric on X), then f is inductively perfect.

Proof. Let $f : X \to Y$ be an s-covering map. Then for every $y \in Y$ there is a nonempty family η_y of open subsets of X such that every $U \in \eta_y$ intersects $f^{-1}(y)$ and

(a) if $U \in \eta_y$ and γ is a cover of $U \cap f^{-1}(y)$ by open subsets of X then there exists a finite number of elements $U_i \in \gamma$, $1 \leqslant i \leqslant k$, such that $\bigcup_{i=1}^k U_i \in \eta_y$;

(b) if $U \in \eta_y$ then there exists a neighborhood $O(y)$ such that $U \in \eta_{y'}$ for every $y' \in O(y)$.

Notice that the condition (b) implies:

(c) if $U \in \eta_y$ then $y \in \operatorname{Int} f(U)$.

[1] It is an open problem whether compact-covering maps of Borel sets are inductively perfect or preserve the property of being Borel [10].

The next part of the proof of Theorem 1 is, to some extent, similar to the proof of Theorem 1 in [8] and Lemma 2 in [11].

It is not hard to prove (see the proof of the Theorem 2 in [8]) that there is a perfect extension $f^*: X^* \to Y$ of f and a G_δ-set $Z \supset X$ in X^* such that $f^{-1}(S)$ is closed in Z for every convergent, including its limit sequence S in Y.

Let $Z = \bigcap_{i\in\omega} O_i$, where the O_i are open in X^*.

Choose for each $y \in Y$ some $U_0^y \in \eta_y$ and for each point $x \in U_0^y$ choose a neighbourhood $U(x) \subset U_0^y$ such that $\mathrm{cl}_{X^*} U(x) \subset O_0$. According to (a) and (b) there are finitely many $U(x_i)$, $i = 1, 2, \ldots, k$ and a neighborhood $O(y)$ such that

$$U_1^y = \bigcup_{i=1}^{k} U(x_i) \in \eta_\xi \quad \text{for every } \xi \in O(y).$$

Let us consider the open cover $\{O(y)\colon\ O(y) \subset f(U_1^y)\}_{y\in Y}$ of Y and let $\gamma = \{U_\alpha\}_{\alpha\in A}$ be its locally finite open refinement. For every $\alpha \in A$ we can choose $\xi(\alpha) \in Y$ and $U_\alpha \subset f(U_1^{\xi(\alpha)})$.

Let us consider the family of sets $W_\alpha^{\xi(\alpha)} = U_1^{\xi(\alpha)} \cap f^{*-1}(U_\alpha)$, $\alpha \in A$ and denote

$$X_0 = \mathrm{cl}_{X^*} \bigcup_{\alpha\in A} W_\alpha^{\xi(\alpha)}.$$

Then obviously $X_0 = \bigcup_{\alpha\in A} \mathrm{cl}_{X^*} W_\alpha^{\xi(\alpha)} \subset O_0$ and $f_0 = f^*|X_0$ is a perfect map onto Y. Since for each $y \in Y$ there is $W_\alpha^{\xi(\alpha)} \in \eta_y$ we can consider O_1, X_0, f_0 instead of O_0, X^*, f^* and repeat the construction and so on.

Thus we have the subspaces $O_i \supset X_i \supset X_{i+1}$, $i \in \omega$, where X_i is closed in X^* and $f^*|X_i = f_i : X_i \to Y$ is perfect. It is obvious that $F = \bigcap_{i\in\omega} X_i \subset Z$ is closed in X^* and hence $f^*|F$ is a perfect map onto Y.

Denote $X' = F \cap X$. By our construction every $f^{-1}(y) \cap X'$ is a nonempty compact set, hence, $f' = f|X'$ is a compact map onto $Y' = Y$.

It is well known that a map $f' : X' \to Y'$ between metric spaces is closed iff for every sequence $S = \{y\} \cup \bigcup\{y_i : y_i \to y\}$ in Y' and for every set $\{x_i\}$, where $x_i \in f'^{-1}(y_i)$, there is a limit point $x \in f'^{-1}(y)$.

