THE STRONG STARK CONJECTURE
FOR TOTALLY ODD CHARACTERS

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ABSTRACT. We prove the p-part of the strong Stark conjecture for every totally odd character and every odd prime p.

Let L/K be a finite Galois CM-extension with Galois group G, which has an abelian Sylow p-subgroup for an odd prime p. We give an unconditional proof of the minus p-part of the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ under certain restrictions on the ramification behavior in L/K.

INTRODUCTION

Let K be a number field and let $\zeta_K(s)$ be the Dedekind zeta function attached to K. If we denote the leading term of its Taylor expansion at $s = 0$ by $\zeta_K^*(0)$, then the analytic class number formula can be rephrased as

$$\zeta_K^*(0) = -\frac{h_K R_K}{w_K},$$

where h_K, R_K and w_K denote the class number, the regulator and the number of roots of unity in K, respectively. Ignoring the sign, this in turn can be restated as follows: the ratio $R_K/\zeta_K^*(0)$ is rational and generates the fractional ideal (w_K/h_K).

Denote the absolute Galois group of K by G_K and let χ be an Artin character of G_K, i.e. a character of G_K with open kernel. Set $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) \mid g \in G_K)$, which is a finite abelian extension of $\mathbb{Q}$. Chinburg [Chi83] has formulated the so-called strong Stark conjecture for χ as a natural refinement of Tate's form of the original Stark conjecture [Tat84]. Roughly speaking, the latter conjecture implies that the ratio of a suitable regulator by the leading term at $s = 0$ of the Artin L-series attached to χ belongs to the character field $\mathbb{Q}(\chi)$. Then the strong Stark conjecture asserts that the principal ideal generated by this ratio coincides with a certain 'q-index' that is defined in purely algebraic terms. For the trivial character this recovers the analytic class number formula (up to sign) as described above.

Suppose that Stark's conjecture holds. Then the strong Stark conjecture naturally decomposes into p-parts, where p runs over all rational primes. More precisely, we say that the p-part of the strong Stark conjecture holds if each prime in $\mathbb{Q}(\chi)$ above p occurs with the same multiplicity in the prime ideal factorizations of the two fractional ideals involved.

Suppose that K is totally real. Then χ is called totally odd if the fixed field L_χ of the kernel of χ is a totally complex finite Galois extension of K and complex conjugation induces a unique automorphism j in the center of $\mathrm{Gal}(L_\chi/K)$ such that $\chi(j) = -\chi(1)$. By a celebrated result of Siegel [Sie70], Stark's conjecture holds for totally odd characters.

The main result of this article is the following.

Theorem 1. *Let K be a totally real number field and let p be an odd prime. Let χ be a totally odd Artin character of G_K. Then the p-part of the strong Stark conjecture holds for χ.*

As an immediate consequence, we obtain the 'weak versions' of non-abelian generalizations of Brumer's conjecture and the Brumer–Stark conjecture due to the author [Nic11a]. This also recovers and generalizes the 'non-abelian Stickelberger theorem' of Burns and Johnston [BJ11].

Corollary 1. *The weak Brumer and the weak Brumer–Stark conjecture of [Nic11a] hold outside their 2-primary parts.*

For results on the 2-primary parts of these conjectures we refer the reader to work of Nomura [Nom14a, Nom14b].

Now let L/K be a finite Galois extension of number fields with Galois group G. We regard $h^0(\mathrm{Spec}(L))$ as a motive defined over K and with coefficients in the semisimple algebra $\mathbb{Q}[G]$. Let $\mathfrak{A}$ be a $\mathbb{Z}$-order in $\mathbb{Q}[G]$ that contains the integral group ring $\mathbb{Z}[G]$. The equivariant Tamagawa number conjecture (ETNC) for the pair $(h^0(\mathrm{Spec}(L)), \mathfrak{A})$ has been formulated by Burns and Flach [BF01] and asserts that a certain canonical element $T\Omega(L/K, \mathfrak{A})$ of the relative algebraic K-group $K_0(\mathfrak{A}, \mathbb{R})$ vanishes. If $\mathfrak{A} = \mathfrak{M}(G)$ is a maximal order, then Burns and Flach [BF03, §3] have shown that the ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathfrak{M}(G))$ is equivalent to the strong Stark conjecture for all irreducible characters of G_K such that L_χ is contained in L.

If v is a place of K, we choose a place w of L above v and write G_w and I_w for the decomposition group and the inertia subgroup at w, respectively. If L/K is a CM-extension we let $j \in G$ denote complex conjugation. As we will see, one may deduce Theorem 1 from (the abelian case of) the following result by adjusting a reduction argument of Ritter and Weiss [RW97].

Theorem 2. *Let p be an odd prime. Let L/K be a Galois CM-extension with Galois group G, which has an abelian Sylow p-subgroup. Suppose that every p-adic place v of K is at most tamely ramified in L or that $j \in G_w$. Then the p-minus part of the ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ holds.*

The main ingredients of the proof are a recent result of Dasgupta and Kakde [DK23] on the strong Brumer–Stark conjecture and a reformulation of the minus part of the ETNC due to the author [Nic11b]. Since the latter result is rather involved and not required if one is only interested in the strong Stark conjecture, we also give a more direct proof of Theorem 1 that almost only relies on [DK23].

By methods of Iwasawa theory, similar results have already been proven by the author under a suitable '$\mu_p = 0$' condition ([Nic09, Theorem 5.1], [Nic11b, Theorem 4] and for not necessarily abelian extensions in [Nic16, Theorem 1.3]). Still assuming that Iwasawa's μ_p-invariant vanishes, Burns [Bur20] presented a strategy for verifying the p-minus part of the ETNC in general. This approach relies, in addition, on a conjecture of Gross [Gro81]. The decisive advantage of our Theorem 2 is that it depends neither on any conjectural vanishing of μ_p-invariants nor any further conjectures. In this direction we also note that, after this article was completed, Theorem 2 has been surpassed if G is abelian by recent work of Atsuta and Kataoka [AK] and Bullach, Burns, Daoud and Seo [BBDS21] using different methods, but still relying on [DK23]. For further applications of the latter we refer the reader to [JN21, DK21, EN22, GP23].

