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STABLE MAPS OF POLISH SPACES

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Abstract. We define the notions of stable and transquotient maps and study the relation between these classes of maps. The class of stable maps contains all closed and open maps and their compositions. The transquotient maps preserve the property of being a Polish space, and every stable map between separable metric spaces is transquotient.

In particular, a composition of closed and open maps (the intermediary spaces may not be metric) preserves the property of being a Polish space. This generalizes the results of Sierpiński and Vainstein for open and closed maps.

In [5] the author introduced a concept of system $(\eta_y)_{y\in Y}$ associated to a map $f : X \to Y$ with the help of which we give here new classes of stable and transquotient maps.[1]

All the maps in this paper are continuous and onto. All the spaces are regular.

Let $f\colon X \to Y$ be a map from a separable, complete metrizable (= Polish) space X onto a metric space Y. Then Y is also Polish in the following cases:

a) if f is an open or a closed map [7], [9];

b) if f is a compact–covering map or an s–covering map[2] [2], [3], [5];

c) if f is a composition of open and perfect maps [3].

First, we will show that the class of stable maps contains the above–mentioned classes. Second, we obtain from Theorem 0 all the above–mentioned statements a)–c) and some new results. In particular, a composition of closed and open maps preserves the property of being a Polish space.

Notice, that in all the cases a)–c) there is a closed subset $X_0 \subset X$ such that the restriction $f|X_0$ is a perfect map onto Y (see [2], [3], [5]).

Theorem 0. *Let $f\colon X \to Y$ be a stable map from a Polish space X onto a metrizable space Y. Then:*

(1) There is a countable set $Y_\sigma \subset Y$ and a Polish space $X_0 \subset X$ such that the restriction $f|X_0$ is a perfect map onto $Y \setminus Y_\sigma$.

(2) Y is a Polish space.

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[1] The system, which was studied in [5, Lemmas 1–3], was triquotient.

[2] A map $f\colon X \to Y$ is compact–covering (resp. s–covering or countable–compact–covering) if every compact (resp. countable compact) $K \subset Y$ is the image of some compact $B \subset X$.

1. Stable and transquotient maps

Our approach will be to establish a characterization of stable, transquotient and harmonious maps in terms of transmittable systems and to give a relation between these classes of maps. We start first with some general notation:

A system associated to a map $f : X \to Y$ is a family $(\eta_y)_{y \in Y}$ where each η_y is a family of open subsets of X satisfying the condition $U \cap f^{-1}(y) \neq \varnothing$ for all $U \in \eta_y$. The system is said to be transmittable if the following condition holds:

(a) for every $U \in \eta_y$ there is a neighborhood $O(y)$ of y such that $U \in \eta_{y'}$ for every $y' \in O(y)$.

It is clear that if $U \in \eta_y$, then $y \in \mathrm{Int} f(U)$. A map may have many transmittable systems and a system is called trivial if $\eta_y = \{X\}$ for every $y \in Y$.

Definition 1. The map $f : X \to Y$ is stable if it admits a transmittable system satisfying the following condition:

(b) If $U \in \eta_y$ and V is open in X such that $V \supset U \cap f^{-1}(y)$, then $V \in \eta_y$.

Definition 2. A map $f : X \to Y$ is transquotient if it admits the transmittable system satisfying the following condition:

(c) If $U \in \eta_y$ and $\gamma = \{U_\alpha\}_{\alpha \in A}$ is a family of open subsets of X such that $\bigcup_{\alpha \in A} U_\alpha \supset f^{-1}(y) \cap U$, then there is a finite number $U_{\alpha_1}, ..., U_{\alpha_n} \in \gamma$ and an open $O(y)$ such that $\bigcup_{i=1}^n U_{\alpha_i} \in \eta_{y'}$ for every $y' \in O(y) \setminus \{y\}$.

Remark 0. Let η_y be a transquotient family for f. Define a new family for f:

$\tilde{\eta}_y = \{V : V$ is open in X and $\exists U \in \eta_y$ such that $V \supset U \cap f^{-1}(y)\}$.

In particular $X \in \tilde{\eta}_y$. Notice that the transmittable property holds for $\tilde{\eta}_y$ since for $V \in \tilde{\eta}_y$ we can consider $\gamma = \{V\}$ in (c) and then, obviously, $V \in \tilde{\eta}_{y'}$ for every $y' \in O(y)$. Since every cover of $f^{-1}(y) \cap V$ is the cover of $f^{-1}(y) \cap U$, we have by (c) that $\tilde{\eta}_y$ is a transquotient family for f.

If V is open in X and there is $U \in \tilde{\eta}_y$ such that $V \supset f^{-1}(y) \cap U$ (in particular, if $V \supset f^{-1}(y)$), then, obviously, V is also an element of $\tilde{\eta}_y$. Hence, a transquotient map f is stable.

It should be noted that the following lemma uses an idea of A.H. Stone's proof [8, Lemma 1].

