From Elliptic Arc Length to Gauss-Krüger Coordinates by Analytical Continuation

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Prologue

The majority of contemporary geodesists[1] considers Eric Grafarend as a most remarkable and outstanding scientist as well as brilliant scholar in the field of geodesy. His strong opinions, founded upon a thorough understanding of mathematical reasoning as applied to geodetic science, his clear views on essentials and needs for a science oriented university education with emphasis on fundamentals, and his openness to other fields and different cultures has gained him many friends throughout the world. In fact, Eric Grafarend's creative and productive power is enormous and can hardly be equaled. His background easily has enabled him investigating new and old problems of our common profession from a purely theoretical, highly mathematical point of view. This is in contrast to the majority of geodesists who consider geodesy as engineering science rather than (geo-) science *per se*, and which should be primarily oriented towards practical applications. Although, partly for that reason, most of Eric Grafarend's scientific publications are beyond the comprehension for those *other* geodesists, I nevertheless consider his work essential for a theoretically sound deepening of our profession.

Considering myself one of those *other* geodesists, I would not even think of attempting to write something that could come close to the quality of the level of Eric Grafarend's publications in terms of mathematical rigor and degree of abstraction. Let alone, I would not succeed anyhow. Yet, both of us seem to have a few convictions in common that encourage me to a contribution dedicated to him on occasion of his 60^{th} birthday even knowing it would not meet his high scientific standards. In my scientific endeavor one of my peculiarities always has been a certain desire, if not longing, for search and investigation of novel ideas and unconventional methods or techniques, even though nothing spectacular would have to be expected from the final results. It is somewhat strange that my main interests were concentrated not so much on the outcome of a certain study but often rather more on the way leading to a solution of the problem. I think Eric Grafarend's way of living for science is not too far off from such an attitude but one level higher, of course. My modest contribution will then not be entirely in vain.

[1] For reason of simplicity and in accordance with Eric Grafarend's understanding, in this paper the definition of *geodesy* is adopted according to the European view, i.e. encompassing the entire spectrum of fields of expertise of *surveying engineering* (a new term is *geomatics*), even though the author, who does not entirely agree with this, has difficulties in finding his own specialization (photogrammetry and remote sensing) properly represented under this name.

1 Introduction

It remains one of the mysteries in geodesy why most of the differential geometric relations on the spheroid (ellipsoid of revolution) always have been developed into truncated power series for numerical computations. Although understandable from a historical point of view when all calculations had to be carried out by hand, there is no reason why this should be done the same way today with computers. A typical example is the computation of Gauss-Krüger coordinates or UTM-coordinates (see, e.g., [Hubeny, 1953]). Virtually all existing software routines employ algorithms derived solely from incomplete power series that were developed ages ago and for regional use only, and nobody asks anymore if there existed more general, universal and mathematically sound algorithms. For the same reason I never really could understand why in geodesy hardly any complex numbers are used and why practically all geodesists, exceptions, of course, prove the rule, prefer to circumvent complex arithmetic despite their claims that conformal mappings on the spheroid are essential. It is a fact, however, that conformal mappings such as the Gauss-Krüger projection are based on and easiest represented by complex numbers, and algorithms written in computer languages containing complex arithmetic turn out to be rather short, effective and transparent.

When we notice departures from this line then mostly those originating from non-geodesists or outright *outsiders*. E.g., in [Klotz, 1993] efforts are undertaken to extend truncated power series from local to global by recursive definitions; the treatise [Lee, 1976] elaborates on conformal projections based on elliptic functions and integrals; and in [Gerstl, 1984] numerical evaluations of (complex) elliptic integrals are performed by Landen transformations. Seemingly

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unnoticed in the geodetic community and despite their innovative character, these research studies have remained in a somewhat dormant state. In a way, this is rather unfortunate because of the knowledge we are carelessly throwing away which, on the other hand, would be of considerable value and help for a better understanding of and insight in a truly geodetic matter. Really disillusioning is, in my opinion, that the mathematical relevance of transitions from real to complex by *analytic continuation,* since inherently having practical consequences, are rarely understood by many geodesists. E.g., who knows that any valid, in this case real, mathematical formulation for the arc length along the meridian of a spheroid as function of the (real) isometric latitude, immediately yields (conformal) Gauss-Krüger coordinates if the quantities used are extended to the complex domain.

