

σ -HOMOGENEITY OF BOREL SETS

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ABSTRACT. We give an affirmative answer to the following question:

Is any Borel subset of a Cantor set \mathbf{C} a sum of a countable number of pairwise disjoint h -homogeneous subspaces that are closed in X ?

It follows that every Borel set $X \subset \mathbf{R}^n$ can be partitioned into countably many h -homogeneous subspaces that are G_δ -sets in X .

We will denote by \mathbf{R} , \mathbf{P} , \mathbf{Q} , and \mathbf{C} the spaces of real, irrational, rational numbers, and a Cantor set, respectively.

Recall that a zero-dimensional topological space X is h -homogeneous if U is homeomorphic to X for each nonempty clopen subset $U \subset X$. More about topological properties of h -homogeneous spaces see, for example, in [5], [6], [7], [12].

We call a zero-dimensional metric space X σ -homogeneous if it is a countable union of h -homogeneous subspaces X_i that are closed in X . It is easily seen that every set $X_i \setminus \bigcup_{j < i} X_j$ is an open subspace in X_i and can be partitioned into countably many pairwise disjoint subsets that are clopen in X_i .

Hence, a space $X \subset \mathbf{C}$ is σ -homogeneous iff it can be partitioned into a countably many pairwise disjoint h -homogeneous subspaces that are closed in X .

According to the Cantor–Bendixson theorem, every closed subset $F \subset \mathbf{C}$ is σ -homogeneous.

The question of whether this assertion holds for all Borel subsets of \mathbf{C} was posed in [8, p.228].

The following theorem gives an affirmative answer to the above question.

Theorem 1. *Every Borel set $X \subset \mathbf{C}$ is a σ -homogeneous space.*

Proof. We proved the following simple proposition in [9, Theorem 7]:

Every Π_2^0 -set (and every Σ_2^0 -set) $X \subset \mathbf{C}$ is representable as a union of countably many disjoint closed copies of following spaces:

- (a) a singleton set;
- (b) a Cantor set \mathbf{C} ;
- (c) irrational numbers \mathbf{P} .

From the topological characterization of \mathbf{C} and \mathbf{P} it follows that they (and obviously a singleton set) are h -homogeneous.

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Recall (for more detail we refer the reader to [2]) that $A \leq_w B$ if for some continuous $f : \mathbf{C} \rightarrow \mathbf{C}$ we have $A = f^{-1}(B)$.

The Borel Wadge class of a Borel set A is $[A] = \{B \subset \mathbf{C} : B \leq_w A\}$.

The Wadge ordering $<$ on dual pairs $\{\Gamma, \check{\Gamma}\}$ (where $\check{\Gamma} = \{\mathbf{C} \setminus A : A \in \Gamma\}$) of Wadge classes that well-orders the pairs of Borel Wadge classes is defined by

$\{\Gamma_0, \check{\Gamma}_0\} < \{\Gamma_1, \check{\Gamma}_1\}$ if and only if $\Gamma_0 \subset \Gamma_1$ and $\Gamma_0 \neq \Gamma_1$.

Γ is self-dual if $\Gamma = \check{\Gamma}$.

Also, for the classes Γ_{u_0} and $\check{\Gamma}_{u_0}$, where Γ_{u_0} is the class of F_σ -sets and $\check{\Gamma}_{u_0}$ is the class of G_δ -sets, Theorem 1 was proved in [9].

We make an induction hypothesis that the theorem is valid for all Γ_α and $\check{\Gamma}_\alpha$ for all $\alpha < \beta$.

Below, we consider two cases **1** and **2**.

1. Suppose Γ_β is not a self-dual class and $X \in \Gamma_\beta \setminus \check{\Gamma}_\beta$.

1.1. If X contains a clopen set U_1 of some class Γ_{α_1} , $\alpha_1 < \beta$, then U_1 falls under the induction hypothesis, and we then consider the set $X_1 = X \setminus U_1$. Obviously, X_1 is closed in X .

If X_1 contains a clopen (in X_1) subset U_2 of some class Γ_{α_2} , $\alpha_2 < \beta$, then it falls under the induction hypothesis, and we then consider the closed set $X_2 = X_1 \setminus U_2$.

Continuing this process as above, we get a chain of closed sets

$$X \supset X_1 \supset \dots \supset X_\gamma \supset \dots$$

($X_\gamma = \bigcap_{\beta < \gamma} X_\beta$ for the limit γ) that, as we know, stabilizes at some countable $\gamma_0 < \omega_1$; i.e., $X_{\gamma_0} = X_{\gamma_0+1} = \dots$

It is clear that X_{γ_0} is a closed set.