Lemma 1. *Let $f : X \to Y$ be a map. If Y is first–countable and if every fiber $f^{-1}(y)$ is hereditarily Lindelöf, then the following conditions are equivalent:*

(i) f is stable;

(ii) f is transquotient.

Proof. By Remark 0 (ii) $\Rightarrow$ (i). We will show (i) $\Rightarrow$ (ii). Suppose the family η_y satisfies Definition 1 and does not satisfy Definition 2. This means that (c) fails for some $U \in \eta_y$ and family $\gamma = \{U_i\}_{i \in \omega}$. It is obvious that there is an open decreasing base $\{O_n(y)\}_{n \in \omega}$ at y such that for every $\gamma_n = \bigcup_{i=0}^n U_i$ there is a $y'_n \in O_n(y) \setminus O_{n+1}(y)$, for which $\gamma_n \notin \eta_{y'_n}$ and, hence, $M_n = (U \cap f^{-1}(y'_n)) \setminus \gamma_n \neq \varnothing$ (we assume that $O_0(y)$ is a neighbourhood of y such that $U \in \eta_{y'}$ for each $y' \in O_0(y)$).

Let $M = \bigcup\{M_n : n \in \omega\}$. Now let us show that $f^{-1}(y) \cap U \cap \mathrm{cl}_X M \neq \varnothing$. Suppose the contrary, then $U' = U \setminus \mathrm{cl}_X M \supset f^{-1}(y) \cap U$ and $U' \in \eta_y$. By the transmittable property, for some n we have $U' \in \eta_{y'_n}$. By our construction, $\gamma_n \supset \gamma_n \cap U \cap f^{-1}(y'_n) = U' \cap f^{-1}(y'_n)$ and, hence, $\gamma_n \in \eta_{y'_n}$. That is impossible.

Choose $x \in f^{-1}(y) \cap U \cap \mathrm{cl}_X M$; then x belongs to some γ_n. It is clear that $x \in T = f^{-1}(y) \cap U \cap \mathrm{cl}_X(\bigcup\{M_k : k > n\})$. Taking into consideration that $M_k \subset U \setminus \gamma_n$ for $k > n$, and hence $T \cap \gamma_n = \varnothing$, we get a contradiction with $x \in \gamma_n$. □

Let us consider some examples of stable and transquotient maps.

Proposition A. *Open maps and closed maps $f\colon X \to Y$ are stable.*

For an open map: $\eta_y = \{U : U \text{ is open and } y \in f(U)\}$.

For a closed map: $\eta_y = \{U \subset X : U \text{ is open in } X, \text{ and } U \supset f^{-1}(y)\}$. It is a well known fact that for every $y \in Y$ and every open $U \supset f^{-1}(y)$ there is an open $O(y)$ such that $f^{-1}(O(y)) \subset U$.

Every open map with $\eta_y = \{U : U \text{ is open and } y \in f(U)\}$ is transquotient but the space $Y = R \bmod Z$, obtained by identifying the integers Z in the real line R to one point y, gives a closed and non–transquotient map $f\colon R \to Y$. Indeed, let γ_1 be a cover of $f^{-1}(y)$ by open intervals of length $< 1/2$. Then for every finite subcover γ_2 of γ_1 and every open $O(y)$ there is $y' \in O(y) \setminus \{y\}$ such that $\bigcup \gamma_2 \cap f^{-1}(y') = \varnothing$.

Proposition B. *Suppose $f\colon X \to Y$ and $\varphi\colon Y \to Z$ are stable maps. Then the composition $g = \varphi \circ f\colon X \to Z$ is also a stable map.*

Let $\eta_y(f)$ and $\eta_z(\varphi)$ be stable families for f and φ, respectively. Let us define $\eta_z(g) = \{U : \exists T \in \eta_z(\varphi) \forall y \in T \cap \varphi^{-1}(z)(U \in \eta_y(f))\}$ and let us show that g is a stable map. It is clear that $X \in \eta_z(g)$ for every $z \in Z$.

(a) Let us assume $U \in \eta_z(g)$ and $y \in T \cap \varphi^{-1}(z)$. By (a) for f there is an open $O(y) \subset T$ such that $U \in \eta_{y'}(f)$ for each $y' \in O(y)$. Let $T' = \bigcup\{O(y) : y \in T \cap \varphi^{-1}(z)\}$. Obviously, $T' \supset T \cap \varphi^{-1}(z)$. Hence, $T' \in \eta_z(\varphi)$.

We choose (according to (a) for φ) an open $O(z)$ such that $T' \in \eta_{z'}(\varphi)$ for every $z' \in O(z)$, and then we have $U \in \eta_{z'}(g)$.

(b) Suppose $V \supset U \cap g^{-1}(z)$ for an open V, where $U \in \eta_z(g)$. Then $V \in \eta_y(f)$ for every $y \in T \cap \varphi^{-1}(z)$ and, hence, $V \in \eta_z(g)$.