Since $f^{-1}(S)$ and F are closed subsets in Z and $T = f^{*-1}(S) \cap F$ is compact, $f'^{-1}(S) = T \cap f^{-1}(S)$ is also compact and by the remark above f is inductively perfect. □

Corollary 1. *Let $f : X \to Y$ be a map from a Polish space X and $M \subset Y$. If the restriction $f|f^{-1}(M)$ is s-covering then it is inductively perfect.*

Since every sequence $S = \{y\} \cup \bigcup\{y_i : y_i \to y\}$, $S \subset M$ is closed in Y we have that $f^{-1}(S)$ is a complete subspace of X and $f^{-1}(M)$. By Theorem 1 the restriction $f|f^{-1}(M)$ is inductively perfect.

According to Hausdorff a space X is called an F_{II}-space if there are no closed subsets in X of the first category in themselves.

Theorem 2. *Let $f : X \to Y$ be an s-covering, uniformly complete map such that for every open set $U \subset X$, $f(U)$ is a Borel (or coanalytic) set in Y. If X is an F_{II}-space then f is inductively perfect.*

Proof. We consider as in the proof of Theorem 1 a perfect extension $f^* : X^* \to Y$ of f and a separable, metrizable G_δ-set $Z \supset X$ in X^* such that every $f^{-1}(y)$ is closed in Z. Denote $h = f^*|Z$. By [8, Lemma 1] (see below the Generalized Novikov's Lemma) for some open base $\{H_n\}_{n\in\omega}$ in Z we have:

$$\bigcup_{y\in Y} \mathrm{cl}_Z \left(h^{-1}(y) \cap X\right) = X = Z \setminus \bigcup_{n\in\omega} \left(H_n \setminus h^{-1}h(H_n \cap X)\right)$$

and hence X is Borel in Z. According to the classical Hurewicz theorem the space X as a Borel F_{II}-space is G_δ in Z, hence in X^*; and, according to (s2), f is inductively perfect. □

At the center of this circle of problems is the following question.

Question 1. Let $f : X \to Y$ be an s-covering map and X be an analytic set in $\mathbb{I} = [0, 1]$ and an F_{II}-space. Must f be inductively perfect?

We conclude this section with the following corollary of Theorem 2, which is similar to classical theorems of Mazurkiewicz and Vainstein for open and closed maps [12, 4.5.14; 4.5.13].

Corollary 2. *Let X, Y be Polish spaces $A \subset X$ and $f : A \to f(A)$ be an s-covering map to the space Y. Then f has an s-covering (even inductively perfect) extension $f^* : A^* \to f^*(A^*)$ where A^* and $f^*(A^*)$ are G_δ-sets in X and Y.*

Proof. Without loss of generality one can assume that $\mathrm{cl}_X A = X$ and hence [12, 4.3.20] f is extendable to a continuous map $F : B \to F(B)$ defined on a G_δ-set $B \subset X$ containing the set A. The set B is a G_δ-set in the complete metric space X. Therefore B is completely metrizable. By Corollary 1 there is a closed $X_0 \subset F^{-1}(f(A))$ such that $F|X_0$ is a perfect map onto $f(A)$. Since $B \supset F^{-1}(f(A))$ and $Y \supset f(A)$ are completely metrizable spaces, by Vainstein's theorem [12, 4.5.13], $F|X_0$ is extendable to a perfect map $g : M \to D$, where $X_0 \subset M \subset B$, $f(A) \subset D \subset Y$ and M, D are G_δ-sets in B, Y.

