On non-abelian Stark-type conjectures

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Abstract

We introduce non-abelian generalizations of Brumer's conjecture, the Brumer-Stark conjecture and the strong Brumer-Stark property attached to a Galois CM-extension of number fields. Moreover, we discuss how they are related to the equivariant Tamagawa number conjecture, the strong Stark conjecture and a non-abelian generalization of Rubin's conjecture due to D. Burns.

Let L/K be a finite Galois CM-extension of number fields with Galois group G. To each finite set S of places of K which contains all the infinite places, one can associate a so-called "Stickelberger element" $\theta_S(L/K)$ in the center of the group ring algebra $\mathbb{C}G$. This Stickelberger element is defined via L-values at zero of S-truncated Artin L-functions attached to the (complex) characters of G. Let us denote the roots of unity of L by μ_L and the class group of L by cl_L. Assume that S contains the set S_{ram} of all finite primes of K which ramify in L/K. Then it was independently shown in [8], [13] and [1] that for abelian G one has

$$\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)\theta_S(L/K) \subset \mathbb{Z}G. \tag{1}$$

Now Brumer's conjecture asserts that $\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)\theta_S(L/K)$ annihilates cl_L. There is a large body of evidence in support of Brumer's conjecture (cf. the expository article [14]); in particular, C. Greither [15] has shown that the appropriate special case of the equivariant Tamagawa number conjecture (ETNC) as formulated by Burns and Flach [6] implies the p-part of Brumer's conjecture for an odd prime p if the p-part of μ_L is a c.t. (short for cohomologically trivial) G-module. A similar result for arbitrary G was recently proven by the author [20], improving an unconditional annihilation result due to D. Burns and H. Johnston [7]. Note that the assumptions made in loc.cit. are adapted to ensure the validity of the strong Stark conjecture. These two results will provide some evidence for our conjecture.

Moreover, we will introduce a non-abelian generalization of the Brumer-Stark conjecture and of the strong Brumer-Stark property. The extension L/K fulfills the latter if certain Stickelberger elements are contained in the (non-commutative) Fitting invariants of corresponding ray class groups; but it does not hold in general, even if G is abelian, as follows from the results in [16]. But if this property happens to be true, this also implies the validity of the (non-abelian) Brumer-Stark conjecture and Brumer's conjecture. We will show that the p-part of this property is implied by the ETNC if the ramification above the odd prime p is at most tame.

D. Burns [3] has introduced a non-abelian analogue of a conjecture formulated by Rubin ([25], Conj. B). It is shown in loc.cit. that this conjecture is implied by the strong Stark conjecture, and we will show that Burns' conjecture implies slightly weaker annihilation results as predicted by the (non-abelian) Brumer-Stark resp. Brumer's conjecture.

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1. Preliminaries

1.0.1 *K-theory* Let Λ be a left noetherian ring with 1 and PMod(Λ) the category of all finitely generated projective Λ-modules. We write $K_0(\Lambda)$ for the Grothendieck group of PMod(Λ), and $K_1(\Lambda)$ for the Whitehead group of Λ which is the abelianized infinite general linear group. If S is a multiplicatively closed subset of the center of Λ which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all finitely generated S-torsion Λ-modules of finite projective dimension by $K_0S(\Lambda)$. Writing Λ_S for the ring of quotients of Λ with denominators in S, we have the following Localization Sequence (cf. [12], p. 65)

$$K_1(\Lambda) \to K_1(\Lambda_S) \xrightarrow{\partial} K_0S(\Lambda) \to K_0(\Lambda) \to K_0(\Lambda_S). \tag{2}$$

In the special case where Λ is an $\mathfrak{o}$-order over a commutative ring $\mathfrak{o}$ and S is the set of all nonzero-divisors of $\mathfrak{o}$, we also write $K_0T(\Lambda)$ instead of $K_0S(\Lambda)$. Moreover, we denote the relative K-group corresponding to a ring homomorphism $\Lambda \to \Lambda'$ by $K_0(\Lambda, \Lambda')$ (cf. [26]). Then we have a Localization Sequence (cf. [12], p. 72)

$$K_1(\Lambda) \to K_1(\Lambda') \xrightarrow{\partial_{\Lambda,\Lambda'}} K_0(\Lambda, \Lambda') \to K_0(\Lambda) \to K_0(\Lambda').$$

It is also shown in [26] that there is an isomorphism $K_0(\Lambda, \Lambda_S) \simeq K_0S(\Lambda)$. For any ring Λ we write $\zeta(\Lambda)$ for the subring of all elements which are central in Λ. Let L be a subfield of either $\mathbb{C}$ or $\mathbb{C}_p$ for some prime p and let G be a finite group. In the case where Λ' is the group ring LG the reduced norm map $\mathrm{nr}_{LG} : K_1(LG) \to \zeta(LG)^\times$ is always injective. If in addition $L = \mathbb{R}$, there exists a canonical map $\hat{\partial}_G : \zeta(\mathbb{R}G)^\times \to K_0(\mathbb{Z}G, \mathbb{R}G)$ such that the restriction of $\hat{\partial}_G$ to the image of the reduced norm equals $\partial_{\mathbb{Z}G,\mathbb{R}G} \circ \mathrm{nr}_{\mathbb{R}G}^{-1}$. This map is called the extended boundary homomorphism and was introduced by Burns and Flach [6].

1.0.2 *χ-twists* We largely adopt the treatment of [3], §1. Let G be a finite group and denote the set of all irreducible characters with values in $\mathbb{C}$ resp. $\mathbb{C}_p$ by $\mathrm{Irr}\,(G)$ resp. $\mathrm{Irr}_p(G)$. Fix an irreducible character $\chi \in \mathrm{Irr}\,(G)$ resp. $\chi \in \mathrm{Irr}_p(G)$ and let E_χ be the minimal subfield of $\mathbb{C}$ resp. $\mathbb{C}_p$ over which χ can be realized and which is both, Galois and of finite degree over $\mathbb{Q}$ resp. $\mathbb{Q}_p$. We put

$$\mathrm{pr}_\chi := \sum_{g \in G} \chi(g^{-1})g, \quad e_\chi := \frac{\chi(1)}{|G|}\mathrm{pr}_\chi.$$