2. Proof of Theorem 0

A map $f\colon X \to Y$ from a metric space X is uniformly sequentially complete if each $f^{-1}(S)$ is complete with the given metric on X for every convergent, including its limit, sequence $S = \{y\} \cup \bigcup\{y_i : y_i \to y\}$ in Y.

Proposition C. *If $f : X \to Y$ is transquotient with appropriate families η_y, $Y_0 \subseteq Y$, $Z = \bigcup_{\alpha \in A} U_\alpha$ and $U_\alpha \in \eta_y$ for each $y \in f(U_\alpha) \setminus Y_0$, then $f|(Z \setminus f^{-1}(Y_0)) : Z \setminus f^{-1}(Y_0) \to f(Z) \setminus Y_0$ is transquotient.*

The new transquotient families are defined as $\tilde{\eta}_y = \{U \setminus f^{-1}(Y_0) \subseteq Z \setminus f^{-1}(Y_0) : U \in \eta_y\}$.

Lemma 2. *Let $f\colon X \to Y$ be a transquotient, uniformly sequentially complete map between separable metric spaces X, Y. Then there exists a G_δ–set $X' \subset X$ and a countable set $Y_\sigma \subset Y$ such that the restriction $f|X'$ is a perfect map onto $Y \setminus Y_\sigma$.*

Proof. There is a perfect extension $f^*\colon X^* \to Y$ of f and $Z \subset X^*$ such that $Z \supset X$, every $f^{-1}(S)$ is closed in Z for every convergent, including its limit sequence $S \subset Y$ and $Z = \bigcap_{i \in \omega} O_i$, where O_i are open in X^*.

Indeed, suppose $B_Y \supset Y$ and $B_{\tilde{X}}$ are compact spaces containing Y and the completion $\tilde{X}$ of X, respectively. Then $\tilde{X}$ is a G_δ–set in $B_{\tilde{X}}$ and the first coordinate projection π on $B_Y \times B_{\tilde{X}}$ is a perfect map. Let $X^* = \pi^{-1}(Y)$, $Z = Y \times \tilde{X}$ and $f^* = \pi|X^*$. Then $f^*\colon X^* \to Y$ is also perfect. Finally, we can identify the graph $G \subset X^*$ of the map f with X and the projection of G with f; then f^* is the extension of f.

Notice that $f^{*-1}(S)$ is always compact for a convergent sequence S including its limit because $f^{*-1}(S) = S \times B_{\tilde{X}}$ and this is a product of two compact sets.

Note also that by the construction of the transmittable system constructed in Remark 0 ($\tilde{\eta}_y$), we can assume that there exists a transmittable system η_y that makes f both transquotient and stable.

Let us choose for each $y \in Y$ some $U_0^y \in \eta_y$, $U_0^y \subset X$ and for each point $x \in U_0^y \cap f^{-1}(y)$ a neighborhood $U(x) \subset U_0^y$ such that $cl_{X^*}U(x) \subset O_0$. According to Definition 2 there are finitely many $U(x_i)$, $i = 1, 2, ..., n$, and a neighborhood $O(y)$ such that $U_1^y = \bigcup_{i=1}^n U(x_i) \in \eta_\xi$ for every $\xi \in O(y) \setminus \{y\}$.

Hence, $\delta_1 = \{O(y) : y \in Y\}$ is an open cover of Y and for every $y \in Y$ we have $f(U_1^y) \supset O(y)$.

Since every separable metric space is a Lindelöf space and every Lindelöf space is paracompact, we can consider the countable, locally finite open refinement $\gamma_1 = \{U_k\}_{k\in\omega}$ of δ_1. We can choose for every $U_k \in \gamma_1$ some y_k such that $O(y_k) \supset U_k$.

Denote $\delta_1(\gamma_1)= \{O(y_k) : k \in \omega\}$ and $Y_0= \{y_k : k \in \omega\}$.

Choose for every $U_k \in \gamma_1$ the corresponding elements $O(y_k) \in \delta_1(\gamma_1)$, $U_1^{y_k} \in \eta_{y_k}$ and denote $W_{1,k} = U_1^{y_k} \cap f^{*-1}(U_k) \subset X$. By our construction

($*$) $W_{1,k} \in \eta_y$ for every point $y \in U_k \setminus Y_0$.

In fact, for every $y' \in O(y_k) \setminus \{y_k\}$ (see the definition $\tilde{\eta}_y$) we obtain ($*$). Let us denote

$$X_0 = (\bigcup_{k\in\omega} W_{1,k}) \setminus f^{-1}(Y_0), \qquad f_0 = f|X_0 : X_0 \to Y \setminus Y_0.$$

According to ($*$) and Proposition C f_0 is transquotient. Let us denote

$$X_0^* = (cl_{X^*} \bigcup_{k\in\omega} W_{1,k}) \setminus f^{*-1}(Y_0), \qquad f_0^* = f^*|X_0^*\colon X_0^* \to Y \setminus Y_0.$$

It is clear that f_0^* is a perfect map. Since $\{W_{1,k}\}_{k\in\omega}$ is a locally finite family, we have: $X_0^* = (\bigcup_{k\in\omega} cl_{X^*}W_{1,k}) \setminus f^{*-1}(Y_0)$ and, hence, $X_0 \subset X_0^* \subset O_0$.

