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Complex Numbers for Block Adjustment

The number of variables is shortened to one-half, and the size of mathematical expressions to one-third or one-fourth.

INTRODUCTION

RIGOROUS LEAST-SQUARES solutions for the simultaneous adjustment of photogrammetric blocks have been discussed by various authors: (Brown, 1967), (Schut, 1967), (Ackermann, 1962), et. al. Within the present stage of development, several thousands of aerial photographs forming a block that mainly extends in two directions, can be adjusted simultaneously even on a medium scale computer (Gyer, et. al., 1969). This has been made possible by the implementation of special compu-

ABSTRACT: *Contrary to other disciplines (e.g., electrical engineering, physics) two-dimensional problems in surveying, geodesy, or photogrammetry have rarely been tackled or solved by means of consistent use of complex numbers. From the theoretical point of view complex numbers give an easier and better insight in the problem structure, the number of mathematical expressions usually being reduced by at least 50 percent. From the practical point of view this topic is worth considering, as e.g., Fortran-IV compilers can process programs written with complex variables. The paper discusses the problem of horizontal block adjustment and its representation in a complex plane. The structure of the matrix of normal equations, here of Hermitian type, is shown, and its solution indicated. A unique, two-dimensional numbering system for block adjustments is introduced, that leads to a special, highly efficient elimination process.*

tational algorithms which take into consideration the regular structure formed by the few non-zero elements of the matrix of normal equations (e.g., Brown, 1967; Elassal, 1969). Independent of the type of units used in photogrammetric networks, this structure essentially is always the same. Whereas Brown (1967), for instance, uses individual photographs as units, the United States Geological Survey (Altenhofen, 1967) takes independent models obtained with analog plotters. Other units are triplets, suitably selected sub-blocks or strips. A profound insight into matrix structures in block adjustments is given by Ackermann (1962).

The fact that in spatial or three-dimensional block triangulation, the degrees of freedom for each photograph (6) or model (7) are quite numerous, and represent arguments of non-linear functions, gave rise to a separation of the vertical coordinate from the planimetric coordinates. This has been particularly useful in densely populated European countries, where planimetric block adjustment has a broad application to large scale cadastral mapping. A special *Anblock*-method has been developed at the ITC in Delft (Van den Hout, 1966). It is assumed that the independent models from which planimetric (or horizontal) coordinates are derived are close enough to level to make negligible horizontal errors due to tilt and relief. Moreover, it is possible

* Presented at the ASP 1970 Symposium on Computational Photogrammetry, Alexandria, Virginia, January 1970.

to approximate successively a rigorous spatial block adjustment by iteratively alternating horizontal and vertical transformations (e.g., Schut, 1967). Treating only planimetric coordinates yields the advantage of directly using linear mathematical expressions. Moreover, as rotations within a plane easily can be described by complex algebra, the formulations become even simpler.

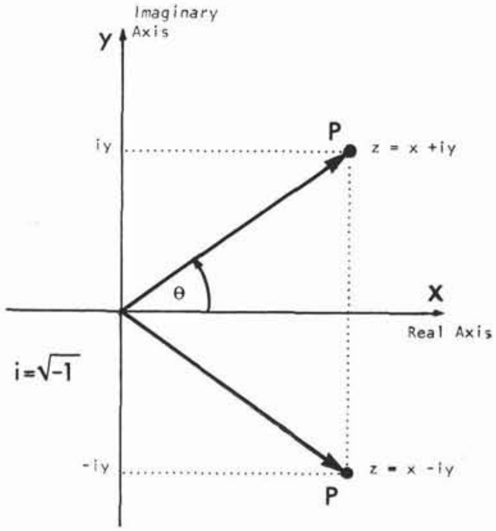

FIG. 1. The complex plane.

That the use of complex arithmetic in computer programs can economize storage and computer time, was mentioned by Schut (1964), (1967), who has been using complex numbers for planimetric block adjustment continuously since 1963. Krijger (1967) has given thought to using complex numbers in the adjustment of survey networks, but, for reasons of the ALGOL-compiler available to him, could not give any practical results. To the knowledge of the author only Krakiwsky (1967) made practical use of complex algebra in surveying computer programs. As modern problem oriented computer languages, such as Fortran-IV or PL/1, contain the basic rules of complex algebra, it seems worthwhile to pursue the idea of consequently and rigorously applying complex numbers in horizontal block adjustments.

The Complex Plane

Any coordinate plane (x,y) can be considered as a complex plane (Figure 1). Complex numbers then are represented geometrically as points in this plane: the real part extends in x-direction, the imaginary part in y-direction. The basic rules of complex algebra are equivalent to those of real algebra. Addition and subtraction of complex numbers is equivalent to adding and subtracting the real parts and the imaginary parts. This is in accordance with the *parallelogram law* between forces.

Multiplication is defined as

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Geometrically, this means a counter clockwise rotation of the vector $OP_1 = z_1$ about the origin O by the argument angle θ_2, and a multiplication of the absolute value r_1 by r_2. It is equivalent to saying the point z_1 has been subject to a similarity transformation z_2. This statement proves to be fundamental for the use of complex numbers in block adjustments.

Division is defined as the inverse operation of multiplication:

$$\frac{z_1}{z_1} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\,.$$

Geometrically, this is a clock-wise rotation of the vector $OP_1 = z_1$ about the origin by the angle θ_2, and a division of the absolute value r_1 by r_2. Of great importance is the definition of a *conjugate* complex number $\bar{z} = x - iy$ of $z = x + iy$. (Figure 1).