Hence e_χ is a central primitive idempotent of $E_\chi G$ and pr_χ is the associated projector. We write $\mathfrak{o}_\chi$ for the ring of integers of E_χ and choose a maximal $\mathfrak{o}_\chi$-order $\mathfrak{M}$ in $E_\chi G$ which contains $\mathfrak{o}_\chi G$. We fix an indecomposable idempotent f_χ of $e_\chi \mathfrak{M}$ and define an $\mathfrak{o}_\chi$-torsionfree right $\mathfrak{o}_\chi G$-module by setting $T_\chi := f_\chi \mathfrak{M}$. Note that this slightly differs from the definition in [3], but follows the notation of [7] and [20]. T_χ is (locally) free of rank $\chi(1)$ over $\mathfrak{o}_\chi$ and the associated right $E_\chi G$-module $V_\chi := E_\chi \otimes_{\mathfrak{o}_\chi} T_\chi$ has character χ. For any left G-module M we set $M[\chi] := T_\chi \otimes_{\mathbb{Z}} M$ resp. $M[\chi] := T_\chi \otimes_{\mathbb{Z}_p} M$, upon which G acts on the left by $t \otimes m \mapsto tg^{-1} \otimes g(m)$ for $t \in T_\chi$, $m \in M$ and $g \in G$. For any integer i we write $H^i(G, M)$ for the Tate cohomology in degree i of M with respect to G. Moreover, we write M^G resp. M_G for the maximal submodule resp. the maximal quotient module of M upon which G acts trivially. We obtain a left exact functor $M \mapsto M^\chi$ and a right exact functor $M \mapsto M_\chi$ from the category of left G-modules to the category of $\mathfrak{o}_\chi$-modules by setting $M^\chi := M[\chi]^G$ and $M_\chi := M[\chi]_G$. The action of $N_G := \sum_{g \in G} g$ induces a homomorphism $t(M, \chi) : M_\chi \to M^\chi$ with kernel $H^{-1}(G, M[\chi])$ and cokernel $H^0(G, M[\chi])$. Thus $M_\chi \simeq M^\chi$ whenever M and hence also $M[\chi]$ is a c.t. G-module.

1.0.3 *Non-commutative Fitting invariants* For the following we refer the reader to [20]. We denote the set of all $m \times n$ matrices with entries in a ring R by $M_{m\times n}(R)$ and in the case $m = n$ the group of all invertible elements of $M_{n\times n}(R)$ by $\mathrm{Gl}_n(R)$. Let A be a separable K-algebra and Λ be an $\mathfrak{o}$-order in A, finitely generated as $\mathfrak{o}$-module, where $\mathfrak{o}$ is a complete commutative noetherian local ring with field of quotients K. Moreover, we will assume that the integral closure of $\mathfrak{o}$ in K is finitely generated as $\mathfrak{o}$-module. The group ring $\mathbb{Z}_pG$ will serve as a standard example. Let N and M be two $\zeta(\Lambda)$-submodules of an $\mathfrak{o}$-torsionfree $\zeta(\Lambda)$-module. Then N and M are called $\mathrm{nr}(\Lambda)$-*equivalent* if there exists an integer n and a matrix $U \in \mathrm{Gl}_n(\Lambda)$ such that $N = \mathrm{nr}(U)\cdot M$, where $\mathrm{nr} : A \to \zeta(A)$ denotes the reduced norm map which extends to matrix rings over A in the obvious way. We denote the corresponding equivalence class by $[N]_{\mathrm{nr}(\Lambda)}$. We say that N is $\mathrm{nr}(\Lambda)$-contained in M (and write $[N]_{\mathrm{nr}(\Lambda)} \subset [M]_{\mathrm{nr}(\Lambda)}$) if for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$ there exists $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N' \subset M'$. Note that it suffices to check this property for one $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$. Moreover, we write $[N]_{\mathrm{nr}(\Lambda)} \subset M$ if $N' \subset M$ for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$. We will say that x is contained in $[N]_{\mathrm{nr}(\Lambda)}$ (and write $x \in [N]_{\mathrm{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$ such that $x \in N_0$.

Now let M be a finitely presented (left) Λ-module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M \tag{3}$$

be a finite presentation of M. We identify the homomorphism h with the corresponding matrix in $M_{a\times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geqslant b$. In the case $a = b$ we call (3) a quadratic presentation. The Fitting invariant of h over Λ is defined to be

$$\mathrm{Fitt}_\Lambda(h) = \begin{cases} [0]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a < b \\ \left[\langle \mathrm{nr}(H) | H \in S(h)\rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a \geqslant b. \end{cases}$$

We call $\mathrm{Fitt}_\Lambda(h)$ a Fitting invariant of M over Λ. One defines $\mathrm{Fitt}^{\max}_\Lambda(M)$ to be the unique Fitting invariant of M over Λ which is maximal among all Fitting invariants of M with respect to the partial order "$\subset$". If M admits a quadratic presentation h, one also puts $\mathrm{Fitt}_\Lambda(M) := \mathrm{Fitt}_\Lambda(h)$ which is independent of the chosen quadratic presentation (cf. also [22]). Finally, we denote by $\mathcal{I} = \mathcal{I}(\Lambda)$ the $\zeta(\Lambda)$-submodule of $\zeta(A)$ generated by the elements $\mathrm{nr}(H)$, $H \in M_{b\times b}(\Lambda)$, $b \in \mathbb{N}$.

For any $\mathbb{Z}_pG$-module M we denote the Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ of M by $M^\vee$ which is equipped with the natural G-action $(gf)(m) = f(g^{-1}m)$ for $f \in M^\vee$, $g \in G$ and $m \in M$. If M is finite, we have

$$\mathrm{Ann}_{\mathbb{Z}_pG}(M^\vee) = \mathrm{Ann}_{\mathbb{Z}_pG}(M)^\sharp, \tag{4}$$

where we denote by $^\sharp : \mathbb{Q}_pG \to \mathbb{Q}_pG$ the involution induced by $g \mapsto g^{-1}$. We will frequently make use of the following proposition.

Proposition 1.1. *Let M, M' and M'' be finitely presented Λ-modules. Then it holds:*

i) *If $M \twoheadrightarrow M'$ is an epimorphism, then $\mathrm{Fitt}^{\max}_\Lambda(M) \subset \mathrm{Fitt}^{\max}_\Lambda(M')$.*

ii) *If $M' \to M \twoheadrightarrow M''$ is an exact sequence of Λ-modules, then*

$$\mathrm{Fitt}^{\max}_\Lambda(M') \cdot \mathrm{Fitt}^{\max}_\Lambda(M'') \subset \mathrm{Fitt}^{\max}_\Lambda(M).$$

iii) *If $\theta \in \mathrm{Fitt}^{\max}_\Lambda(M)$ and $\lambda \in \mathcal{I}$, then also $\lambda \cdot \theta \in \mathrm{Fitt}^{\max}_\Lambda(M)$.*

iv) *If M admits a quadratic presentation, then $\mathrm{Fitt}^{\max}_\Lambda(M) = \mathcal{I} \cdot \mathrm{Fitt}_\Lambda(M)$.*

v) *If $\Lambda = \mathbb{Z}_pG$ and $M \rightarrowtail C \to C' \twoheadrightarrow M'$ is an exact sequence of finite Λ-modules, where C and C' are c.t., then we have an equality*

$$\mathrm{Fitt}^{\max}_\Lambda(M^\vee)^\sharp \cdot \mathrm{Fitt}_\Lambda(C') = \mathrm{Fitt}^{\max}_\Lambda(M') \cdot \mathrm{Fitt}_\Lambda(C).$$