We may consider $O_1, X_0, X_0^*, f_0, f_0^*$ instead of O_0, X, X^*, f, f^* and repeat the construction. In this way we obtain a countable subspace $Y_1 \subset (Y \setminus Y_0)$ and:

a perfect map $f_1^* : X_1^* \to Y \setminus (Y_0 \cup Y_1)$, where $X_1^* \subset O_1$ and $X_1 \subset X_1^* \subset X_0^*$;

a transquotient map $f|X_1$ onto $Y \setminus (Y_0 \cup Y_1)$, where $X_1 \subset X_0$.

Thus, we have the subspaces $O_i \supset X_i^* \supset X_{i+1}^*$, $i \in \omega$, where $X_i^* \supset X_i$ is closed in $X^* \setminus \bigcup_{k=0}^i f^{*-1}(Y_k)$ and every $f_i^*\colon X_i^* \to Y \setminus \bigcup_{k=0}^i Y_k$ is perfect. Denote $Y_\sigma = \bigcup_{i\in\omega} Y_i$. Evidently, Y_σ is countable. It is obvious that $F = \bigcap_{i\in\omega} X_i^* \subset Z$ is closed in $X^* \setminus f^{*-1}(Y_\sigma)$ and, hence, $f^*|F$ is a perfect map onto $Y \setminus Y_\sigma$. Denote $X' = F \cap X$. We shall prove that $f' = f|X'$ is a perfect map onto $Y' = Y \setminus Y_\sigma$.

By inspection of the construction of the spaces X_n^* one can see that they are of the form $X_n^* = T_n \setminus f^{*-1}(\bigcup_{i=0}^n Y_n)$, where the sets T_n are closed in X^*, $T_0 \supseteq T_1 \supseteq ...$ and $T_n \subseteq O_n$. Let $T = \bigcap_n T_n$; then $T \subset Z$, because $T_n \subset O_n$. Let S be a sequence contained in Y' and assume that S is convergent and contains its own limit. $f^{-1}(S)$ is closed in Z. Hence $f^{-1}(S) \cap T$ is closed in T, because $T \subset Z$. Hence $f'^{-1}(S) = f^{-1}(S) \cap F = f^{-1}(S) \cap T$ is closed in $f^{*-1}(S) \cap T$. But the set $f^{*-1}(S)$ is compact. Thus $f'^{-1}(S)$ is compact.

To prove that $f'(X') = Y \setminus Y_\sigma$, note first that for every $y \in Y \setminus Y_\sigma$ the set $L_n = cl_{X^*} f^{-1}(y) \cap X_n^*$ is compact ($n \in \omega$). By our construction $L_n \supset L_{n+1}$ and $L_n \neq \varnothing$ since $X_n^* \supset X_n$ and $f^{-1}(y) \cap X_n \neq \varnothing$. Hence, there is $x \in \bigcap_n L_n = cl_{X^*} f^{-1}(y) \cap F$. Since $f^{-1}(y)$ is closed in Z and $F \subset Z$, we have $x \in f^{-1}(y) \cap F = f'^{-1}(y)$.

To verify that $f'(A)$ is closed in Y' for every closed $A \subset X$, note that if $y_i = f'(x_i)$, $x_i \in A$, and $y_i \to y$, then, as $C = f'^{-1}(\{y_i : i \in \omega\} \cup \{y\})$ is compact, there is a limit point x for the sequence $\{x_i : i \in \omega\}$ of the points of C. But then $x \in A$ because x_i are in A and A is closed. Obviously, $f'(x) = y \in f'(A)$.

Since $f|X'$ is a perfect map onto $Y \setminus Y_\sigma$, X' is a closed subset of $X \setminus f^{-1}(Y_\sigma)$, hence X' is a G_δ–set in X. □

Let X be a topological space. The strong Choquet game C is played as follows. Players α, β take turns, with β playing first, choosing nonempty open subsets of X. Player β also plays a point in his open set at each move and α must play an open set containing this point. Each open set played by any player is contained in the opponent's previous move

β	U_0, x_0		U_1, x_1		...	
α		V_0		U_1		...

where $x_i \in U_i, x_i \in V_i$ and $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq ....$ We say that α wins a run of this game $U_0, V_0, V_1, ...$ if $\bigcap_n V_n (= \bigcap U_n) \neq \varnothing$. One can require without losing generality that at every step the first player plays a different point. We say that X is a strong Choquet space if α has a winning strategy in the strong Choquet game C for X [1].