Obviously we may suppose that $M \subset \mathrm{cl}_B X_0$ and $F|M = g|M$ because $f|X_0 = g|X_0$, X_0 is dense in M and the extension is uniquely determined by g. It follows that $g(M) \subset F(B)$. Now let $A^* = F^{-1}(D)$. Then, obviously, A^* is a G_δ-set in B and X. It is clear that $f^* = F|A^*$ is an inductively perfect map because $f^*|M = F|M = g|M$ is a perfect map onto D. □

Since triquotient maps satisfy conditions (a)–(c) of Theorem 1, we obtain analogously for non-separable X, Y the following sharpening of Corollary 2:

Let X, Y be complete metric spaces, $A \subset X$ and $f : A \to f(A)$ be a triquotient, in particular, an open map to the space Y. Then f has an inductively perfect (*hence triquotient*) *extension $f^* : A^* \to f^*(A^*)$ where A^* and $f^*(A^*)$ are some G_δ-sets in X and Y.*

Notice that Mazurkiewicz's theorem does not hold without the assumption of separability of X [14] and that in Corollary 2 "s-covering" can be replaced by "compact-covering".

2. s_α-covering maps with finite fibers

In connection with (s4) we show in the following Lemma 1 how s-covering maps are related to other maps, which are weaker than being s-covering.

Denote by $(X)^\alpha$ the αth derivative of X and by $S_\alpha(y)$ a countable compact set for which $(S_\alpha(y))^\alpha = \{y\}$. Then we say that $S_\alpha(y)$ is a compact set of order α with the summit y. For example, the sequence $S = \{y\} \cup \bigcup\{y_i : y_i \to y\}$ is a compact set of order 1 with summit y.

A map $f : X \to Y$ is s_α-covering [9] if for every compact $S_\beta(y) \subset Y$ of order $\beta \leqslant \alpha$ there is a compact $B \subset X$ such that $f(B) = S_\beta(y)$.

It is not hard to see that a map is s-covering iff it is s_α-covering for all $\alpha \in \omega_1$ and a map with compact fibers is s_1-covering iff it is quotient.

Theorem 3. *Every n-to-one map*[2] *$f : X \to Y$ is s-covering if and only if it is s_n-covering.*

Lemma 1. *If f is an s_β-covering map and $n \leqslant \beta$ a natural number, then for every $y \in Y$ and open $V \supset f^{-1}(y)$ there is an open $O \ni y$ such that for $V' = f^{-1}(O) \cap V$ we have $f|V' : V' \to O$ is an s_{n-1}-covering map.*

Indeed, suppose it is not true. Then there is an open set $V \supset f^{-1}(y)$, a base $\{O^i\colon i = 1, 2, \ldots\}$ at y and compact sets $S_{n-1}(y_i) \subset O^i$ such that for every compact $B \subset V$,

$$f(B) = S_{n-1}(y_i) \Rightarrow B \setminus V \neq \emptyset.$$

It is clear that $S_n(y) = \{y\} \cup \bigcup\{S_{n-1}(y_i)\colon i = 1, 2, \ldots\}$ is a compact set of order n. By assumption there is a compact $B \subset X$ such that $f(B) = S_n(y)$. Choosing a point $x_i \in (B \cap f^{-1}(S_{n-1}(y_i))) \setminus V$ we obtain a countable set $A = \{x_i\}$. It is clear that $\mathrm{cl}_X A \cap f^{-1}(y) = \emptyset$ and, hence, A is a closed infinite discrete compact subset of B, which is impossible.

Proof of Theorem 3. We shall prove by induction on n that an n-to-one s_n-covering map is s-covering. It is clear that for $n = 1$, f is a homeomorphism.

Now suppose $n > 1$ and the statement is true for all $i < n$. Let $f : X \to Y$ be an n-to-one s_n-covering map. We shall prove that f is s_α-covering for all $\alpha < \omega_1$. For $\alpha = n$ it is true by assumption.

Suppose that for all $n \leqslant \beta < \alpha$ f is s_β-covering. Let S_α be a compact of order α.

[2] I.e., $|f^{-1}(y)| \leqslant n$, where n is some natural number.