If L is abelian over the rationals, the whole ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ is known by work of Burns and Greither [BG03b] and of Flach [Fla11]. These results also rely on Iwasawa theory and use that the relevant Iwasawa μ_p-invariants vanish by a theorem of Ferrero and Washington [FW79]. The case of abelian extensions of an imaginary quadratic field has been considered by Bley [Ble06] and, more recently, by Bullach and Hofer [BH].

Typical examples of wildly ramified extensions L/K where Theorem 2 applies may be constructed as follows. Let F/K be an arbitrary abelian Galois extension of totally real fields and suppose that the degree $[F : \mathbb{Q}]$ is odd. Then for each non-trivial p-power root of unity ζ the field $L := F(\zeta)$ is CM and abelian over K. Moreover, we have $j \in G_w$ (actually $j \in I_w$) for all p-adic places v of L and each such place is wildly ramified whenever the order of ζ is sufficiently large.

Finally, we record the following consequences of Theorem 2.

Corollary 2. *Let p be an odd prime. Let L/K be a Galois CM-extension with Galois group G, which has an abelian Sylow p-subgroup. Suppose that every p-adic place v of K is at most tamely ramified in L or that $j \in G_w$. Then the p-parts of the following conjectures hold:*

   (i) *the (non-abelian) Brumer conjecture [Nic11a, Conjecture 2.1];*
   (ii) *the (non-abelian) Brumer–Stark conjecture [Nic11a, Conjecture 2.6];*
  (iii) *the minus part of the central conjecture (Conjecture 2.4.1) of Burns [Bur01];*
  (iv) *the minus part of the 'lifted root number conjecture' of Gruenberg, Ritter and Weiss [GRW99].*

If L/K is at most tamely ramified at all places, then the p-minus parts of both the central conjecture (Conjecture 3.3) of Breuning and Burns [BB07] and of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}[G])$ hold.

Notation and conventions. All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. Unadorned tensor products will always denote tensor products over $\mathbb{Z}$. For every field F we fix a separable closure F^c of F and write $G_F := \mathrm{Gal}(F^c/F)$ for its absolute Galois group.

For a finite group G and a prime p we let $\mathrm{Irr}(G)$ and $\mathrm{Irr}_p(G)$ be the set of irreducible complex-valued and $\mathbb{Q}_p^c$-valued characters of G, respectively. If U is a subgroup of G and χ is a character of U, we write $\mathrm{Ind}_U^G \chi$ for the induced character. If U is normal and χ is a character of the quotient G/U, then we denote the inflated character of G by $\mathrm{Infl}_{G/U}^G \chi$.

If M is a G-module, we denote the maximal submodule of M upon which G acts trivially by M^G. Similarly, we let M_G denote the maximal quotient of M with trivial G-action. Then $N_G := \sum_{g \in G} g$ induces a map $M_G \to M^G$ which we also denote by N_G.

For a ring R and a positive integer n, we write $M_n(R)$ for the ring of $n \times n$ matrices with entries in R.

1. The conjectures

1.1. General notation.
Let L/K be a finite Galois extension of number fields with Galois group G. For each place v of K we choose a place w of L above v and denote the decomposition group and inertia subgroup of L/K at w by G_w and I_w, respectively. When G is abelian, then both G_w and I_w only depend upon v, and we will occasionally also use the notation G_v and I_v in this case. We let $\phi_w \in G_w/I_w$ be the Frobenius automorphism and denote the cardinality of the residue field $K(v)$ at v by $N(v)$. For a finite place w of L we write $\mathfrak{P}_w$ for the associated prime ideal in L.

For a set S of places of K we let $S(L)$ be the set comprising those places of L that lie above a place in S. For each prime p we denote the set of p-adic places of K by S_p and the set of archimedean places of K by S_∞. When S is finite, we let $Y_{L,S}$ be the free abelian group generated by the places in $S(L)$ and $X_{L,S}$ be the kernel of the augmentation map $Y_{L,S} \to \mathbb{Z}$ which maps each place $w \in S(L)$ to 1. If in addition S contains all archimedean places, then $\mathcal{O}_{L,S}$ denotes the ring of $S(L)$-integers in L. As usual, we abbreviate $\mathcal{O}_{L,S_\infty}$ to $\mathcal{O}_L$.

1.2. Artin L-series and Stark's conjecture.
Let S be a finite set of places of K containing S_∞. For each complex-valued character χ of G we denote the S-truncated Artin L-series attached to χ by $L_S(s,\chi)$ and the leading coefficient of its Taylor expansion at $s = 0$ by $L_S^*(0,\chi)$. We choose a $\mathbb{C}[G]$-module V_χ with character χ and denote the contragredient of χ by $\check{\chi}$. The negative of the usual Dirichlet map induces an isomorphism of $\mathbb{R}[G]$-modules

$$(1.1) \qquad \lambda_S : \mathbb{R} \otimes \mathcal{O}_{L,S}^\times \xrightarrow{\simeq} \mathbb{R} \otimes X_{L,S}, \quad 1 \otimes \epsilon \mapsto -\sum_{w \in S(L)} \log|\epsilon|_w w.$$

By the Noether–Deuring theorem (see [NSW08, Lemma 8.7.1] for instance) there exist (non-canonical) $\mathbb{Q}[G]$-isomorphisms

$$(1.2) \qquad \phi_S : \mathbb{Q} \otimes X_{L,S} \xrightarrow{\simeq} \mathbb{Q} \otimes \mathcal{O}_{L,S}^\times.$$

Each choice of ϕ_S induces a $\mathbb{C}$-linear automorphism

$$(\lambda_S \circ \phi_S)^\chi : \mathrm{Hom}_{\mathbb{C}[G]}(V_{\check{\chi}}, \mathbb{C} \otimes X_{L,S}) \longrightarrow \mathrm{Hom}_{\mathbb{C}[G]}(V_{\check{\chi}}, \mathbb{C} \otimes X_{L,S})$$
$$f \mapsto \lambda_S \circ \phi_S \circ f.$$

We denote its determinant by $R_S(\chi, \phi_S)$ and set

$$A_S(\chi, \phi_S) := \frac{R_S(\chi, \phi_S)}{L_S^*(0,\chi)} \in \mathbb{C}^\times.$$

Conjecture 1.1 (Stark). *For each $\sigma \in \mathrm{Aut}(\mathbb{C})$ one has $A_S(\chi, \phi_S)^\sigma = A_S(\chi^\sigma, \phi_S)$.*

Here we let $\sigma \in \mathrm{Aut}(\mathbb{C})$ act on the left even though we write exponents on the right. Note that Conjecture 1.1 in particular predicts that $A_S(\chi, \phi_S)$ lies in the character field $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) \mid g \in G)$.