Lemma 3. *Let $f\colon X \to Y$ be a transquotient map of a strong Choquet space X onto Y. Then Y is also a strong Choquet space.*

Proof. Denote by Σ a winning strategy for α in C. Denote by C' the strong Choquet game on Y and by α', β' its players. We will describe a winning strategy for α' in C'.

Step 0. Let β' start with (U_0', y_0). For every $x_0 \in f^{-1}(y_0)$ take an open $U_0 \subset f^{-1}(U_0')$, $x_0 \in U_0$. Let β play (U_0, x_0), then α answers, using Σ, by $f_0((x_0, U_0)) = V_0 \subseteq U_0, x_0 \in V_0$.

By Remark 0 to Definition 2 we may suppose $X \in \eta_{y_0}$. Since the elements V_0 cover $X \cap f^{-1}(y_0)$, there is a finite family γ_0 of elements V_0 and an open $V_0' \subseteq U_0'$, $y_0 \in V_0'$ such that $\bigcup \gamma_0 \in \eta_{y'}$ for every $y' \in V_0' \setminus \{y_0\}$. Then α' answers by $f_0'((y_0, U_0')) = V_0' \subset U_0'$, $y_0 \in V_0'$.

Step 1. Let β' play (U_1', y_1), where $y_1 \in U_1' \subseteq V_0'$.

Then β plays (U_1, x_1), where $U_1 \subset V_0 \cap f^{-1}(U_1')$ for every $V_0 \in \gamma_0$ and every $x_1 \in V_0 \cap f^{-1}(y_1)$. Then α, using Σ, answers by $f_1((x_0, U_0), (x_1, U_1)) = V_1 \subset U_1$, $x_1 \in V_1$.

By the definition of transquotient maps, there is a finite family γ_1 of elements V_1 and an open $V_1' \subseteq U_1', y_1 \in V_1'$ such that $\bigcup \gamma_1 \in \eta_{y'}$ for every $y' \in V_1' \setminus \{y_1\}$. Then α' answers by $f_1'((y_0, U_0'), (y_1, U_1')) = V_1' \subset U_1', y_1 \in V_1'$. By construction $U_1' \supset f(\bigcup \gamma_1) \supset V_1'$.

Step $n+1$. Suppose we defined a map $f_n'((y_0, U_0'), ..., (y_n, U_n')) = V_n' \subset U_n'$, $y_n \in V_n'$ for (y_0, U_0'), ..., (y_n, U_n') and finite families γ_n of open sets V_n such that $\bigcup \gamma_n \in \eta_{y'}$ for every $y' \in V_n'$ and $U_n' \supset f(\bigcup \gamma_n) \supset V_n'$. Let β' play (U_{n+1}', y_{n+1}).

Then β plays (U_{n+1}, x_{n+1}) for every $V_n \in \gamma_n$ and every $x_{n+1} \in V_n \cap f^{-1}(y_{n+1})$ with $U_{n+1} \subset V_n \cap f^{-1}(U'_{n+1})$ and α answers, using Σ, $f_{n+1}((x_0, U_0), ..., (x_{n+1}, U_{n+1})) = V_{n+1} \subset U_{n+1}, x_{n+1} \in V_{n+1}$.

By the definition of transquotient maps there is a finite family γ_{n+1} of elements V_{n+1} and an open $V'_{n+1} \subset U'_{n+1}, y_{n+1} \in V'_{n+1}$ such that $\bigcup \gamma_{n+1} \in \eta_{y'}$ for every $y' \in V'_{n+1} \setminus \{y_{n+1}\}$. Then α' answers $f'_{n+1}((y_0, U'_0), ..., (y_{n+1}, U'_{n+1})) = V'_{n+1} \subset U'_{n+1}, y_{n+1} \in V'_{n+1}$.

By our construction

$(*)$ $U'_{n+1} \supset f(\bigcup \gamma_{n+1}) \supset V'_{n+1}$.

By König's theorem an infinite, finitely branching tree must have an infinite branch and we obtain in this way a sequence $V^*_0 \supset V^*_1 \supset ... \supset V^*_i \supset ...$, obtained according to a winning strategy for α, where $V^*_i \in \gamma_i$. Since α wins, there is $x \in \bigcap_n V^*_n$ and, hence, $x \in \bigcup \gamma_n$ for every $n \in \omega$. Since by definition of strong Choquet game $V'_n \supset U'_{n+1}$, we have according to $(*)$: $f(x) \in \bigcap_n V'_n$, and α' wins. □

Proof of Theorem 0. Lemma 2 implies part (1) of Theorem 0. Notice that X is completely metrizable if and only if X is a metrizable strong Choquet space [1] and we obtain from Lemmas 1 and 3 part (2) of Theorem 0 (see also [4, Corollary 7.3]). □

3. Remarks

We begin by recalling the following definition [6]:

Definition 3. A map $f\colon X \to Y$ is harmonious if one can assign to every compact $K \subset Y$ a family η_K of open subsets of X satisfying the following conditions:

(a') If $U \in \eta_K$, then there exists a compact $B \subset U$ such that $f(B) = K$ and for every open $V \supset B$ we have $V \in \eta_K$.