Proof. For (i), (ii) and (v) see [20], Prop. 3.5 (i), (iii) and Prop. 5.3 (ii). For (iii) let h be a finite presentation of M such that $\mathrm{Fitt}_\Lambda(h) = \mathrm{Fitt}_\Lambda^{\max}(M)$. Then $\theta = \sum_H z_H \mathrm{nr}(H)$, $z_H \in \zeta(\Lambda)$, where the sum runs through all the submatrices $H \in S_b(h)$. Hence it suffices to show that $\lambda \cdot \mathrm{nr}(H) \in \mathrm{Fitt}_\Lambda^{\max}(M)$ for any $H \in S_b(h)$. We may assume that $\lambda = \mathrm{nr}(H')$ with $H' \in M_{b' \times b'}(\Lambda)$, and by adding an appropriate identity matrix on H' resp. h we may also assume that $b = b'$. Consider the diagram

$$\begin{tikzcd}
\Lambda^b \arrow[r, "H \circ H'"] \arrow[d, "H'"] & \Lambda^b \arrow[r, two heads] \arrow[d, equal] & \mathrm{cok}\,(H \circ H') \arrow[d] \\
\Lambda^b \arrow[r, "H"] & \Lambda^b \arrow[r, two heads] & \mathrm{cok}\,(H).
\end{tikzcd}$$

Now (i) implies $\mathrm{nr}(H)\mathrm{nr}(H') \in \mathrm{Fitt}_\Lambda^{\max}(\mathrm{cok}\,(H \circ H')) \subset \mathrm{Fitt}_\Lambda^{\max}(\mathrm{cok}\,(H))$, and since there is an epimorphism $\mathrm{cok}\,(H) \twoheadrightarrow M$, also $\mathrm{Fitt}_\Lambda^{\max}(\mathrm{cok}\,(H)) \subset \mathrm{Fitt}_\Lambda^{\max}(M)$. This shows (iii) and the inclusion $\mathcal{I} \cdot \mathrm{Fitt}_\Lambda(M) \subset \mathrm{Fitt}_\Lambda^{\max}(M)$ of (iv). Now let ψ be a quadratic presentation and h be an arbitrary presentation of M. Then it follows from [20], Th. 3.2 resp. its proof that we may assume that $h = (\psi|0) \circ X$, where $X \in \mathrm{Gl}_a(\Lambda)$ for some $a \in \mathbb{N}$. Since ψ is quadratic, each $H \in S(h)$ is the product $\psi \circ \tilde{X}$ for some submatrix $\tilde{X}$ of X and thus $\mathrm{nr}(H) = \mathrm{nr}(\tilde{X}) \cdot \mathrm{nr}(\psi) \in \mathcal{I} \cdot \mathrm{Fitt}_\Lambda(M)$. □

Assume now that $\mathfrak{o}$ is an integrally closed commutative noetherian ring, but not necessarily complete or local. We choose a maximal order Λ' containing Λ. We may decompose the separable K-algebra A into its simple components

$$A = A_1 \oplus \cdots \oplus A_t,$$

i.e. each A_i is a simple K-algebra and $A_i = Ae_i = e_i A$ with central primitive idempotents e_i, $1 \leqslant i \leqslant t$. For any matrix $H \in M_{b \times b}(\Lambda)$ there is a unique matrix $H^* \in M_{b \times b}(\Lambda')$ such that $H^* H = H H^* = \mathrm{nr}(H) \cdot 1_{b \times b}$ and $H^* e_i = 0$ whenever $\mathrm{nr}(H) e_i = 0$ (cf. [20], Lemma 4.1; the additional assumption on $\mathfrak{o}$ to be complete local is not necessary). If $\tilde{H} \in M_{b \times b}(\Lambda)$ is a second matrix, then $(H\tilde{H})^* = \tilde{H}^* H^*$. We define

$$\mathcal{H} = \mathcal{H}(\Lambda) := \{x \in \zeta(\Lambda) | x H^* \in M_{b \times b}(\Lambda) \forall b \in \mathbb{N}\ \forall H \in M_{b \times b}(\Lambda)\}.$$

Since $x \cdot \mathrm{nr}(H) = x H^* H$, we have in particular

$$\mathcal{H} \cdot \mathcal{I} = \mathcal{H} \subset \zeta(\Lambda), \tag{5}$$

where $\mathcal{I}$ is defined as before even if $\mathfrak{o}$ is not complete and local. The importance of the $\zeta(\Lambda)$-module $\mathcal{H}$ will become clear by means of the following result which is [20], Th. 4.2.

THEOREM 1.2. *If $\mathfrak{o}$ is an integrally closed complete commutative noetherian local ring and M is a finitely presented Λ-module, then*

$$\mathcal{H} \cdot \mathrm{Fitt}_\Lambda^{\max}(M) \subset \mathrm{Ann}_\Lambda(M).$$

Now we specialize to $\Lambda = \mathfrak{o}G$, where $\mathfrak{o}$ is either $\mathbb{Z}$ or $\mathbb{Z}_p$. As before, let Λ' be a maximal order containing Λ. The central conductor of Λ' over Λ is defined to be $\mathcal{F} = \mathcal{F}(\Lambda) := \{x \in \zeta(\Lambda') : x\Lambda' \subset \Lambda\}$ and is explicitly given by (cf. [11], Th. 27.13)

$$\mathcal{F} = \bigoplus_\chi \frac{|G|}{\chi(1)} \mathcal{D}^{-1}(K(\chi)/K), \tag{6}$$

where $\mathcal{D}^{-1}(K(\chi)/K)$ denotes the inverse different of the extension $K(\chi) := K(\chi(g) : g \in G)$ over $K = Quot(\mathfrak{o})$ and the sum runs through all the irreducible characters with values in $\mathbb{C}$ resp. $\mathbb{C}_p$ modulo Galois action.

Lemma 1.3. *Let $\Lambda = \mathfrak{o}G$, where $\mathfrak{o}$ is $\mathbb{Z}$ or $\mathbb{Z}_p$. Then it holds:*

i) $\mathcal{F} \subset \mathcal{H}$.

ii) *If G is abelian or if $\Lambda = \mathbb{Z}_p G$ and $p \nmid |G|$, then $\mathcal{H} = \zeta(\Lambda)$.*

Proof. (i) and the case $p \nmid |G|$ of (ii) are clear from the observations above. If G is abelian, we may choose H^* to be the adjoined matrix of H and we get (ii). □

In the sequel we will use the following notation: $\mathcal{F}(G) = \mathcal{F}(\mathbb{Z}G)$, $\mathcal{F}_p(G) = \mathcal{F}(\mathbb{Z}_p G)$ and similarly for $\mathcal{H}$ and $\mathcal{I}$. We denote the normalized valuation at a prime $\mathfrak{P}$ by $v_{\mathfrak{P}}$ and for $x = \sum_{g\in G} x_g g \in \mathbb{Z}G$ resp. $x \in \mathbb{Z}_p G$ we put $v_p(x) := \min_{g\in G} v_p(x_g)$.