Remark 1.2. It is not hard to see that Conjecture 1.1 does not depend on the choices of S and ϕ_S (see [Tat84, Chapitre I, Proposition 7.3 and §6.2]). The truth of the conjecture is invariant under inflation and induction and respects addition of characters [Tat84, Chapitre I, §7.1]. In particular, if we view χ as an Artin character of G_K, then the

conjecture does not depend on the choice of a finite extension L that contains the fixed field L_χ of χ. Moreover, Conjecture 1.1 holds for all characters of G if and only if it holds for all irreducible characters of G. The same observations apply to the strong Stark conjecture (Conjecture 1.5 below).

The following deep result is due to Siegel [Sie70].

Theorem 1.3 (Siegel). *Conjecture 1.1 holds for each totally odd character χ.*

Remark 1.4. We point out that Conjecture 1.1 is known whenever L_χ is abelian over the rationals [RW97]. Using the analytic class number formula, one can show that it holds for the trivial character. By Brauer induction and a theorem of Artin one can deduce the conjecture for all rational-valued characters [Tat84, Chapitre II, Corollaire 7.4]. In fact, even the strong Stark conjecture (Conjecture 1.5 below) is known in theses cases.

1.3. **The q-index and the strong Stark conjecture.** Let $\mathcal{O}$ be a Dedekind domain with field of fractions F. Each finitely generated torsion $\mathcal{O}$-module M is of the form $M \simeq \bigoplus_{i=1}^{n} \mathcal{O}/\mathfrak{a}_i$, where n is a non-negative integer and each $\mathfrak{a}_i$ is an ideal in $\mathcal{O}$. We set $\ell_\mathcal{O}(M) := \prod_{i=1}^{n} \mathfrak{a}_i$. If $f : A \to B$ is a homomorphism of $\mathcal{O}$-modules such that both the kernel and cokernel of f are finitely generated torsion $\mathcal{O}$-modules, then the q-index of f is the fractional ideal

$$q(f) := \ell_\mathcal{O}(\mathrm{cok}(f)) \cdot \ell_\mathcal{O}(\ker(f))^{-1}.$$

Now suppose that $\mathcal{O}$ is the ring of integers in a finite Galois extension F of $\mathbb{Q}$ which is sufficiently large such that we have a ring isomorphism

$$F[G] \simeq \prod_{\chi \in \mathrm{Irr}(G)} M_{\chi(1)}(F).$$

Recall that L/K is a Galois extension of number fields with Galois group G and that S is a finite set of places of K containing S_∞. Choose an injective $\mathbb{Z}[G]$-homomorphism $\phi_S : X_{L,S} \to \mathcal{O}_{L,S}^\times$. Note that ϕ_S then clearly induces a $\mathbb{Q}[G]$-isomorphism as in (1.2). For each $\mathcal{O}[G]$-lattice M we define an $\mathcal{O}$-homomorphism

$$\phi_{S,M} : \mathrm{Hom}_\mathcal{O}(M, \mathcal{O} \otimes X_{L,S})_G \xrightarrow{N_G} \mathrm{Hom}_\mathcal{O}(M, \mathcal{O} \otimes X_{L,S})^G \longrightarrow \mathrm{Hom}_\mathcal{O}(M, \mathcal{O} \otimes \mathcal{O}_{L,S}^\times)^G,$$

where the second arrow is induced by ϕ_S. Since $F \otimes_\mathcal{O} \phi_{S,M}$ is an isomorphism, the q-index $q(\phi_{S,M})$ is defined. If M' is a second $\mathcal{O}[G]$-lattice, then $q(\phi_{S,M}) = q(\phi_{S,M'})$ whenever $F \otimes_\mathcal{O} M \simeq F \otimes_\mathcal{O} M'$ as $F[G]$-modules by [Tat84, Chapitre II, Lemme 6.7]. Let χ be a character of G. By enlarging F if necessary, we may assume that there is an $\mathcal{O}[G]$-lattice M_χ such that $F \otimes_\mathcal{O} M_\chi$ is an $F[G]$-module with character χ. Then the fractional ideal

$$q_{\phi_S}(\chi) := q(\phi_{S,M_\chi})$$

is well-defined, and we note that $q_{\phi_S}(\chi^\sigma) = q_{\phi_S}(\chi)^\sigma$ for each $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. Chinburg [Chi83, Conjecture 2.2] suggested the following refinement of Stark's conjecture 1.1 whenever the set S is sufficiently large.

Conjecture 1.5 (Strong Stark). *Assume that S is a finite set of places of K that contains S_∞ and all places that ramify in L/K. Assume in addition that the finite places in $S(L)$ generate the class group of L. Then we have that $A_S(\chi, \phi_S)^\sigma = A_S(\chi^\sigma, \phi_S)$ for each $\sigma \in \mathrm{Aut}(\mathbb{C})$ and an equality of fractional ideals $q_{\phi_S}(\check{\chi}) = (A_S(\chi, \phi_S))$ in F.*

Suppose that Stark's conjecture holds for χ. Let p be a prime. We say that the p-part of the strong Stark conjecture holds if each prime above p occurs with the same multiplicity in the prime ideal factorizations of $q_{\phi_S}(\check{\chi})$ and $(A_S(\chi, \phi_S))$.

If χ is an Artin character of G_K, we set $G_\chi := \mathrm{Gal}(L_\chi/K)$. If χ is totally odd, then L_χ is a CM-field. We denote its maximal totally real subfield by L_χ^+ and set $G_\chi^+ := \mathrm{Gal}(L_\chi^+/K)$.

Proposition 1.6. *Let p be a prime. Then the p-part of the strong Stark conjecture holds for all totally odd characters if and only if it holds for all totally odd characters χ such that G_χ^+ is cyclic. If p is odd, one may assume in addition that the cardinality of G_χ^+ is prime to p.*

Proof. Assume that Stark's conjecture holds for all characters. Then it is well-known (see [RW97, Proposition 11]) that the p-part of the strong Stark conjecture holds for all characters if and only if it holds for all characters such that G_χ is cyclic of order prime to p. We may adjust the argument for the non-trivial implication of the proposition as follows (see also the proof of [Nic11b, Corollary 2]).