(b') If $U \in \eta_K$, then there exists an open $O(K)$ such that $U \in \eta_{K'}$ for every compact $K' \subset O(K)$.

We obtain the definition of a point–harmonious map by replacing K and K' in the above definition by points y and y'.

By Theorems 1 and 2 from [6] in case of separable metrizable spaces X, Y the class of harmonious (resp. point–harmonious) maps coincides with the class of compact–covering (resp. s–covering) maps.

Recall that triquotient maps may be defined as transquotient maps satisfying condition (c) from Definition 2 for every $y' \in O(y)$.

Proposition D. *Transquotient maps, triquotient maps, point–harmonious maps and harmonious maps are stable.*

Let $f\colon X \to Y$ be a map. Then 1) ⇒ 2) ⇒ 3) ⇒ 4) ⇒ 5).

1) f is harmonious.

2) f is point–harmonious.

3) f is triquotient.

4) f is transquotent.

5) f is stable.

1) ⇒ 2) ⇒ 3) follow from the definition of harmonious and point–harmonious maps.

3) ⇒ 4) ⇒ 5) follow from the definition of transquotient maps and Remark 0.

Proposition E. *Compact–covering maps and s–covering maps onto a metrizable space are stable.*

The corresponding stable family η_y is constructed in Example 1.

Proposition F. *If $f\colon X \to Y$ is a stable map with compact fibres, then f is point–harmonious.*

Let η_y be a stable family for f and let $\varepsilon_y = \{B = cl_X U \cap f^{-1}(y) : U \in \eta_y\}$. Define a transmittable family for f as follows:

$\eta'_y = \{U' : U'$ is open in X and there is $B \in \varepsilon_y$ such that $U' \supset B\}$.

It is clear that the condition (a') is satisfied. Let us consider $U' \in \eta'_y$ and some compact $B \in \varepsilon_y$, $B \subset U'$. Since X is regular, there is an open set $V \supset B$ such that $cl_X V \subset U'$. Since $B \supset U \cap f^{-1}(y)$ where $U \in \eta_y$, there is an open $O(y)$ such that $V \in \eta_{y'}$ for every $y' \in O(y)$. Since $U' \supset B' = cl_X V \cap f^{-1}(y')$ and $B' \in \varepsilon_{y'}$, we have $U' \in \eta'_{y'}$ for every $y' \in O(y)$ and (b') is also satisfied.

A map $f\colon X \to Y$ is feebly open if for every (non-empty) open $V \subset X$ we have $\mathrm{Int} f(V) \neq \varnothing$.

Recall that a space Y is an F_{II}–space if there is no closed subset of Y which is of the first category in itself. The following theorem shows that part (2) of Theorem 0 can be extended onto a larger class of F_{II}–spaces.

Theorem 1. *Let $f\colon X \to Y$ be a transquotient map. If X is an F_{II}–space, then Y is also an F_{II}–space.*

We need the following lemma.

Lemma 4. *Let $f\colon X \to Y$ be a transquotient map. Then there is a closed $Z \subset X$ such that the restriction $f|Z$ is a feebly open, transquotient map onto Y.*

Proof . Suppose f is not feebly open; then there is a non-empty open $V \subset X$ such that $\mathrm{Int} f(V) = \varnothing$. Denote $X_1 = X \setminus V, f_1 = f|X_1, \eta^1_y = \{U^1 = U \cap X_1 : U \in \eta_y\}$. First, $U^1 \cap f_1^{-1}(y) \neq \varnothing$. Suppose not; then for some $U \in \eta_y$, $V \supset U \cap f^{-1}(y)$. According to Remark 0, we may suppose that f is not only transquotient but also stable with respect to η_y, hence there is an open $O(y)$ such that $f(V) \supset O(y)$, which is impossible.

It is clear that the transmittable property (a) for f_1 is satisfied.

Let $U^1 \in \eta^1_y$ and $\gamma^1 = \{U^1_\alpha\}$ be an open cover of $f_1^{-1}(y) \cap U^1$. Let us take an open $U \in \eta_y$ such that $U \cap X_1 = U^1$. Obviously, $\gamma = \{T^1_\alpha = U^1_\alpha \cup (X \setminus X_1)\}$ is an open cover of $U \cap f^{-1}(y)$. Hence, there exist $T^1_{\alpha_1}, ..., T^1_{\alpha_n}$ and $O(y) \subset Y$ such that $T^1 = \bigcup_{i=1}^n T^1_{\alpha_i} \in \eta_{y'}$ for every $y' \in O(y) \setminus \{y\}$. It is clear that $T^1 \cap X_1 = \bigcup_{i=1}^n U^1_{\alpha_i} \in \eta^1_{y'}$ for every $y' \in O(y) \setminus \{y\}$ and, hence, (c) is also satisfied and f_1 is transquotient.