A complex matrix is a matrix consisting of complex elements. It can be divided into a real part matrix and an imaginary part matrix. As has been shown by Gröbner (1966), least-squares adjustments can be expanded to complex quantities. The resulting matrix of the normal equations turns out to be of Hermitian type, i.e., original matrix and its conjugate transposed are equal. This implies that the real part is symmetric, the imaginary part is skew-symmetric.

Linear Conformal Transformation

If a set of points z' in a certain x',y'-coordinate system is supposed to be transformed such that its geometric form remains the same, then only two translations along the coordinate axes, one rotation and one scale change may be accomplished. Assuming the z' as model points, and the z as corresponding ground points, the transformation formula may be written in the form

$$z = cz' + r. \tag{1}$$

Here, r is a complex shift or translation, c is the operator for a similarity transformation. In components, Equation 1 yields the well-known form ($r=p+iq$, $c=a+ib$):

$$x = ax' - by' + p$$
$$y = bx' + ay' + q.$$

The following simple example illustrates the usefulness of complex numbers for a least-squares adjustment of several model points z_i', $i=1, 2, \ldots n \geq 2$, to the same number of ground control points z_i. Each point gives rise to one Equation 1. Additionally, a complex residual $w=u+iv$ must be considered yielding

$$cz_i' + r = z_i + w_i, \tag{2}$$

as the coordinates are subject to errors. The least-squares condition for complex numbers is

$$\sum_{i=1}^{n} \bar{w}_i w_i = \text{Min}. \tag{3}$$

This leads to a Hermitian form,

$$\Phi(c, r, c, r) = \sum_{i=1}^{n} (\bar{c}\bar{z}_i' + \bar{r} - \bar{z}_i)(cz_i' + r - z_i)$$

and to the conditions

$$\frac{\partial \Phi}{\partial \bar{c}} = \frac{\partial \Phi}{\partial \bar{r}} = 0,$$

i.e., to the normal equations:

$$\begin{aligned} \left(\sum_{i=1}^{n} \bar{z}_i' z_i'\right) c + \left(\sum_{i=1}^{n} \bar{z}_i'\right) r &= \sum_{i=1}^{n} \bar{z}_i' z_i \\ \left(\sum_{i=1}^{n} z_i'\right) c \qquad\qquad + nr &= \sum_{i=1}^{n} z_i. \end{aligned} \tag{4}$$

As the product of a complex number with its conjugate is a real number, the coefficient associated with c is real. If the model coordinates were related to their point of gravity as origin, the off-diagonal terms vanish, and

$$c = \frac{\sum_{i=1}^{n} \bar{z}_i' z_i}{\sum_{i=1}^{n} \bar{z}_i' z_i'}, \qquad r = \frac{1}{n} \sum_{i=1}^{n} z_i. \tag{5}$$

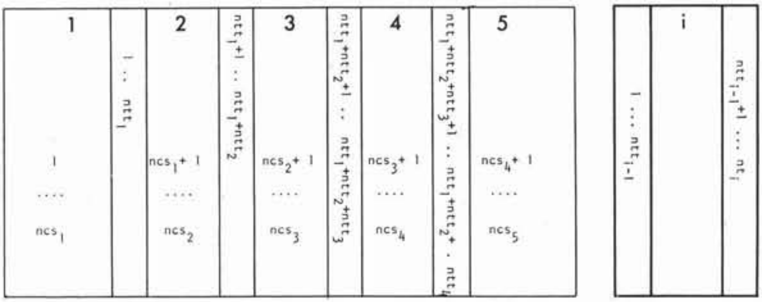

FIG. 2. (*a*) External numbering system (left). (*b*) Internal numbering system.

Complex numbers obviously need less writing, the formulas therefore are more compact and more readable.

HORIZONTAL STRIP ADJUSTMENT

A strip is the natural expansion from one model to a linear sequence of models. Two types of points have to be distinguished: control points and tie points. They may be separated from each other by using different symbols:

ground coordinates of control points	$zc = xc + iyc$
model coordinates of control points	$zc' = xc' + iyc'$
model coordinates of tie points	$z' = x' + iy'$
ground coordinates of tie points	$z = x + iy.$

The efficiency of a numerical treatment of strip or block adjustments depends very much on the ordering or numbering system of the points in use (see e.g., Brown, 1967; Ackermann, 1962).

Whereas control points need to be numbered from an external point of view only, tie points must be numbered both in an external and (model-)-internal system (see Figure 2a, 2b). Denoting with nt_i the number of tie points in model i, with nc_i the number of control points in model i, and with ncs_i the number of control points in all models between the first and the i-th model, then for any model i the following two types of *observation equations* exist. For control points,

$$zc_{ij}'c_i + r_i = zc_{ncs_i - nc_i + j} + wc_{ij} \qquad j = 1, 2, \cdots, nc_i. \tag{6}$$

For tie points,

$$z_{ik}'c_i + r_i - z_{\lambda_i + k} = w_{ik} \qquad k = 1, 2, \cdots, nt_i$$
$$\lambda_1 = \lambda_2 = 0; \qquad \lambda_i = \lambda_{i-2} + nt_{i-2}\ (i > 2). \tag{7}$$

The tie point numbering is consistent if for any model are ordered, firstly, the tie points common with the preceding model, and secondly, tie points common with the following model. The control point numbering is consistent if, where the same control point in two adjacent models, the ground coordinates appear twice and in the correct order. Two kinds of unknown parameters occur: model parameters c_i, r_i, and tie point parameters z_i. Not discussing different weights assigned to the observations, the least-squares condition is given by