Let us consider according to Lemma 1 a decreasing base $V_i \supset f^{-1}(y)$ and a corresponding decreasing base O_i at y such that $f|V_i \to O_i$ is an n-to-one, s_{n-1}-covering map. We may suppose $S_\alpha(y) = Y$, hence X, Y are countable and V_i, O_i are clopen sets. Denote $Y_i = O_i \setminus O_{i+1}$ (we may suppose without loss of generality $V_0 = X$, $O_0 = Y$), $X_i = f^{-1}(Y_i) \cap V_i$. Then $f|X_i = f_i : X_i \to Y_i$ are n-to-one, s_{n-1}-covering maps and X_i, Y_i are clopen sets. Let $S^i_\beta = Y_i \cap S_\alpha(y)$. We may consider S^i_β as a compact of order β with summit y_i. Since f is s_β-covering, there are compacts $B^i_\beta \subset X$ such that $f(B^i_\beta) = S^i_\beta$. It is clear that

$$B^i_\beta = \left(B^i_\beta \setminus X_i\right) \cup \left(B^i_\beta \cap X_i\right).$$

Let $K^i_\beta = f(B^i_\beta \setminus X_i)$, then $f_i|f_i^{-1}(K^i_\beta)$ are $(n-1)$-to-one, s_{n-1}-covering maps. By inductive assumption there are compact sets $T^i_\beta \subset X_i$ such that $f_i(T^i_\beta) = K^i_\beta$, thus we have a compact $R^i_\beta = T^i_\beta \cup (B^i_\beta \cap X_i) \subset X_i$ which is mapped onto S^i_β. It is easy to see that

$$B = f^{-1}(y) \cup \bigcup \left\{R^i_\beta \colon i \in \omega\right\}$$

is a compact for which $f(B) = S_\alpha(y)$. $\square$

Notice that an $(n+1)$-to-one, s_n-covering map can not be s-covering even for $n = 1, 2$ [9].

We showed in [9] that n-to-one quotient maps preserve the property of being Polish and that this statement is generally false for finite-to-one maps. I do not know whether two-to-one quotient maps (or uniformly complete s-covering maps) preserve the property of being a Borel set of the class α even for $F_{\sigma\delta}$-sets in the Cantor set $\mathbb{C}$.

3. Point-harmonious maps of Borel sets

We recall that a map $f : X \to Y$ between separable metric spaces is s-covering if and only if it is point-harmonious [10]. Notice that open, inductively perfect, closed maps are point-harmonious [10] and every point-harmonious map is quotient.

It is clear that every extension $f : X \to Y$ of an inductively perfect map $g = f|T : T \to Y$ is also inductively perfect and hence (see [8] or (s1), (s2)) a map $f : X \to Y$ is inductively perfect if there is a subset $T \subset X$ and an s-covering map $f|T : T \to Y$ such that the following conditions (1) or (2) are satisfied:

(1) there is a perfect extension $(f|T)^* : X^* \to Y$ of $f|T$ such that T is a G_δ-set in X^*;

(2) $f|T : T \to Y$ is uniformly complete and for every open $U \subset T$, $f(U)$ is a G_δ-set in Y.

Notice also that if $f : X \to Y$ is perfect or inductively perfect and X, Y are subsets of the Cantor set $\mathbb{C}$, then the following condition is satisfied (see, for example, Generalized Mazurkiewicz's Lemma in Section 3):

(3) there is a G_δ-section Z for f such that if $U \subset Z$ is open (relative to Z), then $f(U)$ is an F_σ in Y.

The example of an s-covering, finite-to-one and not inductively perfect map [3] shows, that the subspace T with properties from conditions (1) and (2) does not exist in general for an s-covering uniformly complete map $f: X \to Y$. Our approach will be to establish in the following the existence of a similar subspace T if we substitute G_δ for $F_{\sigma\delta}$ in (1), G_δ for F_σ in (2), G_δ and F_σ for G_δ and $F_{\sigma\delta\sigma}$ in (3).