Obviously, X is a countable union of pairwise disjoint closed sets U_α , ($\alpha < \gamma_0$) and X_{γ_0} .

If $X_{\gamma_0} \in \Gamma_\alpha$ with $\alpha < \beta$, then the theorem is proved since the sets U_α and X_{γ_0} fall under the induction hypothesis. Hence we can suppose that X_{γ_0} is nonempty and everywhere $\Gamma_\beta \setminus \check{\Gamma}_\beta$.

1.2. If X_{γ_0} is everywhere of the second category, then we get the theorem since by theorems Keldysh, Harrington and Steel [3], [9], [11] all the spaces everywhere of the second category and everywhere $\Gamma_\beta \setminus \check{\Gamma}_\beta$ (for non- F_σ or non- G_δ classes) are homeomorphic.

1.3. Let X_{γ_0} be not everywhere of the second category and, hence, contains a clopen (in X_{γ_0}) subset T_1 of the first category.

If $Y_1 = Y \setminus T_1$ contains a clopen set U_1 of some class Γ_{α_1} , $\alpha_1 < \beta$, we can repeat the process of **1.1**, etc.

It is clear that we obtain by this way a subspace T that is (everywhere) of the first category and everywhere $\Gamma_\beta \setminus \check{\Gamma}_\beta$, which is h -homogeneous by theorems Keldysh, Harrington and Steel.

2.0. $X \in \Gamma \cap \check{\Gamma}$. Then [2, Lemma 4.4.1] there is a nonempty clopen subset $D \subset X$ such that $[X \cap D] <_w [X]$ and X is decomposed into sets of lower Wadge rank.

We can repeat the process of **1.1**.

□

Since $\mathbf{R} = \mathbf{P} \cup \mathbf{Q}$ and \mathbf{P} is homeomorphic to some G_δ subset in \mathbf{C} and \mathbf{R} , we obtain the following corollary:

Every Borel set $X \subset \mathbf{R}^n$ can be partitioned into countably many h -homogeneous G_δ -subspaces.

Questions on the number of topological types of homogeneous Borel sets have been posed by Aleksandrov and Urysohn, Lusin, Keldysh [1], [4], [3].

By Keldysh's theorem, every Borel set in \mathbf{C} is a countable sum of *canonical elements* that are homeomorphic to \mathbf{P} , \mathbf{C} , a singleton set or h -homogeneous Π_α^0 -sets (which are not Σ_α^0 -sets, $\alpha > 1$) of the first category in themselves. [10] [3].

Since \mathbf{P} , \mathbf{C} , and a singleton set are spaces of the second category in themselves, it would be reasonable to find an analogue of Keldysh's theorem for h -homogeneous subspaces of the second category. Using the following simple observation (see also [9]) we show below that the assertion of Keldysh's theorem holds for h -homogeneous subspaces of the second category in themselves.

Remark. If X is of the first category in itself everywhere $\Gamma_\beta \setminus \check{\Gamma}_\beta$, where $\Gamma_\beta = \Pi_\alpha^0$ ($\alpha > 1$), then X is homeomorphic to the product $Y \times \mathbf{Q}$, where Y is a space everywhere $\Gamma_\beta \setminus \check{\Gamma}_\beta$ of the second category in itself.

Indeed, denote (all embeddings in \mathbf{C} are dense):

$$\begin{aligned} \mathbf{C}_1 &= cl_{\mathbf{C}} X \text{ (it is clear that } \mathbf{C}_1 \text{ is homeomorphic to } \mathbf{C}\text{);} \\ Y &= \mathbf{C} \setminus ((\mathbf{C}_1 \setminus X)) \times \mathbf{Q}. \end{aligned}$$

Obviously, Y is everywhere $\Gamma_\beta \setminus \check{\Gamma}_\beta$ of the second category in itself and $Y \times \mathbf{Q}$ is everywhere $\Gamma_\beta \setminus \check{\Gamma}_\beta$ of the first category in itself.

Finally, $Y \times \mathbf{Q}$ is homeomorphic to X^1 . Hence, every canonical element of Keldysh X is a sum of a countable number of pairwise disjoint h -homogeneous subspaces of the second category (that are closed in X).

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¹Note that S. Medvedev proved that every h -homogeneous space $Y \subset \mathbf{C}$ of the first category in itself is homeomorphic to $Y \times \mathbf{Q}$. For more details we refer the reader to [6].

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