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{nc_i} \bar{w}\bar{c}_{ij} wc_{ij} + \sum_{j=1}^{nt_i} \bar{w}_{ij} w_{ij} \right) = \text{Min}. \tag{8}$$

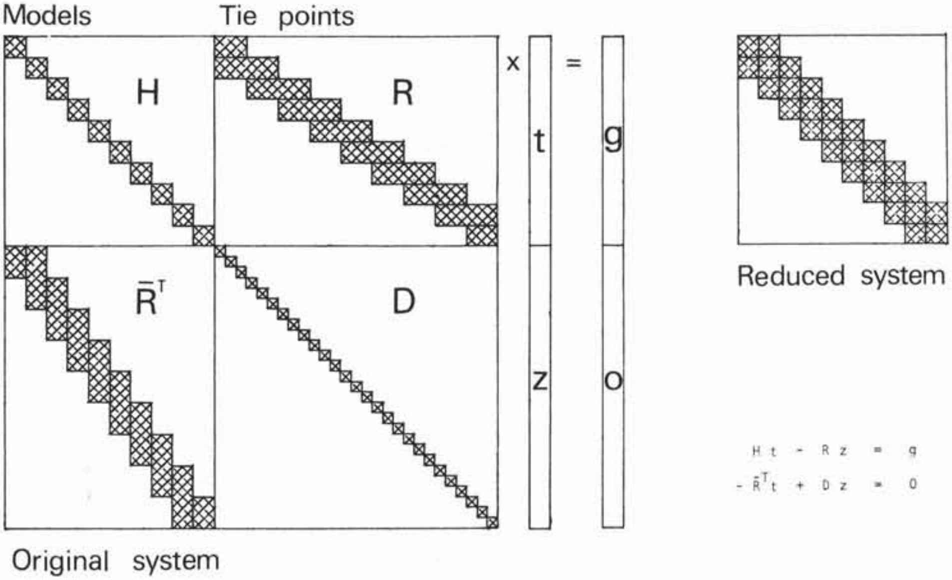

FIG. 3. Block-tridiagonal matrix.

If the model parameters are combined in one (complex) vector

$$t_i = \begin{bmatrix} c_i \\ r_i \end{bmatrix}, \tag{9}$$

the resulting system of normal equations can be subdivided into a model part, and a tie point part (Figure 3). The model sub-matrix H is a block-diagonal Hermitian matrix, each block representing one model, and consisting of 2×2 components. The tie point sub-matrix $D=2I$ is a real scalar matrix. The rectangular matrix R which correlates models and tie points is also of a block-diagonal form. The matrix normal equations are:

$$Ht - Rz = g \qquad -\bar{R}^T t + Dz = 0. \tag{10}$$

Its components are given by the following expressions:

$$\begin{aligned} h_{2i-1,2i-1} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}' z c_{ik}' + \sum_{k=1}^{nt_i} \bar{z}_{ik}' z_{ik}', \\ h_{2i-1,2i} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}' + \sum_{k=1}^{nt_i} \bar{z}_{ik}' \\ h_{2i,2i-1} &= \bar{h}_{2i-1,2i} \\ h_{2i,2i} &= nc_i + nt_i, \end{aligned} \tag{11}$$

all other components being identically zero. Also,

$$\begin{aligned} r_{2i-1,l} &= \begin{cases} \bar{z}'_{i,l-\lambda_i}, & \lambda_i < l \le \lambda_i + nt_i \\ 0, & \text{else} \end{cases} \\ r_{2i,l} &= \begin{cases} 1, & \lambda_i < l \le \lambda_i + nt_i \\ 0, & \text{else} \end{cases} \end{aligned} \tag{12}$$

where

$$d_{kl} = \begin{cases} 2, & k = l \\ 0, & \text{else.} \end{cases} \tag{13}$$

Furthermore, the absolute right-hand terms are:

$$\begin{aligned} g_{2i-1} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}{}' zc_{ncs_i - nc_i + k} \\ g_{2i} &= \sum_{k=1}^{nc_i} zc_{ncs_i - nc_i + k} \end{aligned} \tag{14}$$

and

$$t_{2i-1} = c_i,\ t_{2i} = r_i.$$

If a computer program uses these subscripted variables as shown above, a tremendous amount of storage locations would be required, because of the dimensions of the arrays h, r, d. In order to take advantage of the sparsely filled sub-matrices, arrays with three subscripts have turned out to be of great help and convenience. The first subscript indicates the model to which the variable refers, the second and third indicate the location within the proper sub-matrix. Equations 11 to 14 then read

$$\begin{aligned} h_{i,1,1} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}{}' zc_{ik}{}' + \sum_{k=1}^{nt_i} \bar{z}_{ik}{}' z_{ik}{}' \\ h_{i,1,2} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}{}' + \sum_{k=1}^{nt_i} \bar{z}_{ik}{}' \\ h_{i,2,1} &= \bar{h}_{i,1,2}, \qquad h_{i,2,2} = nc_i + nt_i \end{aligned} \tag{15}$$

$$r_{i,l,1} = \bar{z}'_{i,l}, \qquad r_{i,l,2} = 1 \quad l = 1, 2, \cdots, nt_i \tag{16}$$

$$d_j = 2, \qquad j = 1, 2, \cdots, \sum_{i=j}^{m} nt_i/2 \tag{17}$$

$$\begin{aligned} g_{i,1} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}{}' zc_{ncs_i - nc_i + k} \\ g_{i,2} &= zc_{ncs_i - nc_i + k} \end{aligned} \tag{18}$$

and

$$t_{i,1} = c_i,\ t_{i,2} = r_i.$$

The original Equation 10 can be further reduced to a more compact form. From the second matrix of Equation 10, expression of z in terms of t is easy as $D = 2I$. Substitution of

$$z = D^{-1}\overline{R}^T t = \tfrac{1}{2}\overline{R}^T t \tag{19}$$

into the first of Equation 10 leads to the so-called reduced system of normal equations (Figure 3).