Lemma 1.4. *Let p be a prime and let G be a finite group. Then $\mathcal{H}(G)$ is dense in $\mathcal{H}_p(G)$ with respect to the p-adic topology.*

Proof. Let $x \in \mathcal{H}_p(G)$ and choose $x' \in \zeta(\mathbb{Z}G)$ close to x. Then for any $H \in M_{n\times n}(\mathbb{Z}_p G)$ we have

$$x'H^* = xH^* + (x'-x)H^* \in M_{n\times n}(\mathbb{Z}_p G)$$

if $v_p(x'-x) \geqslant n$, where $|G| = m \cdot p^n$ with $p \nmid m$. Thus $x' \in \mathcal{H}_p(G)$. Now let $N \in \mathbb{N}$ be large; since p does not divide m, we can choose a multiple m' of m such that $m' \equiv 1 \bmod p^N$. Then $m'x'$ is also close to x, since $v_p(x - m'x') \geqslant \min\{v_p(x-x'), v_p((1-m')x')\}$. But since $m|m'$, we have $m'x' \in \mathcal{H}_q(G)$ for all primes q. Now let $H \in M_{n\times n}(\mathbb{Z}G)$. Then

$$m'x'H^* \in \bigcap_q M_{n\times n}(\mathbb{Z}_q G) \cap M_{n\times n}(\mathbb{Q}G) = M_{n\times n}(\Lambda)$$

and hence $m'x' \in \mathcal{H}(G)$. □

1.0.4 *Equivariant L-values* Let us fix a finite Galois extension L/K of number fields with Galois group G. For any prime $\mathfrak{p}$ of K we fix a prime $\mathfrak{P}$ of L above $\mathfrak{p}$ and write $G_{\mathfrak{P}}$ resp. $I_{\mathfrak{P}}$ for the decomposition group resp. inertia subgroup of L/K at $\mathfrak{P}$. Moreover, we denote the residual group at $\mathfrak{P}$ by $\overline{G_{\mathfrak{P}}} = G_{\mathfrak{P}}/I_{\mathfrak{P}}$ and choose a lift $\phi_{\mathfrak{P}} \in G_{\mathfrak{P}}$ of the Frobenius automorphism at $\mathfrak{P}$.
If S is a finite set of places of K containing the set S_∞ of all infinite places of K, and χ is a (complex) character of G, we denote the S-truncated Artin L-function attached to χ and S by $L_S(s,\chi)$ and define $L_S^*(0,\chi)$ to be the leading coefficient of the Taylor expansion of $L_S(s,\chi)$ at $s=0$. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}G) = \prod_{\chi\in\mathrm{Irr}\,(G)} \mathbb{C}$. We define the equivariant Artin L-function to be the meromorphic $\zeta(\mathbb{C}G)$-valued function

$$L_S(s) := (L_S(s,\chi))_{\chi\in\mathrm{Irr}\,(G)}.$$

We put $L_S^*(0) = (L_S^*(0,\chi))_{\chi\in\mathrm{Irr}\,(G)}$ and abbreviate $L_{S_\infty}(s)$ by $L(s)$. If T is a second finite set of places of K sucht that $S \cap T = \emptyset$, we define $\delta_T(s) := (\delta_T(s,\chi))_{\chi\in\mathrm{Irr}\,(G)}$, where $\delta_T(s,\chi) = \prod_{\mathfrak{p}\in T} \det(1 - N(\mathfrak{p})^{1-s}\phi_{\mathfrak{P}}^{-1} | V_\chi^{I_{\mathfrak{P}}})$, and put

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\sharp.$$

These functions are the so-called (S,T)-modified G-equivariant L-functions and we define Stickelberger elements

$$\theta_S^T := \Theta_{S,T}(0) \in \zeta(\mathbb{C}G).$$

If T is empty, we abbreviate θ_S^T by θ_S. Note that the χ-part of θ_S^T vanishes for a non-trivial character χ if there is an (infinite) prime $\mathfrak{p} \in S$ such that $V_\chi^{G_{\mathfrak{P}}} \neq 0$. This is the main reason why we will assume henceforth that L/K is a CM-extension, i.e. L is a CM-field, K is totally real and complex conjugation induces an unique automorphism j of L which lies in the center of G. If R is a subring of either $\mathbb{C}$ or $\mathbb{C}_p$ for a prime p such that 2 is invertible over R, we put $RG_- := RG/(1+j)$ which is a ring, since the idempotent $\frac{1-j}{2}$ lies in RG. For any RG-module M we define $M^- = RG_- \otimes_{RG} M$ which is an exact functor since $2 \in R^\times$. If M is a $\mathbb{Z}G$-module, we define M^- to be $\mathbb{Z}[\frac{1}{2}]G/(1+j) \otimes_{\mathbb{Z}G} M$. This notation is nonstandard, but practical, since taking minus parts is again an exact functor. Now Stark's conjecture (which is a theorem for odd characters, see [28], Th. 1.2, p. 70) implies

$$\theta_S^T \in \zeta(\mathbb{Q}G_-). \tag{7}$$

Note that we actually have to exclude the special case $|S_\infty(L)| = 1$ (cf. the proof of [19], Prop. 3, where (7) is shown in the relevant case $S = S_\infty$ and $T = \emptyset$), but in this situation the extension L/K is abelian. Let us fix an embedding $\iota : \mathbb{C} \rightarrowtail \mathbb{C}_p$; then the image of θ_S^T in $\zeta(\mathbb{Q}_p G_-)$ via the canonical embedding

$$\zeta(\mathbb{Q}G_-) \rightarrowtail \zeta(\mathbb{Q}_p G_-) = \bigoplus_{\substack{\chi \in \mathrm{Irr}_p(G)/\sim \\ \chi \text{ odd}}} \mathbb{Q}_p(\chi),$$

is given by $\sum_\chi (\delta_T(0, \chi^{\iota^{-1}}) \cdot L_S(0, \check{\chi}^{\iota^{-1}}))^\iota$. Here the sum runs over all $\mathbb{C}_p$-valued irreducible odd characters of G modulo Galois action. Note that we will frequently drop ι and ι^{-1} from the notation.