The author remembers with horror the lectures on "Landesvermessung" when his teacher, with a relatively high degree of dilettantism, tried to explain both nature and background of the Gauss-Krüger projection and derive their mathematical relations. Instead of having kept to the simple essentials, the matter submerged into a sea of obscurity, and it would take the author many years of own search until he became sufficiently confident in comprehending the subject. The following paragraphs are to present the findings and results of the author's work as an outsider to the whole matter. By virtue of his understanding of belonging to an engineering science, main emphasis will ultimately be placed upon practical applicability rather than theoretical rigor. This leads to the presentation of not only general formulations and algorithms derived therefrom but also genuine yet simple and immediately applicable computer programs. Although not new in mathematical literature, the numerical evaluation of elliptic integrals of the second and third kind, essentially defining the arc length on the meridian of a spheroid, will be based entirely on the highly convergent Landen transformation. Whether the geodetic community finally will appreciate this or not remains to be seen. While this topic forms the kernel of the paper, the transition to Gauss-Krüger coordinates represents but a mere extension from real to complex numbers without modification of the algorithms.

2 Elliptic Integrals

The radius of curvature μ of the spheroid with semi-major axis $a = 1$ in the direction of the meridian at a point of geographic latitude φ is given by

$$\mu = \mu(k, \varphi) = \frac{1-k^2}{(1-k^2\sin^2\varphi)^{3/2}} \tag{1}$$

where k is the (first) numerical eccentricity of the meridian (often denoted e or ε in geodetic literature). An element of length $d\zeta$ along the meridian is then given by

$$d\zeta = \mu(k, \varphi)\, d\varphi\,. \tag{2}$$

Hence by integration we get

$$\zeta = \int_0^{\varphi} \mu(k, \varphi)\, d\varphi = \int_0^{\varphi} \frac{1-k^2}{(1-k^2\sin^2\varphi)^{3/2}} d\varphi \tag{3}$$

for the arc length normalized to $a = 1$ (denoted *arc latitude* here). If, instead, we use the *reduced latitude* τ defined by

$$\tan\tau \;=\; \sqrt{1-k^2}\, \tan\varphi = k' \tan\varphi, \tag{4}$$

where k' is termed complementary modulus, then arc latitude is given by

$$\zeta = \int_0^{\tau} \sqrt{1-k^2\cos^2\tau}\, d\tau = \tag{5a}$$

$$= \int_0^{\frac{\pi}{2}-\tau} \sqrt{1-k^2\sin^2\left(\frac{\pi}{2}-\tau\right)}\, d(-\tau) = \tag{5b}$$

$$= \int_{\tau}^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\tau}\, d\tau. \tag{5c}$$

Equation (5b) is equivalent to Legendre's normal (incomplete) elliptic integral of the second kind [Korn et al., 1968] defined by

$$E_{(2)}(k, \theta) = \int_0^{\theta} \sqrt{1-k^2\sin^2\theta}\, d\theta \tag{6}$$

thus yielding, together with (5c), the simple relation

$$\boxed{\begin{aligned} \zeta &= E_{(2)}(k, \tfrac{\pi}{2}) - E_{(2)}(k, \tau) = \\ &= C_{(2)}(k) - E_{(2)}(k, \tau) \end{aligned}} \tag{7}$$

for arc latitude as function of reduced latitude, viz. as difference between complete and incomplete elliptic integral of the second kind.

Equation (3) is proportional to Legendre's normal (incomplete) elliptic integral of the third kind, generally defined by

$$E_{(3)}(n, k, \theta) = \int_0^{\theta} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \;, \tag{8}$$