Suppose we obtained for some α a strictly decreasing sequence of closed subsets $X_\beta \subset X$ $(\beta < \alpha)$ and transquotient maps $f_\beta = f|X_\beta$ onto Y with hereditary transquotient family $\eta^\beta_y = \{U^\beta = U \cap X_\beta : U \in \eta_y\}$.

If α is not a limit ordinal, that is, if $\alpha = \beta + 1$, and $f|X_\beta$ is not feebly open, we may consider X_α, X_β instead of X_1, X and repeat the construction above.

For the limit ordinal α define: $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$, $f_\alpha = f|X_\alpha$, $\eta^\alpha_y = \{U^\alpha = U \cap X_\alpha : U \in \eta_y\}$ and prove that for every $y \in Y$ and every $U^\alpha \in \eta_y$ we have $U^\alpha \cap f_\alpha^{-1}(y) \neq \varnothing$.

Suppose the contrary. Then for some $U \in \eta_y$ and $U^\alpha = U \cap X_\alpha$, we have $U \cap X_\alpha \cap f^{-1}(y) = \varnothing$. It is clear that $\{O_\beta = X \setminus X_\beta\}_{\beta<\alpha}$ is an open cover of

$f^{-1}(y) \cap U$. Since $X_\beta \supset X_{\beta+1}$ and f is transquotient, there is $\beta < \alpha$ and $y' \in O(y)$ such that $O_\beta \in \eta_{y'}$ and, hence, $O_\beta \cap X_\beta \in \eta^\beta_{y'}$. This contradicts the condition $\varnothing \notin \eta^\beta_{y'}$ because $O_\beta \cap X_\beta = \varnothing$.

Just like for X_1 we can prove that $f|X_\alpha$ is a transquotient map with hereditary transquotient family. We may repeat the above construction if $f|X_\alpha$ is not feebly open, etc. Since $\{X_\alpha\}$ is a strictly decreasing sequence of closed sets, there is α_0 such that $Z = X_{\alpha_0} = X_{\alpha_0+1} = \dots$. Let us denote $f_{\alpha_0} = f|Z$. Then f_{α_0} is a feebly open, transquotient map onto Y. □

To prove Theorem 1 suppose the contrary and let $Y_0 \subset Y$ be a closed subspace of the first category and, hence, Y_0 can be represented as a countable union of nowhere dense and closed in Y subsets F_i. It is not hard to prove (see Proposition C) that $f_0 = f|X_0$, where $X_0 = f^{-1}(Y_0)$, is a transquotient map. By Lemma 4 we can suppose that f_0 is feebly open and, hence, every $f^{-1}(F_i)$ is a closed nowhere dense subset. Then X_0 is a subspace of the first category in itself, which is impossible by definition of F_{II}–space.

Let X be a (non–separable) complete metric space, and let Y be a metric space. I don't know whether the map f in the following cases 1 and 2 must be transquotient (or Y completely metrizable):

1. $f\colon X \to Y$ is a countable–inductively perfect map (i.e., for every countable closed $Y_0 \subset Y$ there is $X_0 \subset X$ such that $f|X_0$ is a perfect map onto Y_0).
2. $f\colon X \to Y$ is a composition of closed and open maps (the domain and the range of all maps are regular).

4. Examples

The assumption that X is separable cannot be omitted in Theorem 0:

Example 1. A stable, non-transquotient map $f\colon X \to Y$ from a completely metrizable space X onto metric space Y which does not satisfy conditions (1) and (2) of Theorem 0.

In fact, let Y not be a union of a countable set and an absolutely G_δ–set. For example, $Y = \mathbb{Q}^\omega$ (where $\mathbb{Q}$ is the space of rational numbers) and let X be a topological sum of all compact subsets of Y with the obvious map $f\colon X \to Y$. It is clear that X is completely metrizable and, by Lemma 3, f is not transquotient. Since a perfect image of a completely metrizable space is completely metrizable, f does not satisfy the conditions of Theorem 0.

We'll prove that f is stable. In fact, let us define:

$\eta_y = \{U : U$ is open in X and for every compact $K \subset Y$ containing y, there is a compact $B \subset X$ such that $f^{-1}(y) \cap B \subset U$ and $f(B) = K\}$.