We first note that Stark's conjecture holds for totally odd characters by Siegel's theorem (Theorem 1.3). Fix a prime p. Let χ be an arbitrary totally odd character. If m is a positive integer, then (the p-parts of) two fractional ideals in $\mathbb{Q}(\chi)$ agree if and only if their m-th powers agree. Thus it suffices to show the p-part of the strong Stark conjecture for $m\chi$ for an appropriate integer $m > 0$. Let $\mathcal{F}$ be the family of all subgroups U of G_χ such that U contains j and $U/\langle j \rangle$ is cyclic. Then G_χ is the union of all subgroups U in $\mathcal{F}$. Thus by Artin's induction theorem [Ser77, Theorem 17] there is a positive integer m such that

$$m\chi = \sum_{U \in \mathcal{F}} a_U \mathrm{Ind}_U^{G_\chi} \phi_U$$

for appropriate $a_U \in \mathbb{Z}$ and linear characters ϕ_U of U. The corresponding intermediate extensions L/L^U are again CM-extensions thanks to the condition $j \in U$. Hence each irreducible character of U is either totally odd or totally even. Since the induction of an even (odd) character is again even (odd), the even characters in the above expression must add up to 0. In other words, we may assume that all ϕ_U are odd. Since the strong Stark conjecture behaves well under addition and induction of characters, we may assume that G_χ^+ is cyclic. This proves the first claim of the proposition.

Let $\mathcal{I}_p$ be the inertia subgroup of the (abelian) extension $\mathbb{Q}(\chi)/\mathbb{Q}$ at p. Since each $\sigma \in \mathcal{I}_p$ fixes the primes above p and Stark's conjecture holds for χ, it suffices to show the p-part of Conjecture 1.5 for the character $\chi' = \sum_{\sigma \in \mathcal{I}_p} \chi^\sigma$. We may and do therefore assume that p is unramified in $\mathbb{Q}(\chi)$. Let P be the Sylow p-subgroup of G_χ. Let ψ be an irreducible constituent of $\mathrm{Res}_P^{G_\chi} \chi$. Then $\mathbb{Q}(\psi)/\mathbb{Q}$ is totally ramified at p so that ψ^σ is also an irreducible constituent of $\mathrm{Res}_P^{G_\chi} \chi$ for each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q})$. Thus $\mathrm{Res}_P^{G_\chi} \chi$ is a finite sum of characters of the form $\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q})} \psi^\sigma$. As the latter is rational-valued (indeed a permutation character) and the strong Stark conjecture is known for those characters [Tat84, Chapitre II, Corollaire 7.4], we may assume that $\mathrm{Res}_P^{G_\chi} \chi$ is trivial whenever p is odd. Note that the last step indeed fails for $p = 2$ because the resulting character would be even. $\square$

1.4. **The equivariant Tamagawa number conjecture.** Let L/K be a finite Galois extension of number fields with Galois group G. Let S be a finite set of places of K that contains all archimedean places and all places that ramify in L. In [Bur01] Burns defines a canonical element $T\Omega(L/K, 0)$ in the relative algebraic K-group $K_0(\mathbb{Z}[G], \mathbb{R})$. The definition involves the refined Euler characteristic of a trivialized complex and the leading term of an equivariant S-truncated L-series at $s = 0$. The occurring (perfect) complex is essentially Tate's canonical class in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{L,S}, \mathcal{O}_{L,S}^\times)$ and the trivialization

is given by the negative of the Dirichlet map (1.1). We will not give any further details, as we will mainly work with an explicit reformulation of the author for the case at hand.

The ETNC has been formulated by Burns and Flach [BF01] in vast generality, but for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ simply asserts the following [Bur01, Theorem 2.4.1].

Conjecture 1.7. *The element* $T\Omega(L/K, 0)$ *in* $K_0(\mathbb{Z}[G], \mathbb{R})$ *vanishes.*

We point out that this assertion is equivalent to the 'lifted root number conjecture' of Gruenberg, Ritter and Weiss [GRW99] by [Bur01, Theorem 2.3.3]. For our purposes, however, it is more important to note that $T\Omega(L/K, 0)$ belongs to the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if Stark's conjecture holds for all irreducible characters of G by [Bur01, Theorem 2.2.4]. In this case the ETNC decomposes into conjectures at each prime p by means of the canonical isomorphism

$$K_0(\mathbb{Z}[G], \mathbb{Q}) \simeq \bigoplus_p K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$$

Let $T\Omega(L/K, 0)_p$ be the image of $T\Omega(L/K, 0)$ in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$. Then $T\Omega(L/K, 0)_p$ is torsion if and only if the p-part of the strong Stark conjecture holds by [Bur01, Theorem 2.2.4] (or rather [Bur01, Lemmas 2.2.6 and 2.2.7]).

Now assume that L/K is a CM-extension. Then for each odd prime p there is a canonical isomorphism

$$K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \simeq K_0(\mathbb{Z}_p[G]_+, \mathbb{Q}_p) \oplus K_0(\mathbb{Z}_p[G]_-, \mathbb{Q}_p),$$

where $\mathbb{Z}_p[G]_\pm := \mathbb{Z}_p[G]/(1 \mp j)$. By Theorem 1.3 we therefore obtain a well-defined element $T\Omega(L/K, 0)_p^-$ in $K_0(\mathbb{Z}_p[G]_-, \mathbb{Q}_p)$. We say that the p-minus part of the ETNC holds if the latter element vanishes. In particular, the p-part of the strong Stark conjecture then holds for all odd characters of G.

Lemma 1.8. *Theorem 2 for abelian extensions* L/K *implies Theorem 1.*

Proof. This follows from the above considerations and Proposition 1.6. $\square$

2. Ray class groups

2.1. **A natural exact sequence.** Let L/K be a finite Galois extension of number fields with Galois group G. If T is a finite set of finite places of K, we write cl_L^T for the ray class group of L for the modulus $\mathfrak{M}_T := \prod_{w \in T(L)} \mathfrak{P}_w$. Let S be a second finite set of places of K such that $S \cap T$ is empty. We denote the subset of S comprising all finite places in S by S_f. Then there is a natural map $Y_{L, S_f} \to \mathrm{cl}_L^T$ which sends each place $w \in S_f(L)$ to the class $[\mathfrak{P}_w] \in \mathrm{cl}_L^T$ of the associated prime ideal. We denote the cokernel of this map by $\mathrm{cl}_{L,S}^T$. If T is empty, we usually remove it from the notation. We set $E_{L,S} := \mathcal{O}_{L,S}^\times$ and define $E_{L,S}^T$ to be the subgroup of those units in $E_{L,S}$ which are congruent to 1 modulo each place in $T(L)$. We have an exact sequence of $\mathbb{Z}[G]$-modules

(2.1) $$0 \to E_{L,S}^T \to E_{L,S} \to (\mathcal{O}_{L,S}/\mathfrak{M}_T)^\times \xrightarrow{\nu} \mathrm{cl}_{L,S}^T \to \mathrm{cl}_{L,S} \to 0,$$

where the map ν lifts an element $\overline{x} \in (\mathcal{O}_{L,S}/\mathfrak{M}_T)^\times$ to $x \in \mathcal{O}_{L,S}$ and sends it to the class of the principal ideal (x) in $\mathrm{cl}_{L,S}^T$.