Definition 1. A map $f: X \to Y$ is point-harmonious if one can assign to every point $y \in Y$ a nonempty family η_y of open subsets of X satisfying the following conditions:

(a) If $U \in \eta_y$ then there is a nonempty compact $B \subset f^{-1}(y) \cap U$ such that for every open $V \supset B$ we have $V \in \eta_y$.

(b) If $U \in \eta_y$ then there is an open $O(y)$ such that $U \in \eta_{y'}$ for every $y' \in O(y)$.

It is not hard to see that the above definition is equivalent to the following:

Definition 1′. A map $f: X \to Y$ is point-harmonious if for every $y \in Y$ there is a family ε_y of nonempty compact subsets of $f^{-1}(y)$ such that the condition $B \in \varepsilon_y$ implies:

($*$) for every open $U \supset B$ there is a neighborhood $O(y)$ such that for every $y' \in O(y)$ there is $B' \in \varepsilon_{y'}$ for which $B' \subset U$.

We now study point-harmonious maps.

Theorem 4. *Let $f: X \to Y$ be a point-harmonious map. Then there is a subset $Z(X) \subset X$ such that $f|Z(X)$ is a point-harmonious map onto Y and if U is open in X and $W = U \cap Z(X)$, then $f(W)$ is an F_σ-set in Y.*[3]

Proof. Define: $\tilde{\varepsilon}_y = \{B\colon B$ is a nonempty compact set in $f^{-1}(y)$ satisfying condition ($*$) of Definition 1′$\}$.

It is not hard to see that f is point-harmonious relative to $\tilde{\varepsilon}_y$.

Define:

$$\hat{\varepsilon}_y = \bigl\{B \in \tilde{\varepsilon}_y\colon \forall B' \in \tilde{\varepsilon}_y\ (B' \subset B \Rightarrow B' = B)\bigr\}.$$

Lemma 2. *For every $B_0 \in \tilde{\varepsilon}_y$ there is a nonempty compact $B_m \subset B_0$, $B_m \in \hat{\varepsilon}_y$.*

Proof. This follows from the following statements.

(1) If $B_\alpha \in \tilde{\varepsilon}_y$ and for every compact $B \subset B_\alpha$, $B \neq B_\alpha$ we have $B \notin \tilde{\varepsilon}_y$, then by the definition of $\hat{\varepsilon}_y$, $B_\alpha \in \hat{\varepsilon}_y$.

(2) If $B_0 \supset \cdots \supset B_\beta \supset B_{\beta+1} \supset \cdots$ and $B_\beta \in \tilde{\varepsilon}_y$, then $\bigcap_\beta B_\beta \in \tilde{\varepsilon}$.

It is clear that $B_\alpha = \bigcap_\beta B_\beta$ is a nonempty compact set. Indeed, it is easy to see that for every open $U \supset B_\alpha$ there is β_0 for which $U \supset B_{\beta_0} \supset B_\alpha$. Hence the condition ($*$) is satisfied for every open U and it follows from definition of $\tilde{\varepsilon}_y$ that $B_\alpha \in \tilde{\varepsilon}_y$. □

Lemma 3. *Denote $Z(X) = \bigcup_{y \in Y} \bigcup \hat{\varepsilon}_y$. Then the restriction $f|Z(X)$ is a point-harmonious map.*

[3] It would be interesting to extend Theorem 4 for the case of compact-covering or harmonious maps.