$$(H - \tfrac{1}{2}R\overline{R}^T) \cdot t = g. \tag{20}$$

This is a block-tridiagonal Hermitian matrix, consisting of 2×2 sub-matrices. Denoting the Hermitian sub-matrices along the main diagonal by H_i, and the sub-

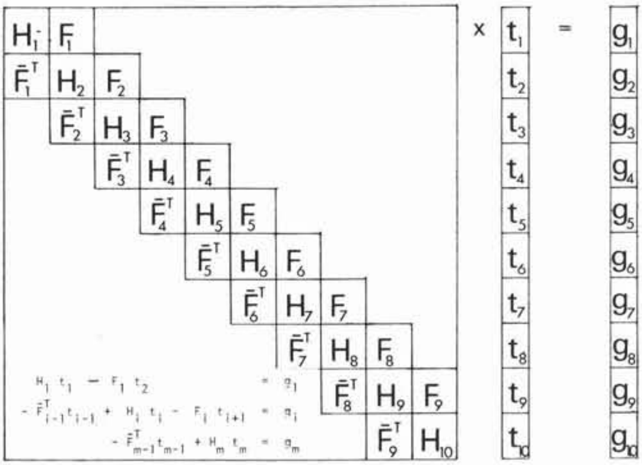

FIG. 4. Block-tridiagonal matrix.

matrices along the upper adjacent diagonal by F_i, the reduced system (Figure 4) reads

$$
\begin{aligned}
H_1 t_1 - F_1 t_2 &= g_1 \\
-\overline{F}^T{}_{i-1} t_{i-1} + H_i t_i - F_i t_{i+1} &= g_i \\
-\overline{F}^T{}_{m-1} t_{m-1} + H_m t_m &= g_m \qquad (i = 2, 3, \cdots, m-1).
\end{aligned}
\tag{21}
$$

In component form:

$$
\begin{aligned}
h_{i,1,1} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}' zc_{ik}' + \frac{1}{2} \sum_{k=1}^{nl_i} \bar{z}_{ik}' z_{ik}' \\
h_{i,1,2} &= \sum_{k=1}^{nc_i} \bar{z}\bar{c}_{ik}' + \frac{1}{2} \sum_{k=1}^{nl_i} \bar{z}_{ik}' \\
h_{i,2,1} &= \bar{h}_{i,1,2} \\
h_{i,2,2} &= nc_i + \tfrac{1}{2}\, nl_i.
\end{aligned}
\tag{22}
$$

Also, if $i < m$:

$$
\begin{aligned}
f_{i,1,1} &= \frac{1}{2} \sum_{k=1}^{ntt_i} \bar{z}'_{i,k+nll_{i-i}} z'_{i+1,k} \\
f_{i,1,2} &= \frac{1}{2} \sum_{k=1}^{ntt_i} \bar{z}'_{i,k+nll_i-1} \\
f_{i,2,1} &= \frac{1}{2} \sum_{k=1}^{ntt_i} \bar{z}'_{i+1,k} \\
f_{i,2,2} &= \frac{1}{2}\, ntt_i,
\end{aligned}
\tag{23}
$$

where ntt_i is equal to the number of tie points common to the two adjacent models i and $i+1$.

An interesting feature of the individual H-matrices is that the off-diagonal elements $h_{i,1,2}$ and $h_{i,2,1}$ are identical to a weighted sum of all pass point coordinates

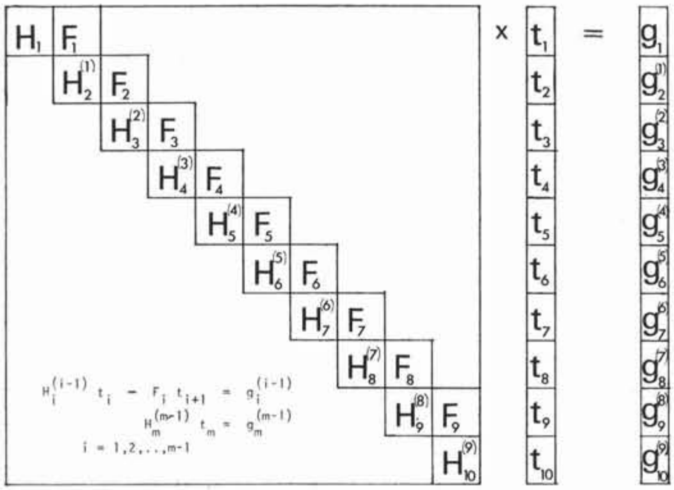

FIG. 5. Block-triangular matrix.

in the particular model: control points weight 1, tie points weight 1/2. For one particular model coordinate system, the origin of which coincides with the correspondingly weighted point of gravity of all pass points in this model, $h_{i,1,2}$ and $h_{i,2,1}$ would be equal to zero, i.e., all sub-matrices H_i would degenerate to real diagonal matrices. Though attractive, this property does not bear any computational advantage, as it will be lost again during the subsequent matrix partitioning.