Obviously, the condition (b) of Definition 1 is fulfilled. We will verify the transmittable property (a). Let us assume the contrary: for some $U \in \eta_y$ there is a base $\{O_i\}$ at the point y, points $y_i \in O_i$ and compacts $K_i \subset O_i$, which contain y_i such that there is no compact subset B_i of X satisfying the conditions: $f^{-1}(y_i) \cap B_i \subset U$ and $f(B_i) = K_i$. It is clear that $S = \{y\} \cup \bigcup\{K_i : i \in \omega\}$ is a compact set and $y \in S$. By definition of η_y there is a compact $B \subset X$ for which $f(B) = S$ and $f^{-1}(y) \cap B \subset U$. Then compact $B \setminus U$ contains a closed countable discrete family of non-empty sets $(f^{-1}(y_i) \cap B) \setminus U$. This is impossible. □

Example 2. A transquotient (and stable) map $f\colon X \to Y$ from a countable, Polish space X onto a compact, metric space Y and the following conditions are satisfied:
1) $f\colon X \to Y$ is not s–covering (= not point–harmonious) map.
2) $f = f_2 \circ f_1$, where f_1 is a closed map and f_2 is an open map.[3]

Let $p_0 < p_1 < ... < p_n < ...$ be a sequence of prime numbers. Denote

$$Y_n = \{1/p_n^{i+1} : i \in \omega\}, \quad Y_n' = \{0\} \cup Y_n \subset \mathbb{I} = [0,1].$$

For the prime numbers p_n, p_k we have $p_n^{i+1} = p_k^{j+1}$ iff $p_k = p_n$. Hence $Y_n' \cap Y_k' = \{0\}$ ($n \neq k$). Denote also:

$$X_n^1 = Y_n' \times \{n+1\}, \quad X_n^2 = Y_n \times \{1/(n+1)\},$$

$$X = \{0,0\} \cup \bigcup \{X_n^1 \cup X_n^2 : n \in \omega\} \subset \mathbb{I} \times \mathbb{R}, \quad Y = \bigcup \{Y_n' \ : n \in \omega\}.$$

Let $\pi\colon \mathbb{I} \times \mathbb{R} \to Y$ be the projection onto the first coordinate and $f = \pi|X$.

1. It is easy to see that f is stable (define η_y as a family of all open in X subsets U such that $y \in f(U)$, if $y \neq 0$ and $U \supset f^{-1}(0)$, if $y = 0$).

2. If f is an s–covering map, then there is a compact $X_0 \subset X$ for which $f(X_0) = Y$ and $X_0 \cap X_n^1 \neq \varnothing$ (since X_n^1 are open and closed in X) for a finite number of n only. Then there is an n for which the set $X_n^2 \cap X_0$ has cardinality $\aleph_0$. Since X_n^2 is closed in X, $X_n^2 \cap X_0$ has an accumulation point in X_n^2. This is impossible.

3. Define f_1 by factorising $f^{-1}(0)$ to one point z_0. Then f_1 is a closed map. The map f_2 from Z onto Y is defined as follows: $f_2(z) = 0$ if $z = z_0$, and $f_2(z) = f(f_1^{-1}(z))$ if $z \neq z_0$. Then f_2 is an open map and $f = f_2 \circ f_1$.

The product of two stable (even two transquotient) maps cannot be stable. Indeed, let us consider the product $t = f \times id_{\mathbb{I}}\colon X \times \mathbb{I} \to Y \times \mathbb{I}$ of the stable, transquotient map f from Example 2 and identity map $id_{\mathbb{I}}$. If t is stable, then by Theorem 0 there is a countable set $Y_\sigma \subset Y \times \mathbb{I}$ and $X_0 \subset X \times \mathbb{I}$ such that $t|X_0$ is a perfect map onto $(Y \times \mathbb{I}) \setminus Y_\sigma$. Hence, there is $x_0 \in \mathbb{I}$ such that $(Y \times \{x_0\}) \cap Y_\sigma = \varnothing$ and $t|t^{-1}(Y \times \{x_0\}) = f\colon X \to Y$ is s–covering. This is impossible.

References

1. G. Choquet, *Lectures on Analysis, Volume* I, Math. Lecture Notes Series–W.A. Benjamin, 1969. MR **40:**3252
2. G. Debs and J.Saint Raymond, *Compact Covering and Game Determinacy*, Topology and Appl. **68** (1996), 153–185. MR **96m:**54067
3. E. Michael, *Complete spaces and tri–quotient maps*, Illinois J. Math. **21(3)** (1977), 716–733. MR **57:**7543
4. E. Michael, *Partition-complete spaces and their preservation by tri-quotient and related maps*, Top. and Appl. **73** (1996), 121–131. MR **97h:**54016
5. A. Ostrovsky, *On compact–covering mappings*, Soviet Math.Dokl. **17** (1976), 606–610.
6. A. Ostrovsky, *New classes of maps related to k-covering maps*, Moskow University Mathematics Bulletin **49** (1994), 20–23.
7. W. Sierpiński, *Sur une propriété des ensembles* G_δ, Fund. Math. (1930), 173–180.
8. A.H. Stone, *Metrizability of decomposition spaces*, PAMS **7** (1956), 690–700. MR **19:**299b
9. I.A. Vainstein, *On closed mappings of metric spaces*, Dokl.Acad.Nauk USSR **57** (1947), 319–321.

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[3] 2) was remarked by V. Popov. Note that f is not a triquotient map.