Proof. Indeed, if $B \in \hat{\varepsilon}_y$ and $V \supset B$ is an open set in $Z(X)$ then there is an open $V' \subset X$ for which $V' \cap Z(X) = V$. By definition of $\hat{\varepsilon}_y$, $B \in \tilde{\varepsilon}_y$, hence there is $O(y)$ such that for every $y' \in O(y)$ there is $B' \in \varepsilon_{y'} \subset \tilde{\varepsilon}_{y'}$, and $B' \subset V'$. Hence, by Lemma 2, there is $B_m \subset B'$, $B_m \in \hat{\varepsilon}_{y'}$, $B_m \subset V'$. Since $B_m \subset Z(X)$ we have $B_m \subset V$. □

Remark 1. We can suppose in Theorem 4 that X is a subset of the Cantor set $\mathbb{C}$. Indeed, let $B \supset X$ be a metrizable compact. It is a well-known fact that there is a continuous map $t:\mathbb{C} \to B$. Denote $X_0 = t^{-1}(X)$, then $t|X_0$ is perfect. Since f is s-covering and invers image under a perfect map of a compact set is compact, the composition $g = f \circ (t|X_0)$ is an s-covering and point-harmonious map from X_0 to Y.

Let $Z(X_0) \subset X_0$ be assumed to satisfy condition of Theorem 4. Denote $Z(X) = t(Z(X_0)) \subset X$. Then $f|Z(X)$ is the point-harmonious map onto Y. In fact, if $K \subset Y$ and $B \subset Z(X_0)$ are compact sets for which $f(B) = K$, then $t(B) \subset Z(X)$ and $f(t(B)) = K$.

If U is open in X and $W = U \cap Z(X)$, then $f(W) = g(t^{-1}(U) \cap Z(X_0))$ is an F_σ in Y.

Lemma 4. *Let $B \in \hat{\varepsilon}_y$ and U be a clopen set in X for which $U \cap B \neq \emptyset$. Then there is an open $V \supset B$ such that*:

(i) *for every neighborhood $O(y)$ of y there is $y' \in O(y)$ such that if $B' \subset V$ and $B' \in \varepsilon_{y'}$ then $U \cap B' \neq \emptyset$;*

(ii) *if $B_1 \in \hat{\varepsilon}_y$ and $B_1 \subset V$ then $B_1 \cap U \neq \emptyset$.*

Proof. (i) By definition of $\hat{\varepsilon}_y$ we have $B \setminus U \notin \tilde{\varepsilon}_y$ and, hence, there is an open $V' \supset B \setminus U$ such that for every open $O(y)$ there is $y' \in O(y)$ such that for every $B' \in \varepsilon_{y'}$ we have $B' \not\subset V'$. Define: $V = V' \cup U \supset B$.

(ii) If $B_1 \cap U = \emptyset$ then $V \setminus U$ is an open set containing B_1. Since $B_1 \in \hat{\varepsilon}_y$ and hence $B_1 \in \tilde{\varepsilon}_y$ we have by definition of $\tilde{\varepsilon}_y$ a contradiction to (i). □

We turn to the proof of Theorem 4. Let $W = Z(X) \cap U$ then

$$f(W) = \bigcup \bigl\{ f(B_\alpha):\ B_\alpha \cap U \neq \emptyset,\ B_\alpha \in \hat{\varepsilon}_y,\ y \in Y \bigr\}.$$

Since every open set in X is a countable union of clopen sets, we can suppose that the set U in Theorem 4 is clopen.

For every $B = B_\alpha \in \hat{\varepsilon}_y$ for which $B \cap U \neq \emptyset$ choose, according to Lemma 4, $V = V_\alpha \supset B_\alpha$.

Since B_α are compact sets and X has a countable base we can suppose that $\{V_\alpha\}$ is a countable family. Define $F_\alpha = \{y_1 \in Y:$ for every B_1 the conditions $B_1 \in \hat{\varepsilon}_{y_1}$ and $B_1 \subset V_\alpha$ imply that $B_1 \cap U \neq \emptyset\}$. By (ii) of Lemma 4, $y \in F_\alpha$.

Denote

$$V_\alpha^* = \bigl\{ y_1 \in Y:\ \exists B_1 \in \hat{\varepsilon}_{y_1} \text{ such that } B_1 \subset V_\alpha \bigr\}.$$

By Lemma 3 and according to the definition of point-harmonious map V_α^* is open in Y and $V_\alpha^* \supset F_\alpha$.