Similar to a method named of recurrent partitioning (Brown, 1968), the block-tridiagonal matrix of the reduced normal equations can be reduced further. This reduction consists of successively eliminating the unknown model parameter vectors, beginning with the first model. In matrix notation, this means expressing t_1 in the first model in terms of t_2 from the first of Equation 21:

$$t_1 = H_1^{-1}g_1 + H_1^{-1}F_1t_2 \tag{24}$$

and substituting it in the second of Equation 21, yielding

$$H_2^{(1)}t_2 - F_2t_3 = g_2^{(1)} \tag{25}$$

where

$$H_2^{(1)} = H_2 - \overline{F}_1^TH_1^{-1}F_1,\ g_2^{(1)} = g_2 + \overline{F}_1^TH_1^{-1}g_1.$$

The process has to be continued until a last equation

$$H_m^{(m-1)}t_m = g_m^{(m-1)} \tag{26}$$

is obtained, from which t_m can easily be determined. Thus, System 21 has been changed into a *block-triangular matrix* with two block-diagonals (Figure 5).

$$H_i^{(i-1)}t_i - F_it_{i+1} = g_i^{(i-1)}\ (i < m) \qquad H_m^{(m-1)}t_m = g_m^{(m-1)} \tag{27}$$

where

$$\begin{aligned} H_i^{(i-1)} &= H_i = \overline{F}_{i-1}^T[H_{i-1}^{(i-2)}]^{-1}F_{i-1}, \\ g_i^{(i-1)} &= g_i + \overline{F}_{i-1}^T[H_{i-1}^{(i-2)}]^{-1}g_{i-1} \qquad (i = 1, 2, \cdots, m). \end{aligned} \tag{28}$$

By a procedure analog to ordinary back-substitution, all model parameters may be determined:

$$
\begin{aligned}
t_m &= [H_m{}^{(m-1)}]^{-1} g_m{}^{(m-1)} \\
t_{m-i} &= [H_{m-i}{}^{(m-i-1)}]^{-1} (g_{m-i}{}^{(m-i-1)} + F_{m-i} t_{m-i+1}) \qquad (i = 1, 2, \cdots, m-1).
\end{aligned}
\tag{29}
$$

The model parameters are by far the most important ones, as now any point in any model can be transformed to the ground system by using Equation 1. Although the adjusted ground coordinates of the tie points can be determined from the matrix Equation 19, a closer inspection of the resulting relations shows that the same answer is obtained if Equation 1 is used. As with Equation 1, tie points are computed twice, their arithmetic mean is identical to the value obtained from Equation 19.

Horizontal Block Adjustment

The previous section showed the usefulness of complex numbers and of an intelligent numbering system for single strip adjustments. It can be expected that, as any block may be considered consisting of various strips, for a simultaneous adjustment of total blocks, even more attention has to be paid to the numbering. It has been shown (e.g., Brown, 1967) that the structure of the matrix of normal equations depends on the numbering of models and tie points. Main emphasis has been put on rearranging the adjustment parameters such as to obtain a matrix whose elements are concentrated around the main diagonal with as small a band width as possible. The numbering and ordering so far seems to bear a great deal of subjectivity, although efforts have been made for optimization. A natural order scheme is given by grouping the model tie points along rows (or strips) or along columns. It still, however, bears some arbitrariness particularly in cases of irregular tie point numbers. Within this context an attempt is made to develop the most natural numbering system realizing the two-dimensional extension of the block. This has been made possible by introducing tie zones rather than individual tie points. Models as well as tie zones are numbered with double subscripts, the first subscript indicating the strip, the second indicating the column. Figure 6 represents a block of three strips each consisting of 6 models. The models are numbered from (1,1) through (1,6), (2,1), $\cdots$, (2,6), and (3,1) to (3,6). Tie zones together with the remaining model zones are numbered according to the same principle. These numbers provide an instant indication of the type of zone: if both subscripts are even, four models are overlapping; if one is even and the other odd, only two models overlap. If both subscripts are odd numbers, the zone belongs just to one model and may be called control zone, because all ground control can be considered lying within it. Connection between model and tie zone numbers is given by the fact that a model j,k corresponds to the zone $2j-1$, $2k-1$.

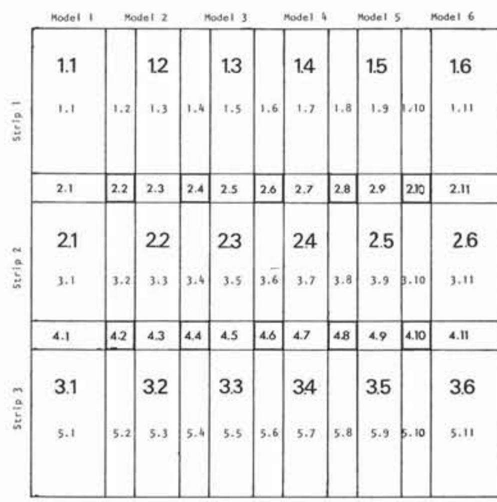

FIG. 6. Unbiased numbering system for models and tie zones.