1.0.5 *Ray class groups* For any set S of places of K, we write $S(L)$ for the set of places of L which lie above those in S. Now let T and S be as above. We write cl_L^T for the ray class group of L to the ray $\mathfrak{M}_T := \prod_{\mathfrak{P} \in T(L)} \mathfrak{P}$ and $\mathfrak{o}_S$ for the ring of $S(L)$-integers of L. Let S_f be the set of all finite primes in $S(L)$; then there is a natural map $\mathbb{Z}S_f \to \mathrm{cl}_L^T$ which sends each prime $\mathfrak{P} \in S_f$ to the corresponding class $[\mathfrak{P}] \in \mathrm{cl}_L^T$. We denote the cokernel of this map by $\mathrm{cl}_S^T(L) =: \mathrm{cl}_S^T$. Further, we denote the $S(L)$-units of L by E_S and define $E_S^T := \{x \in E_S : x \equiv 1 \bmod \mathfrak{M}_T\}$. All these modules are equipped with a natural G-action and we have the following exact sequences of G-modules

$$E_{S_\infty}^T \rightarrowtail E_S^T \xrightarrow{v} \mathbb{Z}S_f \to \mathrm{cl}_L^T \twoheadrightarrow \mathrm{cl}_S^T, \tag{8}$$

where $v(x) = \sum_{\mathfrak{P} \in S_f} v_{\mathfrak{P}}(x)\mathfrak{P}$ for $x \in E_S^T$, and

$$E_S^T \rightarrowtail E_S \to (\mathfrak{o}_S/\mathfrak{M}_T)^\times \xrightarrow{\nu} \mathrm{cl}_S^T \twoheadrightarrow \mathrm{cl}_S, \tag{9}$$

where the map ν lifts an element $\overline{x} \in (\mathfrak{o}_S/\mathfrak{M}_T)^\times$ to $x \in \mathfrak{o}_S$ and sends it to the ideal class $[(x)] \in \mathrm{cl}_S^T$ of the principal ideal (x). Note that the G-module $(\mathfrak{o}_S/\mathfrak{M}_T)^\times$ is c.t. if no prime in T ramifies in L/K. We define

$$A_S^T := (\mathrm{cl}_S^T)^-.$$

If $S = S_\infty$, we also write A_L^T and E_L^T instead of $A_{S_\infty}^T$ and $E_{S_\infty}^T$. Finally, we suppress the superscript T from the notation if T is empty. If M is a finitely generated $\mathbb{Z}$-module and p is a prime, we put $M(p) := \mathbb{Z}_p \otimes_{\mathbb{Z}} M$. In particular, $A_L(p)$ is the p-part of the minus class group if p is odd.

2. Statement of the conjectures

Let L/K be a Galois CM-extension with Galois group G. Let S and T be two finite sets of places of K such that

- S contains all the infinite places of K and all the places which ramify in L/K, i.e. $S \supset S_{\mathrm{ram}} \cup S_\infty$.
- $S \cap T = \emptyset$.
- E_S^T is torsionfree.

We refer to the above hypotheses as $Hyp(S,T)$. We put $\Lambda = \mathbb{Z}G$ and choose a maximal order Λ' containing Λ. For a fixed set S we define $\mathfrak{A}_S$ to be the $\zeta(\Lambda)$-submodule of $\zeta(\Lambda')$ generated by the elements $\delta_T(0)$, where T runs through the finite sets of places of K such that $Hyp(S,T)$ is satisfied. The following is a non-abelian generalization of Brumer's conjecture.

Conjecture 2.1 $B(L/K,S)$. Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Then $\mathfrak{A}_S\theta_S \subset \mathcal{I}(G)$ and for each $x \in \mathcal{H}(G)$ we have

$$x \cdot \mathfrak{A}_S\theta_S \subset \mathrm{Ann}_\Lambda(\mathrm{cl}_L).$$

Before we make some clarifying remarks, we state the following lemma.

Lemma 2.2. *Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Then the elements $\phi_{\mathfrak{P}} - N(\mathfrak{p})$, where $\mathfrak{p}$ runs through all the finite places of K such that the sets S and $T_{\mathfrak{p}} = \{\mathfrak{p}\}$ satisfy $Hyp(S,T_{\mathfrak{p}})$, generate $\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)$. Moreover, if we restrict to the primes $\mathfrak{p}$ such that $\phi_{\mathfrak{P}} = 1$, the greatest common divisor of the integers $1 - N(\mathfrak{p})$ is $|\mu_L|$.*

Proof. The proof of [28], Lemma 1.1, p. 82 (where G is assumed to be abelian) remains unchanged. □

Remark 1. i) Since $\mathfrak{A}_S$ is generated by the elements $\delta_T(0)$ such that $Hyp(S,T)$ holds, Conjecture 2.1 is equivalent to the assertion that for all these sets T the Stickelberger element θ_S^T lies in $\mathcal{I}(G)$ and $x\theta_S^T$ annihilates the class group for each $x \in \mathcal{H}(G)$. Note that $\theta_S^T \in \mathcal{I}(G)$ implies $x\theta_S^T \in \zeta(\Lambda)$.

ii) Lemma 2.2 implies that in fact $[\mathfrak{A}_S(p)]_{\mathrm{nr}(\mathbb{Z}_pG)}$ is a Fitting invariant of $\mu_L(p)$ over $\mathbb{Z}_pG$. Moreover, we have

$$\mathcal{I}_p(G) \cdot [\mathfrak{A}_S(p)]_{\mathrm{nr}(\mathbb{Z}_pG)} \subset \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}(\mu_L(p))$$

by Proposition 1.1. So it is reasonable to ask if this inclusion might be an equality (at least if $S = S_{\mathrm{ram}} \cup S_\infty$).

iii) If G is abelian, Lemma 2.2 implies that the module $\mathfrak{A}_S$ equals $\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)$. In this case the inclusion $\mathfrak{A}_S\theta_S \subset \mathcal{I}(G) = \Lambda$ follows from (1), and since $\mathcal{H}(G) = \Lambda$ by Lemma 1.3, Conjecture 2.1 recovers Brumer's conjecture.

Since $\mathcal{H}(G)$ always contains the central conductor $\mathcal{F}(G)$, we can state the following weaker form of Conjecture 2.1.

Conjecture 2.3 $B_w(L/K,S)$. Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Then $\mathfrak{A}_S\theta_S \subset \zeta(\Lambda')$ and for each $x \in \mathcal{F}(G)$ we have

$$x \cdot \mathfrak{A}_S\theta_S \subset \mathrm{Ann}_\Lambda(\mathrm{cl}_L).$$

Lemma 2.4. *Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Then*

$$B(L/K,S) \Longrightarrow B_w(L/K,S).$$

If $S \subset S'$, then

$$B(L/K,S) \Longrightarrow B(L/K,S'),$$
$$B_w(L/K,S) \Longrightarrow B_w(L/K,S').$$

Proof. The first assertion is obvious. Now assume that $B(L/K,S)$ resp. $B_w(L/K,S)$ holds. Since $\theta_{S'} = \mathrm{nr}(\lambda)\theta_S$, where $\lambda = \prod_{\mathfrak{p}\in S'\setminus S}(1 - \phi_{\mathfrak{P}}^{-1}) \in \Lambda$, we see that also $\mathfrak{A}_{S'}\theta_{S'} \subset \mathfrak{A}_S\mathrm{nr}(\lambda)\theta_S$ lies in $\mathcal{I}(G)$ resp. $\zeta(\Lambda')$. Moreover $\tilde{x} := x \cdot \mathrm{nr}(\lambda)$ lies in $\mathcal{H}(G)$ resp. $\mathcal{F}(G)$ if x does. Hence we find that $x\mathfrak{A}_{S'}\theta_{S'} \subset \tilde{x}\mathfrak{A}_S\theta_S$ annihilates cl_L. □

Replacing the class group cl_L by its p-parts $\mathrm{cl}_L(p)$ for each rational prime p, Conjecture $B(L/K,S)$ resp. Conjecture $B_w(L/K,S)$ naturally decomposes into local conjectures $B(L/K,S,p)$ resp. $B_w(L/K,S,p)$.