The set F_α is closed in V_α^*, since if $y_1^0 \in V_\alpha^*$ is an accumulation point for $y_1^i \in F_\alpha$ $(i = 1, 2, \ldots)$ and there is $B^0 \in \hat{\varepsilon}_{y_1^0}$ such that $B^0 \subset V_\alpha$ and $B^0 \cap U = \emptyset$, then the open set $V_\alpha \setminus U$ contains B^0 and by definition of point-harmonious maps for some i it contains $B^i \in \hat{\varepsilon}_{y_1^i}$ with the properties $B^i \subset V_\alpha$ and $B^i \cap U = \emptyset$, which is impossible by definition of F_α.

It is clear that each F_α is an intersection of a closed and an open set and hence is an F_σ-set in Y. Since $\{V_\alpha\}$ is the countable family, $f(W) = \bigcup_\alpha F_\alpha$ is an F_σ in Y. □

Theorem 5. *Let $f : X \to Y$ be a uniformly complete, point-harmonious map. Then*:

(1) *There is $T \subset X$ such that $f|T$ is a uniformly complete point-harmonious map onto Y and if $W \subset T$ is open* (*relative to T*) *then $f(W)$ is an F_σ-set in Y.*
(2) *There is a perfect extension $f^* : X^* \to Y$ of f over a metric space X^* such that T is a $F_{\sigma\delta}$-set in X^*.*
(3) *There is a G_δ-section Z for f such that if $U \subset Z$ is open* (*relative to Z*) *then $f(U)$ is an $F_{\sigma\delta\sigma}$ in Y.*

We start first with some general lemmas.

Generalized Novikov's Lemma. *Let $f_0^* : X_0^* \to Y$ be a map, $\{H_n\}$ be a countable base in X_0^*, $Z(X) \subset X_0^*$ is an arbitrary subset such that for $h = f_0^*|Z(X)$ we have*: $h(Z(X)) = Y$. *Denote $T = \bigcup_{y \in Y} \mathrm{cl}_{X_0^*} h^{-1}(y)$. Then*

$$T = X_0^* \setminus \bigcup_{n \in \omega} \left(H_n \setminus f_0^{*-1} h\left(H_n \cap Z(X)\right)\right).$$

In particular,

(i) *if for every open $W \subset Z(X)$ the image $h(W)$ is an F_σ in Y, then T is an $F_{\sigma\delta}$ in X_0^*,*
(ii) *for every open $W \subset T$ we have $h(W) = h(W \cap Z(X))$.*

Proof. See [1, Lemma 1 of §14]; [8, Lemma 1]. □

Generalized Mazurkiewicz's Lemma. *Let $g : T \to Y$ be a map with each $g^{-1}(y)$ complete in the given metric on T and let $\{U_n\}$ be a countable base for T. Then there is a section $Z \subset T$ for g such that*:

(a) *$Z = \bigcap_{n \in \omega} Z_n$ where Z_n is a finite union of sets $U_i \setminus g^{-1}g(U_r)$,*
(b) *for every open $V \subset T$ the set $g(Z \cap V)$ is a union of countably many of $g(U_i) \setminus g(U_r)$.*

In particular, if the image $g(W)$ of every open (*in T*) *subset W is an F_σ-set in Y, then Z is a G_δ in T and $g(Z \cap V)$ is an $F_{\sigma\delta\sigma}$ in Y.*

Proof. See [1, lemma of §5]; [7, Theorem 1].[4] □

Proof of Theorem 5. Let $\widetilde{X}, \widetilde{Y}$ be completions of X and Y, respectively. Then every fiber $f^{-1}(y)$ is closed in $\widetilde{X}$. Let $\tilde{f} : X_0 \to \tilde{f}(X_0)$ be an extension of f, where X_0 is a G_δ-set in $\widetilde{X}$. It is clear that every fiber $f^{-1}(y)$ is closed in X_0.