Besides this *external* numbering system, tie zones surrounding a model advantageously are numbered *internally* by double subscripts $r,s = 1,2,3$ (see Figure 7). Similar to Equation 6 and 7, each control point or tie point measured in model k of strip j, and situated within the (internal) zone r,s gives rise to one observation equation ($j = 1,2, \ldots q$; $k = 1,2, \ldots m$):

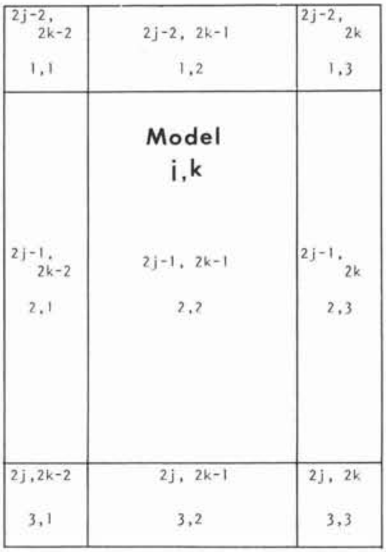

FIG. 7. Relations around Model j, k. External and internal tie-zone numbering.

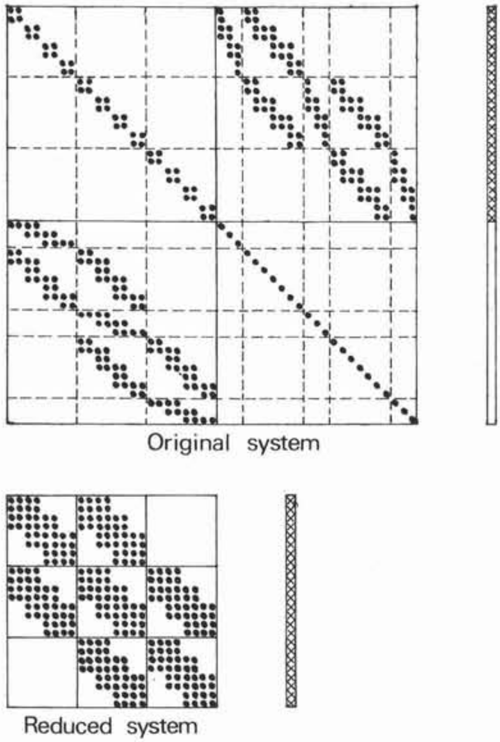

FIG. 8. Horizontal block of three strips with four models each. Ordered row-wise.

$$z'_{jkrs}c_{jk} + r_{jk} - z_{2j-3+r,2k-3+s} = w_{jkrs}. \tag{30}$$

This equation comprises of both control and tie points;* but note that control points can be located in the control zone $r=s=2$ only. Because of the fact that four dimensional arrays are used, a simple graphical representation of the distribution of all non-zero elements in the design matrix is impossible on a two-dimensional sheet of paper. It is this difficulty that makes it necessary to arrange the models and tie zones in a certain (however consistent) sequence in order to obtain a two-dimensional matrix display. Such a sequence may be done according to strips or according to columns. The normal equations in matrix form are equivalent to the System 10, except that the connecting matrix R consists of several diagonal strips (Figure 8).

Denoting the (2×2) submatrices of H by H_{jk}, the elements on the main diagonal in D by d_{il}, and the collapsed sub-matrices in R by R_{jkrs}, the normal equations can be written as

$$\begin{aligned} H_{jk}t_{jk} - \sum_{r,s=r',s'}^{r'',s''}\!\!\circ\!\!\sum R_{jkrs}z_{2j-3+r\ 2k-3+r} &= g_{jk}{}^{\dagger} \\ \sum_{u=j'}^{j''}\sum_{v=k'}^{k''}\overline{R}^{T}_{u,v,i+3-2u,l+3-2v}t_{uv} + d_{il}z_{il} &= 0 \end{aligned} \tag{31}$$

where

* Zones rather than points; but this does not affect the principles of the method. For simplicity one may think of each zone consisting of one point only.

† The connected two sigma symbols $\Sigma\circ\Sigma$ have the meaning that the sum must be taken exclusively around (*circular sum*) the central zone $r=s=2$.

$$r' = \begin{cases} 2, & \text{if } j = 1 \\ 1, & \text{if } j > 1 \end{cases} \qquad r'' = \begin{cases} 3, & \text{if } j < q \\ 2, & \text{if } j = q \end{cases}$$
$$s' = \begin{cases} 2, & \text{if } k = 1 \\ 1, & \text{if } k > 1 \end{cases} \qquad s'' = \begin{cases} 3, & \text{if } k < m \\ 2, & \text{if } k = m \end{cases} \tag{32}$$

$$j' = \begin{cases} j, & \text{if } j = 1 \\ j-1, & \text{if } j > 1 \end{cases} \qquad j'' = \begin{cases} j+1, & \text{if } j < q \\ j, & \text{if } j = q \end{cases}$$
$$k' = \begin{cases} k, & \text{if } k = 1 \\ k-1, & \text{if } k > 1 \end{cases} \qquad k'' = \begin{cases} k+1, & \text{if } k < m \\ k, & \text{if } k = m. \end{cases} \tag{33}$$

The components are

$$h_{jk11} = \sum_{r=r'}^{r''} \sum_{s=s'}^{s''} \bar{z}'_{jkrs} z'_{jkrs}$$
$$h_{jk12} = \sum_{r=r'}^{r''} \sum_{s=s'}^{s''} \bar{z}'_{jkrs}, \qquad h_{jk21} = \bar{h}_{jk12} \tag{34}$$
$$h_{jk22} = \sum_{r=r'}^{r''} \sum_{s=s'}^{s''} n_{jkrs}$$

$$r_{jkrs1} = \bar{z}'_{j,k,2j-3+r,2k-3+s}$$
$$r_{jkrs2} = 1 \tag{35}$$

$$d_{il} = \begin{cases} 2, & \text{if } (i+l) \text{ is odd} \\ 4, & \text{if } i \text{ and } l \text{ are even} \end{cases} \tag{36}$$

$$g_{jk1} = \bar{z}'_{jk22} z_{2j-1,2k-1}$$
$$g_{jk2} = z_{2j-1,2k-1}. \tag{37}$$