In the following we assume that no non-trivial root of unity in L is congruent to 1 modulo each place in $T(L)$. In other words, we assume that $E_{L,S}^T$ is torsion-free. For instance, this condition holds if T contains two places of different residue characteristic or one place of sufficiently large norm.

Let L/K be a CM-extension. For any G-module M, we define submodules

$$M^{\pm} := \{m \in M \mid jm = \pm m\}.$$

In particular, we have that $\mu_L := E_{L,S_\infty}^-$ is the group of roots of unity in L, and that $(E_{L,S_\infty}^T)^-$ vanishes. We set

$$A_{L,S}^T := (\mathrm{cl}_{L,S}^T)^-.$$

If p is a prime, we set $M(p) := M \otimes \mathbb{Z}_p$ for any finitely generated $\mathbb{Z}$-module M. If p is odd, then taking p-minus-parts is an exact functor so that (2.1) induces an exact sequence of $\mathbb{Z}_p[G]_-$-modules

$$(2.2) \qquad 0 \to \mu_L(p) \to (\mathcal{O}_L/\mathfrak{M}_T)^\times(p)^- \to A_{L,S_\infty}^T(p) \to A_{L,S_\infty}(p) \to 0.$$

2.2. **Stickelberger elements.** Let L/K be an *abelian* extension of number fields. Let S and T be two disjoint finite sets of places of K with $S_\infty \subseteq S$. For each irreducible character χ of G we define the S-truncated T-modified L-series of χ by

$$L_S^T(s, \chi) := L_S(s, \chi) \prod_{v \in T} (1 - \chi(\phi_w)N(v)^{1-s}).$$

We adopt the convention that $\chi(\phi_w) = 0$ if χ is ramified at v. The Stickelberger element attached to S and T is the unique element $\theta_{L/K,S}^T \in \mathbb{C}[G]$ such that for all irreducible characters χ of G one has

$$\chi(\theta_{L/K,S}^T) = L_S^T(0, \check{\chi}).$$

It follows from Theorem 1.3 that $\theta_{L/K,S}^T$ actually belongs to the rational group ring $\mathbb{Q}[G]$. In fact, we will need the following finer result.

Proposition 2.1. *Let p be a prime and let L/K be an abelian Galois extension of number fields with Galois group G. Let S and T be two disjoint finite sets of places of K such that the following conditions are satisfied:*

   (i) *The union of S and T contains all non-p-adic ramified places;*
   (ii) *the set S contains all p-adic places that ramify wildly;*
   (iii) *all archimedean places lie in S;*
   (iv) *the group $E_{L,S}^{T_0}$ is torsion-free, where T_0 denotes the set of all unramified places in T.*

Then we have that $\theta_{L/K,S}^T \in \mathbb{Z}_p[G]$.

Proof. This follows from [Nic16, Theorem 5.2] (take $H = 1$) and heavily relies on results of Pi. Cassou-Noguès [CN79] and of Deligne and Ribet [DR80]. See also [Nic11b, Lemma 1] for an important special case. $\square$

Let $x \mapsto x^\sharp$ be the (anti-) involution on $\mathbb{Z}[G]$ induced by $g \mapsto g^{-1}$ for each $g \in G$. Let T_0 be a finite set of places of K that do not ramify in L and are such that $E_{L,S}^{T_0}$ is torsion-free (recall that this condition does not depend on S). Following [Gre07] and [DK23, §1] we define the *Sinnott–Kurihara ideal* to be the fractional $\mathbb{Z}[G]$-ideal

$$(2.3) \qquad \mathrm{SKu}^{T_0}(L/K) := (\theta_{L/K,S_\infty}^{T_0})^\sharp \prod_{v \in S_{\mathrm{ram}}} (N_{I_v}, 1 - \phi_w|I_v|^{-1}N_{I_v})$$

and note that this is actually contained in $\mathbb{Z}[G]$ (see [DK23, Lemma 3.4]).

2.3. **Fitting ideals and the strong Brumer–Stark conjecture.** If M is a finitely presented module over a commutative ring R, we denote the (initial) Fitting ideal of M over R by $\mathrm{Fitt}_R(M)$. For basic properties of Fitting ideals including the following well-known lemma we refer the reader to Northcott's excellent book [Nor76].

Lemma 2.2. *Let R be a commutative ring and let M and M' be finitely presented R-modules. Then the following hold.*

   (i) *We have that $\mathrm{Fitt}_R(M \oplus M') = \mathrm{Fitt}_R(M) \cdot \mathrm{Fitt}_R(M')$.*
   (ii) *If $M \to M'$ is an epimorphism, then we have an inclusion*

$$\mathrm{Fitt}_R(M) \subseteq \mathrm{Fitt}_R(M').$$

Let p be prime and let G be a finite abelian group. If M is a $\mathbb{Z}_p[G]$-module, we denote the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ endowed with the contragredient G-action by $M^{\vee}$. The following lemma is certainly well-known.