[4] In the proof of this theorem $V_0 = \emptyset$.

We can suppose without loss of generality that $\widetilde{X} \subset \mathbb{C}_X$, $\widetilde{Y} \subset \mathbb{C}_Y$ where $\mathbb{C}_X, \mathbb{C}_Y$ are two copies of the Cantor set $\mathbb{C}$. Let $\pi : \mathbb{C}_X \times \mathbb{C}_Y \to \mathbb{C}_Y$ be the projection and G_0 be the graph of the map $\tilde{f}$. It is well known that $G_0 \approx X_0$ and we may associate $\pi | G_0$ with $\tilde{f}$.

Denote $\pi^{-1}(Y) = X^*$. It is obvious that $f^* = \pi | X^*$ is a perfect extention of $\pi | X = f$, $X_0^* = G_0 \cap X^*$ is a G_δ-set in X^*.

Let us consider the set $Z(X) \subset X$ (see Theorem 4). Denote $f_0^* = \pi | X_0^*$. It is not hard to see that every fiber $f^{-1}(y)$ is closed in X_0^* and for the set T, defined by the Generalized Novikov's Lemma we have:

$$X_0^* \supset X \supset T \supset Z(X).$$

Theorem 4 implies that the map $f^*|T : T \to Y$ is s-covering and point-harmonious. It is clear that $f|T$ is uniformly complete. By definition of T the set $f^{*-1}(y) \cap Z(X)$ is dense in $f^{*-1}(y) \cap T$ and for every open $U \subset X^*$ and $h = f_0^*$ we have

$$f^*(U \cap T) = h\big(U \cap Z(X)\big)$$

and according to the Theorem 4, $f^*(U \cap T)$ is an F_σ in Y. Hence, the condition (1) of Theorem 5 is satisfied.

Since X_0^* is a G_δ in X^* and according to Theorem 4 every $h(H_n \cap Z(X))$ is an F_σ in Y, we obtain condition (2) from the Generalized Novikov's Lemma.

Condition (3) of Theorem 5 follows from the Generalized Mazurkiewicz's Lemma (where $g = f|T$) and condition (1). □

Corollary 3. *Let $f : X \to Y$ be an s-covering map. Then there exists an $F_{\sigma\delta}$-subset $T \subset X$ such that $f|T$ is an s-covering map onto Y and for every open $W \subset T$, $f(W)$ is an F_σ in Y.*

Proof. Consider according to the Theorem 4 the set $Z(X) \subset X$ and apply Generalized Novikov's Lemma for $f_0^* = f$, $X_0^* = X$. □

Corollary 4. *Let X, Y be subset of a Polish space P. Let $f : X \to Y$ be a point-harmonious map such that every fiber $f^{-1}(y)$ is closed in P. If X is a Borel set of the multiplicative class α then Y is a Borel set of the multiplicative class $3 + \alpha$.*

Indeed, for $\alpha = 1$ X is a G_δ in P and, hence, X is a Polish space. Since an s-covering image of a Polish space is also Polish [6], Y is of the multiplicative class 1.

Suppose $\alpha > 1$, then G_δ-set Z in the condition (3) of Theorem 5 is also of the multiplicative class α. According to the condition (3) of Theorem 5 for every closed $F \subset Z$ the set $f(F)$ is a $G_{\delta\sigma\delta}$ and of the multiplicative class 3. Hence, $f|Z$ is a homeomorhism of class $0, 3$ and Y is of the multiplicative class $3 + \alpha$ [5, 36,VII, Corollaries 2]. □

Notice that every point-harmonious map is quotient, but for quotient maps a Borel section does not exist in general. Indeed, as Michael and Stone showed [13] there is a quotient $f : \mathbb{P} \to Y$ from the space of irrational numbers $\mathbb{P}$ onto an arbitrary non-Borel set $Y \subset \mathbb{P}$.

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