Note that it is possible to replace $\mathcal{H}(G)$ by $\mathcal{H}_p(G)$ by means of Lemma 1.4. Taking Lemma 1.3 into account, a similar proof shows the following lemma.

Lemma 2.5. *Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$ and let p be a prime. Then*

$$B(L/K,S,p) \Longrightarrow B_w(L/K,S,p).$$

If $p \nmid |G|$ then

$$B(L/K,S,p) \iff B_w(L/K,S,p).$$

If $S \subset S'$, then

$$B(L/K,S,p) \Longrightarrow B(L/K,S',p),$$

$$B_w(L/K,S,p) \Longrightarrow B_w(L/K,S',p).$$

For $\alpha \in L^\times$ we define

$$S_\alpha := \{\mathfrak{p} \text{ finite prime of } K : \mathfrak{p} | N_{L/K}(\alpha)\}$$

and we call α an *anti-unit* if $\alpha^{1+j} = 1$. Let $\omega_L := \mathrm{nr}(|\mu_L|)$. The following is a non-abelian generalization of the Brumer-Stark conjecture.

Conjecture 2.6 $BS(L/K,S)$. Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Then $\omega_L \cdot \theta_S \in \mathcal{I}(G)$ and for each $x \in \mathcal{H}(G)$ and each fractional ideal $\mathfrak{a}$ of L, there is an anti-unit $\alpha = \alpha(x,\mathfrak{a},S) \in L^\times$ such that

$$\mathfrak{a}^{x\cdot\omega_L\cdot\theta_S} = (\alpha)$$

and for each finite set T of primes of K such that $Hyp(S \cup S_\alpha, T)$ is satisfied there is an $\alpha_T \in E_{S_\alpha}^T$ such that

$$\alpha^{z\cdot\delta_T(0)} = \alpha_T^{z\cdot\omega_L} \tag{10}$$

for each $z \in \mathcal{H}(G)$.

Remark 2. If G is abelian, we have $\mathcal{I}(G) = \mathcal{H}(G) = \mathbb{Z}G$ and $\omega_L = |\mu_L|$. Hence it suffices to treat the case $x = z = 1$. Then [28], Prop. 1.2, p. 83 states that the condition (10) on the anti-unit α is equivalent to the assertion that the extension $L(\alpha^{1/\omega_L})/K$ is abelian.

As above we can state the following weaker conjecture.

Conjecture 2.7 $BS_w(L/K,S)$. Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Then $\omega_L \cdot \theta_S \in \zeta(\Lambda')$ and for each $x \in \mathcal{F}(G)$ and each fractional ideal $\mathfrak{a}$ of L, there is an anti-unit $\alpha = \alpha(x,\mathfrak{a},S) \in L^\times$ such that

$$\mathfrak{a}^{x\cdot\omega_L\cdot\theta_S} = (\alpha)$$

and for each finite set T of primes of K such that $Hyp(S \cup S_\alpha, T)$ holds there is an $\alpha_T \in E_{S_\alpha}^T$ such that

$$\alpha^{z\cdot\delta_T(0)} = \alpha_T^{z\cdot\omega_L} \tag{11}$$

for each $z \in \mathcal{F}(G)$.

Remark 3. i) Since $E_{S_\alpha}^T$ is torsionfree, we may replace the equalities (10) and (11) by the equality $\alpha^{\delta_T(0)} = \alpha_T^{\omega_L}$ in $\mathbb{Q} \otimes E_{S_\alpha}^T$.

ii) If $\mathfrak{a}$ and $\mathfrak{b}$ are two fractional ideals of L for which Conjecture $BS(L/K,S)$ resp. $BS_w(L/K,S)$ holds, then it is easy to see that this conjecture is also true for the product $\mathfrak{a}\cdot\mathfrak{b}$. Since it is also true for principal ideals, it suffices to verify these conjectures for totally decomposed primes, as these primes generate the class group.

iii) If we restrict Conjecture $BS(L/K,S)$ resp. $BS_w(L/K,S)$ to ideals whose class in cl_L has p-power order, we get local conjectures $BS(L/K,S,p)$ resp. $BS_w(L/K,S,p)$.

iv) If the prime p is odd, we may omit the condition that the generator α is an anti-unit by the following reason (cf. [17], remark preceding Prop. 1.1): Let $\mathfrak{a}$ be a fractional ideal whose class in cl_L has p-power order. Since squaring is invertible on $\mathrm{cl}_L(p)$ we find $\mathfrak{b}$ such that $\mathfrak{a} = (u)\mathfrak{b}^2$ for some $u \in L^\times$. Now let $x \in \mathcal{H}(G)$ resp. $x \in \mathcal{F}(G)$ and assume that $\mathfrak{b}^{x\cdot\omega_L\theta_S}$ is generated by $\beta \in L^\times$ such that (10) holds (with α replaced by β). But since $(1-j)\theta_S = 2\theta_S$, we have $\mathfrak{a}^{x\cdot\omega_L\theta_S} = (u^{x\cdot\omega_L\theta_S})\mathfrak{b}^{x\cdot\omega_L 2\theta_S} = (u^{x\cdot\omega_L\theta_S}\cdot\beta^{1-j})$ and this generator is an appropriate anti-unit.

v) Burns [4] has meanwhile formulated a new conjecture which generalizes many refined Stark conjectures to the non-abelian situation. In particular, it implies our generalization of Brumer's conjecture (cf. loc.cit., Prop. 3.5.1). Since it also implies the Brumer-Stark conjecture (cf. loc.cit., Remark 3.5.2) in the abelian case, the author expects that it also implies Conjecture $BS(L/K,S)$. If this is true, loc.cit., Th. 4.1.1 would give a different proof of Theorem 5.1 below.

We have the following implications which are either obvious or which are proved by a similar argument as in Lemma 2.4.

Lemma 2.8. *Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$ and p be a prime. Then*

$$BS(L/K,S) \Longrightarrow BS_w(L/K,S),\ BS(L/K,S,p) \Longrightarrow BS_w(L/K,S,p).$$

If $p \nmid |G|$, then

$$BS(L/K,S,p) \iff BS_w(L/K,S,p).$$

If $S \subset S'$, then

$$BS(L/K,S) \Longrightarrow BS(L/K,S'),\ BS(L/K,S,p) \Longrightarrow BS(L/K,S',p),$$

$$BS_w(L/K,S) \Longrightarrow BS_w(L/K,S'),\ BS_w(L/K,S,p) \Longrightarrow BS_w(L/K,S',p),$$

We have the following relation to the non-abelian Brumer Conjectures:

Lemma 2.9. *Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$ and p be a prime. Then*

$$BS(L/K,S) \Longrightarrow B(L/K,S),\ BS(L/K,S,p) \Longrightarrow B(L/K,S,p),$$

$$BS_w(L/K,S) \Longrightarrow B_w(L/K,S),\ BS_w(L/K,S,p) \Longrightarrow B_w(L/K,S,p).$$