Lemma 2.3. *Let p be an odd prime and let L/K be a finite abelian CM-extension with Galois group G. Let M be a finite cohomologically trival $\mathbb{Z}_p[G]_{-}$-module. Then the following hold.*

   (i) *The Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_p[G]_{-}}(M)$ is principal.*
   (ii) *We have an equality*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_{-}}(M^{\vee}) = \mathrm{Fitt}_{\mathbb{Z}_p[G]_{-}}(M)^{\sharp}$$

   (iii) *The index of the Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_p[G]_{-}}(M)$ in $\mathbb{Z}_p[G]_{-}$ is finite and*

$$[\mathbb{Z}_p[G]_{-} : \mathrm{Fitt}_{\mathbb{Z}_p[G]_{-}}(M)] = |M|.$$

Proof. A finite $\mathbb{Z}_p[G]_{-}$-module is cohmologically trivial if and only if its projective dimension is at most 1. Since $\mathbb{Z}_p[G]_{-}$ is semilocal, any first syzygy of M is free. This proves (i). For (ii) see [BG03a, Lemma 6] (note that we have to add the $\sharp$ on the right because of our different conventions on the action of G on Pontryagin duals). Also (iii) is a standard result (see the proof of [Gre00, Theorem 4.11], for instance). $\square$

The following deep result has recently been shown by Dasgupta and Kakde [DK23, Theorem 3.5]. In particular, it settles the so-called strong Brumer–Stark conjecture away from 2.

Theorem 2.4. *Let L/K be an abelian CM-extension of number fields with Galois group G. Let T_0 be a finite set of places of K that do not ramify in L and are such that $E_S^{T_0}$ is torsion-free. Let p be an odd prime. Then we have that*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_{-}}(A_{L,S_\infty}^{T_0}(p)^{\vee}) = \mathrm{SKu}^{T_0}(L/K)(p).$$

2.4. **The strong Stark conjecture for totally odd characters.** For the proof of Theorem 1 it suffices to verify Theorem 2 for abelian extensions (see Lemma 1.8). However, the proof of the latter result will build upon a reformulation of the ETNC due to the author [Nic11b] which is rather involved. Since Theorem 1 can be obtained by more elementary means, we include a proof here for convenience. The reader who is mainly interested in Theorem 2 or already familiar with the results in [Nic11b] may jump directly to the next subsection.

Proof of Theorem 1. Let K be a totally real number field and let p be an odd prime. By Proposition 1.6 it suffices to prove the p-part of the strong Stark conjecture for totally odd characters of G_K that factor through an abelian CM-extension L/K of degree prime

to p. We set $G := \mathrm{Gal}(L/K)$ and observe that in this case every $\mathbb{Z}_p[G]$-module M is cohomologically trivial. In particular, the map $N_G : M_G \to M^G$ is an isomorphism and taking G-(co)-invariants is an exact functor.

Let χ be a totally odd irreducible complex character of G. Each choice of isomorphism $\iota : \mathbb{C} \simeq \mathbb{C}_p$ restricts to an embedding $\mathbb{Q}(\chi) \hookrightarrow \mathbb{Q}_p(\chi)$, which corresponds to a choice of prime in $\mathbb{Q}(\chi)$ above p. We henceforth fix such an isomorphism. This allows us to view complex characters as $\mathbb{C}_p$-valued and vice versa. The covariant functor

$$M \mapsto M^\chi := \mathrm{Hom}_{\mathbb{Z}_p(\chi)}(\mathbb{Z}_p(\chi), \mathbb{Z}_p(\chi) \otimes M)^G$$

is exact on finitely generated $\mathbb{Z}[G]$-modules. Moreover, if $f : A \to B$ is a homomorphism of finitely generated $\mathbb{Z}[G]$-modules with finite kernel and cokernel, then one has an equality

$$q(f^\chi) = \mathrm{Fitt}_{\mathbb{Z}_p(\chi)}(\mathrm{cok}(f)^\chi) \cdot \mathrm{Fitt}_{\mathbb{Z}_p(\chi)}(\ker(f)^\chi)^{-1}.$$

In particular, for every finite set S containing the archimedean places and every injective $\mathbb{Z}[G]$-homomorphism $\phi_S : X_{L,S} \to \mathcal{O}_{L,S}^\times$ we have that

$$(2.4) \qquad\qquad \iota(q_{\phi_S}(\chi)) = \mathrm{Fitt}_{\mathbb{Z}_p(\chi)}(\mathrm{cok}(\phi_S)^\chi).$$

Now assume that S also contains all ramified primes and that $S(L)$ is large enough to generate the class group of L. We choose an arbitrary ϕ_{S_∞} as above. As cl_L is finite and $\mathbb{Q}[G]$ is semisimple, there is always an integer N and a (necessarily injective) $\mathbb{Z}[G]$-homomorphism ϕ_S such that the following diagram commutes

$$(2.5) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & X_{S_\infty} & \longrightarrow & X_S & \longrightarrow & \mathbb{Z}[S_f(L)] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_{S_\infty}} & & \downarrow{\scriptstyle \phi_S} & & \downarrow{\scriptstyle \cdot N} & & \downarrow \\ 0 & \longrightarrow & E_{S_\infty} & \longrightarrow & E_S & \longrightarrow & \mathbb{Z}[S_f(L)] & \longrightarrow & \mathrm{cl}_L & \longrightarrow & 0. \end{array}$$

Since χ is odd and $X_{S_\infty}^-$ vanishes, we obtain the following exact sequence of $\mathbb{Z}_p(\chi)$-modules

$$0 \to \mu_L^\chi \to \mathrm{cok}(\phi_S)^\chi \to (\mathbb{Z}/N\mathbb{Z}[S_f(L)])^\chi \to \mathrm{cl}_L^\chi \to 0.$$

Now we choose a second finite set T_0 of places of K such that $S \cap T_0 = \emptyset$ and $E_{S_\infty}^{T_0}$ is torsion-free. We deduce from (2.1) an exact sequence

$$0 \to \mu_L^\chi \to ((\mathcal{O}_L/\mathfrak{M}_{T_0})^\times)^\chi \to (\mathrm{cl}_L^{T_0})^\chi \to \mathrm{cl}_L^\chi \to 0.$$

The last two exact sequences and (2.4) imply that
$$(2.6)$$
$$\iota(q_{\phi_S}(\chi)) = \mathrm{Fitt}_{\mathbb{Z}_p(\chi)}((\mathbb{Z}/N\mathbb{Z}[S_f(L)])^\chi) \cdot \mathrm{Fitt}_{\mathbb{Z}_p(\chi)}\left(((\mathcal{O}_L/\mathfrak{M}_{T_0})^\times)^\chi\right) \cdot \mathrm{Fitt}_{\mathbb{Z}_p(\chi)}((\mathrm{cl}_L^{T_0})^\chi)^{-1}.$$