By eliminating the tie points z_{il} in the original System 31 of normal equations, reduced normal equations are obtained. As indicated in Figure 8, the matrix of the reduced system consists of several block-tri-diagonal sub-matrices which themselves as units are arranged in block-tridiagonal form. The structure therefore is very simple: If the ordering takes place according to strips, the total matrix H of the reduced system is a block-tridiagonal Hermitian matrix. The size of the sub-matrices is given by the number of strips. The sub-matrices are also of block-tridiagonal form. The number of (2×2) sub-sub-matrices is given by the number of models per strip which is equivalent to the number of columns.

From the second of Equation 31, z_{il} can be expressed in terms of t_{uv} and substituted into the first equation. The most general form of the system of reduced normal equations therefore reads as

$$H_{jk}t_{jk} - \sum_{r,s=r',s'}^{r'',s''}\circ\sum \frac{R_{jkrs}}{d_{2j-3+r,2k-3+s}} \sum_{u=j'}^{j''} \sum_{v=k'}^{k''} \overline{R}^{T}_{u\ v,2j-2u+r,2k-2v+s} t_{uv} = g_{jk}. \tag{34}$$

For a model completely surrounded by adjacent models, i.e., for any non-marginal model, r and s take the values $1,2,3,u$ takes the values $j-1$, $j+1$ and v the values $k-1$, k, $k+1$, which ensues from Equations 32 and 33. Extracting from the second compound term on the left-hand side of Equation 34 the term for which $u=j$, $v=k$, and combining it with the first term $H_{jk}t_{jk}$, yields the reduced matrices of all block-diagonal Hermitian matrices H_{jk}. As any non-marginal model j,k is surrounded by

H_{1122}	H_{1123}			H_{1132}	H_{1133}						
H_{1221}	H_{1222}	H_{1223}		H_{1231}	H_{1232}	H_{1233}					
	H_{1321}	H_{1322}	H_{1323}		H_{1331}	H_{1332}	H_{1333}				
		H_{1421}	H_{1422}			H_{1431}	H_{1432}				
H_{2112}	H_{2113}			H_{2122}	H_{2123}			H_{2132}	H_{2133}		
H_{2211}	H_{2212}	H_{2213}		H_{2221}	H_{2222}	H_{2223}		H_{2231}	H_{2232}	H_{2233}	
	H_{2311}	H_{2312}	H_{2313}		H_{2321}	H_{2322}	H_{2323}		H_{2331}	H_{2332}	H_{2333}
		H_{2411}	H_{2412}			H_{2421}	H_{2422}			H_{2431}	H_{2432}
				H_{3112}	H_{3113}			H_{3122}	H_{3123}		
				H_{3211}	H_{3212}	H_{3213}		H_{3221}	H_{3222}	H_{3223}	
					H_{3311}	H_{3312}	H_{3313}		H_{3321}	H_{3322}	H_{3323}
						H_{3411}	H_{3412}			H_{3421}	H_{3422}

FIG. 9. Arrangement and numbering of sub-matrices for the reduced system of normal equations of a block consisting of three rows and four columns.

eight adjacent models, eight off-block-diagonal matrices which indicate the correlation between neighboring models can be set up. As they essentially have the same structure, all (2×2) submatrices are distinguishable by introducing two more subscripts u,v which can take the values 1,2,3. Figure 9 shows the entire matrix of the reduced system for a block of three strips and four columns ordered according to strips. Only the sub-matrices on and above the main block-diagonal are linearly independent, the reason being the Hermitian type of the matrix. The most general form expressing the throughout block-tridiagonality of the matrix of reduced normal equations then can be written as

$$
\begin{aligned}
H_{jk22} &= R_{jk22}\overline{R}^T{}_{jk22} + \sum_{r,s=r',s'}^{r'',s''}\!\circ\!\sum \left(1 - \frac{1}{d_{2j-3+r\ 2k-3+s}}\right) R_{jkrs}\overline{R}^T{}_{jkrs} \\
H_{jkuv} &= -\sum_{r,s=r',s'}^{r'',s''}\!\circ\!\sum \frac{R_{jkrs}\overline{R}^T{}_{j+u-2,k+v-2,r-2(u-2),s-2(v-2)}}{d_{2j-3+r,2k-3+s}}
\end{aligned}
\tag{35}
$$

(u and v are not equal to 2 simultaneously).

It is obvious that one of the most important features of the above representation is the large amount of algebraic manipulations occurring amongst the subscripts. This is the reason for the compactness of all equations and expressions. By a process of recurrent partitioning analogous to that one described in strip adjustment, the entire matrix can be altered to a block-triangular matrix. Except for the very first sub-matrix on the main block-diagonal, all other Hermitian sub-matrices (i.e., those situated on the main block-diagonal) lose their block-tridiagonal character. During the sequential reduction procedure, they turn out to be general Hermitian matrices with all elements filled. From the numerical point of view, as they have to be inverted, it is therefore desirable to obtain such sub-matrices as small as possible. This means that the models and tie zones preferably are ordered accoiding to the larger number of either strips (rows) or columns. For a block of three strips and four columns, Figures 8 and 10 show the difference. Ordering according to columns here results in smaller Hermitian matrices (Figures 11a, 11b) along the main block-diagonal, hence is preferable. Independent of that, however, is the fact that during recurrent partitioning and during back-substitution, general Hermitian matrices are to be inverted. Because of a close relationship between Hermitian and real symmetric matrices, the algorithm is quite similar to the standard Gaussian algorithm. The principal characteristics of the solution of a system of reduced normal equations derived from a planimetric block is best understood through the extremely regular interlocking of (2×2)-model sub-matrices and $(q \times q)$-strip sub-matrices or $(m \times m)$-column sub-matrices. It is this regular structure that enables the setting up of highly efficient algorithms for general solutions of block adjustments.