Proof. Let $\mathfrak{a}$ be a fractional ideal of L whose class in cl_L is assumed to have p-power order if we are in the local case. Let $x \in \mathcal{H}(G)$ resp. $x \in \mathcal{F}(G)$. Then $\mathfrak{a}^{x\cdot\omega_L\theta_S} = (\alpha)$ and $(\alpha)^{z\cdot\delta_T(0)} = (\alpha_T)^{z\cdot\omega_L}$ for all $z \in \mathcal{F}(G)$. Hence

$$\mathfrak{a}^{x\cdot z\cdot\omega_L\cdot\theta_S^T} = (\alpha)^{z\cdot\delta_T(0)} = (\alpha_T)^{z\cdot\omega_L}. \tag{12}$$

Since $\omega_L \in \zeta(\mathbb{Q}G)^\times$, we find $N \in \mathbb{N}$ such that $N\cdot\omega_L^{-1} \in \zeta(\Lambda)$. Moreover, $|G|\cdot\zeta(\Lambda) \subset \mathcal{F}(G)$ such that we may choose $z = |G|\cdot N\cdot\omega_L^{-1}$. But the group of fractional ideals has no $\mathbb{Z}$-torsion such that equation (12) implies $\mathfrak{a}^{x\cdot\theta_S^T} = (\alpha_T)$. ∎

3. Burns' Conjecture and the strong Brumer-Stark property

We first recall a non-abelian generalization of the Rubin-Stark conjecture due to D. Burns [3]. Note that our slightly different definition of χ-twist will lead to some minor changes. Let L/K be an extension of number fields with Galois group G and fix a non-trivial irreducible complex character χ of G. For any finite non-empty set S of places of K we write Y_S for the free abelian group on $S(L)$

and X_S for the kernel of the augmentation map $Y_S \to \mathbb{Z}$ which sends each element of $S(L)$ to 1. If S contains S_∞, the Dirichlet map

$$\lambda_S : \mathbb{R} \otimes_{\mathbb{Z}} E_S \to \mathbb{R} \otimes_{\mathbb{Z}} X_S,\ \varepsilon \mapsto - \sum_{\mathfrak{P} \in S(L)} \log |\varepsilon|_{\mathfrak{p}} \mathfrak{P}$$

is an isomorphism of $\mathbb{R}G$-modules. The Noether-Deuring Theorem combines with the fact that X_S is torsionfree to imply the existence of G-invariant embeddings $\phi : X_S \rightarrowtail E_S$. For any such ϕ we set

$$R_\phi^S(\chi) := \det(\lambda_S \circ \phi | \mathbb{C} \otimes_{E_\chi} (V_{\check{\chi}} \otimes_{\mathbb{Z}G} X_S)) \in \mathbb{C}^\times,$$

where $\check{\chi}$ denotes the character contragredient to χ. Then Stark's conjecture (as interpreted in [28], Conj. 5.1,p. 27, but see [3], §2) states that for each $\alpha \in \mathrm{Aut}(\mathbb{C})$ one has

$$\frac{L_S^*(0,\chi^\alpha)}{R_\phi^S(\chi^\alpha)} = \alpha \left(\frac{L_S^*(0,\chi)}{R_\phi^S(\chi)} \right),$$

where $\chi^\alpha := \alpha \circ \chi$. We put

$$r_S = r_S(\chi) := \sum_{\mathfrak{p} \in S} \dim_{E_\chi}(V_\chi^{G_{\mathfrak{P}}}).$$

Then, since χ is non-trivial, one has

$$r_S = \dim_{E_\chi}(V_{\check{\chi}} \otimes_{\mathbb{Z}G} X_S) = \dim_{E_\chi}(E_\chi \otimes_{\mathfrak{o}_\chi} X_{S,\check{\chi}})$$

and the function $L_S(s,\chi)$ vanishes to order r_S at $s = 0$ by [28], Prop. 3.4, p. 24. Further, if we denote by

$$\lambda_S^{(\chi)} : \mathbb{C} \otimes_{E_\chi} (\wedge_{E_\chi}^{r_S} (V_{\check{\chi}} \otimes_{\mathbb{Z}G} E_S)) \xrightarrow{\sim} \mathbb{C} \otimes_{E_\chi} (\wedge_{E_\chi}^{r_S} (V_{\check{\chi}} \otimes_{\mathbb{Z}G} X_S))$$

the isomorphism of $\mathbb{C}$-spaces induced by λ_S, one finds that Stark's conjecture implies

$$L_S^*(0,\chi) \cdot \wedge_{E_\chi}^{r_S} (V_{\check{\chi}} \otimes_{\mathbb{Z}G} X_S)) = \lambda_S^{(\chi)}(\wedge_{E_\chi}^{r_S} (V_{\check{\chi}} \otimes_{\mathbb{Z}G} E_S)). \tag{13}$$

Let $L_{S,T}^*(0,\chi) := \delta_T(0,\check{\chi}) \cdot L_S^*(0,\chi)$ if S and T satisfy $Hyp(S,T)$. For any G-module resp. $\mathfrak{o}_\chi$-module M we write M_{tor} for the $\mathbb{Z}$-torsion submodule of M and set $\overline{M} := M/M_{\mathrm{tor}}$ which we identify as a submodule of $\mathbb{Q} \otimes_{\mathbb{Z}} M$ resp. $E_\chi \otimes_{\mathfrak{o}_\chi} M$ in the natural way. Now Burns' conjecture ([3], Conj. 2.1) asserts the following refinement of (13):

Conjecture 3.1 *$RS(L/K,S,T,\chi)$* (Burns). Let S and T be two finite sets of places of K such that $Hyp(S,T)$ is satisfied and let χ be a non-trivial irreducible complex character. Then Stark's conjecture holds for χ and in $\mathbb{C} \otimes_{E_\chi} (\wedge_{E_\chi}^{r_S} (V_{\check{\chi}} \otimes_{\mathbb{Z}G} X_S))$ one has

$$|G|^{r_S} L_{S,T}^*(0,\chi) \cdot \wedge_{\mathfrak{o}_\chi}^{r_S} \overline{X_{S,\check{\chi}}} \subset \mathrm{Fitt}_{\mathfrak{o}_\chi}(H^{-1}(G, X_S[\check{\chi}])) \cdot \lambda_S^{(\chi)}(\wedge_{\mathfrak{o}_\chi}^{r_S} E_S^{T,\check{\chi}}).$$

Moreover, we will say that $RS(L/K,S,\chi)$ holds if $RS(L/K,S,T,\chi)$ holds for all finite sets of places T such that $Hyp(S,T)$ is satisfied.

Remark 4. It is reasonable to expect that the inclusion in Conjecture 3.1 is an equality for sufficiently large S (cf. [3], Prop. 4.8).