We next compute the Fitting ideals on the right-hand side. The first two are easily determined. We have

$$\mathrm{Fitt}_{\mathbb{Z}_p(\chi)}((\mathbb{Z}/N\mathbb{Z}[S_f(L)])^\chi) = (N^{d_S(\chi)}),$$

$$\mathrm{Fitt}_{\mathbb{Z}_p(\chi)}\left(((\mathcal{O}_L/\mathfrak{M}_{T_0})^\times)^\chi\right) = \left(\prod_{v \in T_0}(1 - \chi(\phi_w)N(v))\right),$$

where $d_S(\chi)$ denotes the number of places v in S_f such that the restriction of χ to G_v is trivial. The analytic class number formula implies that (see [DK23, Lemma 2.1])

$$(2.7) \qquad\qquad |A_L^{T_0}(p)| = |\mathbb{Z}_p[G]_-/(\theta_{L/K,S_\infty}^{T_0})|.$$

Let v be an arbitrary finite place of K. Since $\chi(\phi_w)$ is either zero or a root of unity of order coprime to p, we see that $1 - \chi(\phi_w) \in \mathbb{Z}_p(\chi)^\times$ unless G_v acts trivially on V_χ. In that

case we have $\chi(N_{I_v}) = |I_v| \in \mathbb{Z}_p^\times$ so that $(\theta_{L/K,S_\infty}^{T_0})^\sharp$ lies in $\mathrm{SKu}^{T_0}(L/K)(p)$. Therefore (2.7), Lemma 2.3 and Theorem 2.4 imply that the Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(A_L^{T_0}(p))$ is principal and generated by $\theta_{L/K,S_\infty}^{T_0}$. Since the cardinality of G is prime to p, the ring $\mathbb{Z}_p[G]_-$ decomposes into a product $\prod_{\chi(j)=-1} \mathbb{Z}_p(\chi)$. So the former statement is equivalent to

$$\mathrm{Fitt}_{\mathbb{Z}_p(\chi)}((\mathrm{cl}_L^{T_0})^\chi) = (L_{S_\infty}^{T_0}(0, \check{\chi}))$$

for all odd χ. So we deduce from (2.6) and the above that

(2.8) $$\iota(q_{\phi_S}(\chi)) = (N^{d_S(\chi)}/L_{S_\infty}(0, \check{\chi})).$$

Finally, we have the following commutative diagram whose rows are those of (2.5) tensored with $\mathbb{R}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} \otimes E_{S_\infty} & \longrightarrow & \mathbb{R} \otimes E_S & \longrightarrow & \mathbb{R}[S_f(L)] & \longrightarrow & 0 \\
& & \downarrow{\lambda_{S_\infty}} & & \downarrow{\lambda_S} & & \downarrow{\mathrm{Log}_S} & & \\
0 & \longrightarrow & \mathbb{R} \otimes X_{S_\infty} & \longrightarrow & \mathbb{R} \otimes X_S & \longrightarrow & \mathbb{R}[S_f(L)] & \longrightarrow & 0.
\end{array}
$$

Here the first two vertical arrows are the negative Dirichlet maps (1.1) and Log_S maps $w \in S_f(L)$ to $-\log(N(w))$. For odd χ one has $R_{S_\infty}(\chi, \phi_{S_\infty}) = 1$ and hence

$$R_S(\chi, \phi_S) = N^{d_S(\chi)} \cdot \prod_{\substack{v \in S_f \\ V_\chi^{G_v} = V_\chi}} (-\log(N(w))).$$

By [Wei96, Proposition 6, p. 50] we likewise have that

$$L_S^*(0, \chi) = L_{S_\infty}(0, \chi) \cdot \prod_{\substack{v \in S_f \\ V_\chi^{G_v} = V_\chi}} \log(N(w)) \cdot \prod_{\substack{v \in S_f \\ V_\chi^{G_v} = 0}} (1 - \chi(\phi_w)).$$

Since $d_S(\chi) = d_S(\check{\chi})$ and $\iota(1 - \chi(\phi_w)) \in \mathbb{Z}_p(\chi)^\times$ whenever $V_\chi^{G_v}$ vanishes, the last two displayed equalities and (2.8) complete the proof. $\qquad\square$

2.5. The relation to the equivariant Tamagawa number conjecture.
In order to relate the results of Dasgupta and Kakde to the p-minus-part of the ETNC, we use the following reformulation due to the author. Roughly speaking, its proof is an elaborate refinement of the argument given in §2.4.

Theorem 2.5. *Let L/K be an abelian CM-extension of number fields with Galois group G and let p be an odd prime. Assume that each $v \in S_p$ is at most tamely ramified or that we have $j \in G_v$. Let S_1 be the set of places of K comprising all archimedean places and all wildly ramified p-adic places. Choose an unramified place v_0 of K such that $E_{L,S_1}^{T_0}$ is torsion-free, where we set $T_0 := \{v_0\}$. Moreover, we let T be the union of T_0 and the set of all non-p-adic ramified places of K. Then the following holds.*

    (i) *The attached Stickelberger element is p-integral, i.e. $\theta_{L/K,S_1}^T \in \mathbb{Z}_p[G]$;*

    (ii) *the G-module $A_{L,S_\infty}^T(p)$ is cohomologically trivial;*

    (iii) *the following are equivalent:*

        (a) *the p-minus-part of the ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ holds;*

        (b) *the Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(A_{L,S_\infty}^T(p))$ is generated by $\theta_{L/K,S_1}^T$;*

        (c) *we have that $(\theta_{L/K,S_1}^T)^\sharp \in \mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(A_{L,S_\infty}^T(p)^\vee)$.*

Proof. Part (i) is a special case of Proposition 2.1. Part (ii) is [Nic11b, Theorem 1] and the equivalence of (a) and (b) in part (iii) is a reformulation in terms of Fitting ideals of [Nic11b, Theorem 2]. Note that (ii) implies that the Fitting ideal of $A_{L,S_\infty}^T(p)$ is principal by Lemma 2.3 (i). Part (ii) of the same lemma shows that (b) implies (c). Now suppose that (c) holds. Then the inclusion $(\theta_{L/K,S_1}^T) \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(A_{L,S_\infty}^T(p))$ must be an equality, since we have that

$$|\mathbb{Z}_p[G]_- : (\theta_{L/K,S_1}^T)| = |A_{L,S_\infty}^T(p)| = |\mathbb{Z}_p[G]_- : \mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(A_{L,S_\infty}^T(p))|.$$

Here the first equality is [Nic11b, Proposition 4] (a variant of (2.7) and likewise a consequence of the analytic class number formula), whereas the second follows from Lemma 2.3 (iii).   □

3. Proof of the main results

We have provided all the ingredients that we need for the proof of our main results. We first claim that it suffices to prove Theorem 2 for abelian extensions.