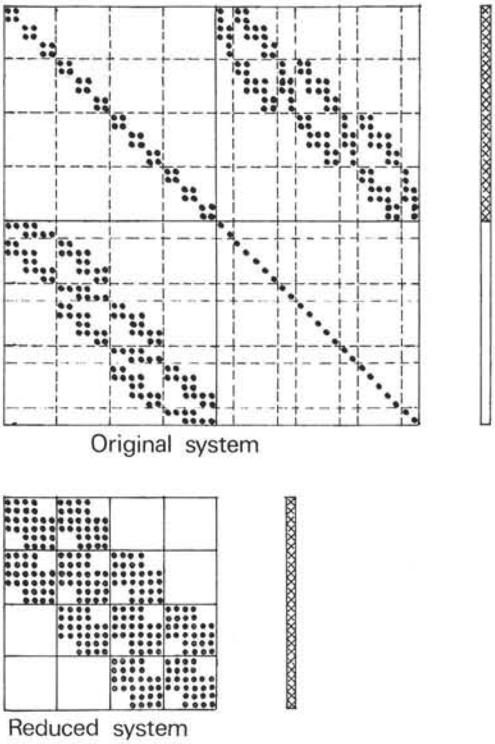

FIG. 10. Horizontal block of three strips with four models each. Ordered column-wise.

CONCLUSIONS

This paper has been intended to show the feasibility of incorporating complex algebra in planimetric block adjustments. Use of complex numbers shortens the number of variables to one-half, and the size of mathematical expressions to one-third or even one-fourth, as compared with real variables. This is not only of theoretical interest, but also desirable from the programming point of view, since several computer languages contain complex arithmetic. An important by-product of this investigation has been the evaluation of objective and unbiased numbering systems. By using variables with several subscripts, a natural connection between model numbers and tie point or tie zone numbers can be obtained. The characteristic block-tridiagonal structure of the matrix of reduced normal equations, containing only model parameters, has been fully utilized. Extremely compact formulations have been obtained by algebraic manipulations of subscripts. Future work will be concentrated on taking advantage of these results and writing corresponding computer programs.

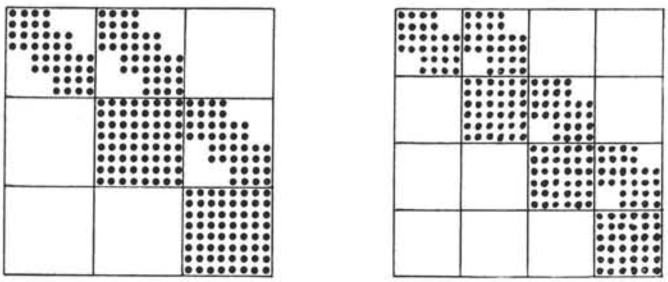

FIG. 11. (*a*) (left) Block-triangular matrix, ordered row-wise. (*b*) Block-triangular matrix, ordered column-wise.

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RESUMEN

RESUMEN DE LOS ARTÍCULOS TÉCNICOS PUBLICADOS EN EL MES PREVIO*

Vol. XXXVI No. 12 Diciembre 1970

Dispersión de los Desperdicios en el Mar

Por *Wesley James* y *Fred J. Burgess*

La fotografía aérea puede ser un método para analizar la dispersión de los desperdicios que se arrojan al mar. Se describe uno de los procedimientos usados para determinar las concentraciones de desperdicios por medio de la aerofotografía.

Es posible que esta técnica rinda mejores resultados que los métodos convencionales de estudio por medio de barcos. Las discrepancias entre las concentraciones por barco y las fotográficas, se deben principalmente al cambio y variación de los desperdicios en este ambiente dinámico. La técnica fotográfica es un método de estudio que permitirá conocer el problema de la dispersión, lo que ha sido imposible de efectuar con métodos convencionales de pruebas.

Photo. Engr., Diciembre 1970, página 1241

Pseudoradar: Fotografía Aérea de Alto Contraste por Angulos Solares Bajos

Por *R. J. P. Lyon*, *José Mercado* y *Robert Campbell, Jr.*

El análisis de imágenes de radar de toma lateral, banda K, indicó que la mayor parte de su utilidad geológica proviene de: a) su presentación a escala pequeña (cerca de 1:100,000) y b) sus sombras fuertes, negras como el azabache, que recalcaron enormemente el relieve topográfico. Varios ensayos publicados subrayaron el efecto de los ángulos solares bajos en la fotografía aérea vertical, y por eso ideamos esta técnica para simular el radar de toma lateral (SLAR) por medio de la fotografía aérea convencional, pero con el sol a unos 20–30° sobre el horizonte. Se propone que este tipo de aerofotografía poco convencional se denomine fotografía por ángulos solares bajos (LSAP).

Photo. Engr., Diciembre 1970, página 1257

* *Nota:* Traducido por la Sección de Traducciones de la Escuela Cartográfica del Servicio Geodésico Interamericano (IAGS).