By Lemma 1.8, this also finishes the proof of Theorem 1. Moreover, Theorem 1 implies Corollary 1 by [Nic11a, Theorem 4.1, Proposition 3.9 and Lemma 2.12].

By the discussion in §1.4 we then also know that $T\Omega(L/K,0)_p^-$ is torsion for each Galois CM-extension L/K. Hence it follows from [Nic16, Proposition 6.2] that $T\Omega(L/K,0)_p^-$ vanishes if and only if $T\Omega(L'/K',0)_p^-$ vanishes for all intermediate Galois CM-extensions L'/K' whose Galois group is either p-elementary or a direct product of a p-elementary group and a cyclic group of order 2 (generated by j). The full strength of Theorem 2 follows since the hypotheses on the extension L/K are inherited by each intermediate Galois CM-extension L'/K' and a p-elementary group with abelian Sylow p-subgroup is itself abelian. The main point here is that each subquotient of a finite group G with abelian Sylow p-subgroup P also has an abelian Sylow p-subgroup. To see this, let U be a subgroup of G and let N be a normal subgroup of U. Then by [Hup67, Aufgabe 26, p. 36] there is a $g \in G$ such that $Q := gPg^{-1} \cap U$ is a Sylow p-subgroup of U. Then Q is abelian and its image in the quotient U/N is a Sylow p-subgroup of U/N.

Moreover, Theorem 2 and [Nic11a, Theorem 5.3] directly imply the result on the non-abelian Brumer and Brumer–Stark conjecture listed in Corollary 2. We have already observed that the vanishing of $T\Omega(L/K,0)$, the 'lifted root number conjecture' of Gruenberg, Ritter and Weiss and the ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ all are equivalent. Hence Theorem 2 implies (iii) and (iv) of Corollary 2. The final claims of Corollary 2 on the conjecture of Breuning and Burns and on the ETNC for the pair $(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}[G])$ follow from a result of Bley and Burns [BB03, Corollary 6.3(i)] as in the proof of [Nic16, Corollary 1.6].

We are left with the following.

Proof of Theorem 2 for abelian extensions. Let L/K be an abelian CM-extension with Galois group G and fix an odd prime p. Suppose that for each p-adic place v of K we have that the cardinality of I_v is prime to p or that $j \in G_v$. Let S_1 be the set of places of K comprising all archimedean places and all wildly ramified p-adic places. Choose an unramified place $v_0 \notin S_p$ such that $E_{L,S_1}^{T_0}$ is torsion-free, where we set $T_0 := \{v_0\}$. Moreover, we let T be the union of T_0 and the set of all non-p-adic ramified places of K. By Theorem 2.5 we have to show that $(\theta_{L/K,S_1}^T)^\sharp$ belongs to the Fitting ideal of $A_{L,S_\infty}^T(p)^\vee$.

By Theorem 2.4 it suffices to check that $(\theta^T_{L/K,S_1})^\sharp$ lies in the Sinnott–Kurihara ideal $\mathrm{SKu}^{T_0}(L/K)(p)$. We write

$$(\theta^T_{L/K,S_1})^\sharp = (\theta^{T_0}_{L/K,S_\infty})^\sharp \cdot \prod_{v\in S_{\mathrm{ram}}\setminus S_p}(1-\phi_w N(v)|I_v|^{-1}N_{I_v}) \cdot \prod_{v\in S_{\mathrm{ram}}\cap S_p}(1-\phi_w|I_v|^{-1}N_{I_v}).$$

If we compare this to (2.3), we see that we are left with showing

$$(3.1) \qquad 1 - \phi_w N(v)|I_v|^{-1}N_{I_v} \in (N_{I_v}, 1-\phi_w|I_v|^{-1}N_{I_v})$$

for each place $v \nmid p$ that ramifies in L/K. Let $\ell \neq p$ be the prime below v. By local class field theory [Ser79, Chapter XV, §2] the local units at v surject under the reciprocity map onto I_v. The subgroup of principal units is mapped onto the Sylow ℓ-subgroup of I_v. As the factor group of the local units modulo the principal units has cardinality $N(v) - 1$, the ramification index $|I_v|$ divides $N(v) - 1$ up to a p-adic unit. Hence (3.1) follows from the equality

$$1 - \phi_w N(v)|I_v|^{-1}N_{I_v} = 1 - \phi_w|I_v|^{-1}N_{I_v} - \phi_w\frac{N(v)-1}{|I_v|}N_{I_v}.$$

If we jump deeper into the results of Dasgupta and Kakde, we can alternatively argue as follows. Let $\mathrm{Sel}^T_{S_1}(L)$ be the Selmer module considered by Burns, Kurihara and Sano in [BKS16]. We will not recall its definition as we only need the following two facts. First there is an epimorphism of $\mathbb{Z}_p[G]_-$-modules

$$\mathrm{Sel}^T_{S_1}(L)(p)^- \twoheadrightarrow A^T_{L,S_\infty}(p)^\vee$$

by [DK23, Lemma 3.1]. Second we have that

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\mathrm{Sel}^T_{S_1}(L)(p)^-) = ((\theta^T_{L/K,S_1})^\sharp)$$

by [DK23, Theorem 3.3]. Hence Lemma 2.2 implies that we have

$$(\theta^T_{L/K,S_1})^\sharp \in \mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(A^T_{L,S_\infty}(p)^\vee).$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.1. One may ask whether it is possible to deduce the p-minus part of the ETNC for further non-abelian CM-extensions from Theorem 2 by considering 'hybrid' cases as in [JN16]. We claim that this is not the case. To see this, assume that the p-adic group ring $\mathbb{Z}_p[G]$ is 'N-hybrid' for a normal subgroup N of G in the sense of [JN16, Definition 2.5]. Then p does not divide the cardinality of N by [JN16, Proposition 2.8(i)]. Hence each Sylow p-subgroup of G is mapped isomorphically onto a Sylow p-subgroup of G/N under the natural quotient map $G \to G/N$. Thus if G/N has an abelian Sylow p-subgroup so does G.

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