If Stark's conjecture holds, the quotient $L_S^*(0,\chi)/R_\phi^S(\chi)$ belongs to E_χ. The strong Stark conjecture as formulated by T. Chinburg ([9], Conj. 2.2) further predicts that

$$\frac{L_S^*(0,\chi)}{R_\phi^S(\chi)} \mathfrak{o}_\chi = q(\psi_{\check{\chi}})^{-1}, \tag{14}$$

where ψ_χ denotes the composite homomorphism of $\mathfrak{o}_\chi$-modules

$$X_{S,\chi} \xrightarrow{\phi_\chi} E_{S,\chi} \xrightarrow{t(E_S,\chi)} E_S^\chi,$$

and for each irreducible character χ we use the following general notion: if $f : M \to N$ is a homomorphism of finitely generated $\mathfrak{o}_\chi$-modules with finite kernel and finite cokernel, then $q(f)$ denotes the frational $\mathfrak{o}_\chi$-ideal $\mathrm{Fitt}_{\mathfrak{o}_\chi}(\mathrm{cok}\,(f)) \cdot \mathrm{Fitt}_{\mathfrak{o}_\chi}(\ker(f))^{-1}$.

THEOREM 3.2 ([3], Th. 4.1). *If the strong Stark conjecture (14) holds for the character χ, then $RS(L/K, S, \chi)$ is valid for all admissible sets S.*

We will need the following result which is [3], Prop. 3.2. We set $\mathfrak{c}_S(\chi) := \mathrm{Fitt}_{\mathfrak{o}_\chi}(H^{-1}(G, X_S[\check{\chi}]))$.

PROPOSITION 3.3. *Assume that $r_S = 1$ and $|S| > 1$. Let $\mathfrak{p}_1$ be the unique place in S with $V_\chi^{G_{\mathfrak{P}_1}} \neq 0$ and set $S_1 := S_\infty \cup \{\mathfrak{p}_1\}$. If $RS(L/K, S, T, \chi)$ is valid, then for any element d of $\mathfrak{c}_S(\chi)^{-1}\mathcal{D}(E_\chi/\mathbb{Q})^{-1}$ there exists an element $u(d) \in E_{S_1}^T$ which at each place $\mathfrak{P}$ of L satisfies*

$$-\log|u(d)|_{\mathfrak{P}} = \begin{cases} 0, & \text{if } \mathfrak{P} \nmid \mathfrak{p}_1 \\ \sum_{\gamma \in \mathrm{Gal}(E_\chi/\mathbb{Q})} \sum_{h \in G_{\mathfrak{P}_1}} \gamma(d)\chi^\gamma(gh) L_{S,T}^*(0, \chi^\gamma), & \text{if } \mathfrak{P} = \mathfrak{P}_1^g,\ g \in G. \end{cases}$$

THEOREM 3.4. *Let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. If $RS(L/K, S \cup \{\mathfrak{q}\}, \chi)$ is valid for all characters χ such that $r_S = 0$ and all primes $\mathfrak{q}$ which are totally split in L/K, then $BS_w(L/K, S)$ and $B_w(L/K, S)$ hold. In particular, if the strong Stark conjecture (14) holds for these characters, then $BS_w(L/K, S)$ and $B_w(L/K, S)$ hold for all admissible sets S.*

Proof. By means of Lemma 2.9 and Theorem 3.2, it suffices to show that the relevant case of Burns' conjecture implies $BS_w(L/K, S)$. Since $e_\chi \cdot \theta_S^T = 0$ if $r_S > 0$, we only have to treat characters χ with $r_S = 0$. Let T be a finite set of primes of K such that $Hyp(S, T)$ is satisfied and let $\mathfrak{q}$ be a totally decomposed prime not in $S \cup T$. We claim that $\frac{1}{2} \in \mathfrak{c}_{S \cup \{\mathfrak{q}\}}(\chi)^{-1}$. Taking this for granted for the moment, let $x \in \mathcal{F}(G)$ and write

$$\tfrac{1}{2} \cdot x \cdot \theta_S^T = \sum_{\chi} \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \gamma(\tfrac{1}{2}x_\chi) L_{S,T}^*(0, \chi^\gamma) \cdot \mathrm{pr}_{\check{\chi}^\gamma},$$

where the first sum runs over all irreducible characters with $r_S(\chi) = 0$ modulo Galois action, and where $x_\chi \in \mathcal{D}^{-1}(\mathbb{Q}(\chi)/\mathbb{Q})$ according to the description (6) of the central conductor. Since $L_{S,T}^*(0, \chi) = L_{S \cup \{\mathfrak{q}\}, T}^*(0, \chi)$ and the trace maps $\mathcal{D}^{-1}(E_\chi/\mathbb{Q})$ onto $\mathcal{D}^{-1}(\mathbb{Q}(\chi)/\mathbb{Q})$, we can apply Proposition 3.3 to the set $S \cup \{\mathfrak{q}\}$ and obtain

$$\mathfrak{Q}^{\frac{1}{2}x\theta_S^T} = (\alpha_T),$$

where $\alpha_T \in E_{S_\infty \cup \{\mathfrak{q}\}}^T$ and $\mathfrak{Q}$ is a prime in L above $\mathfrak{q}$. Since the ray class group cl_L^T is generated by totally decomposed primes, we have for any fractional ideal $\mathfrak{a}$ of L, coprime to the ideals in T that

$$\mathfrak{a}^{\frac{1}{2}x\theta_S^T} = (\alpha_T(\mathfrak{a})) \tag{15}$$

with $\alpha_T(\mathfrak{a}) \in E_{S(\mathfrak{a})}^T$, where $S(\mathfrak{a})$ contains all the primes of K below the primes dividing $\mathfrak{a}$. Now let p be a prime and let $\mathfrak{a}$ be a fractional ideal of L such that its class $[\mathfrak{a}] \in \mathrm{cl}_L$ has p-power order . Then Lemma 2.2 implies that there is a totally decomposed prime $\mathfrak{p}_0$ such that $|\mu_L| = (1 - N(\mathfrak{p}_0)) \cdot c$, where $c \in \mathbb{Q}$ with $v_p(c) = 0$. We may assume that $\mathfrak{p}_0$ is coprime to $\mathfrak{a}$ and that $Hyp(S, \{\mathfrak{p}_o\})$ is satisfied. The observations above imply that for any $x \in \mathcal{F}(G)$ we have

$$\mathfrak{a}^{x\omega_L\theta_S} = \mathfrak{a}^{\frac{1}{2}x\omega_L 2\theta_S} = \mathfrak{a}^{\frac{1}{2}x\mathrm{nr}(c)(1-j)\theta_S^{\{\mathfrak{p}_o\}}} = (\alpha')^{(1-j)} = (\alpha)$$

for an appropriate $\alpha' \in L^\times$ and an anti-unit $\alpha = \alpha'^{(1-j)}$. Note that there is a natural number N with $v_p(N) = 0$ such that $N \cdot x\mathrm{nr}(c) \in \mathcal{F}(G)$ and raising to the N^{th} power is a bijection on $\mathrm{cl}_L(p)$. Moreover, if T is a finite set of primes such that $Hyp(S \cup S_\alpha, T)$ holds, then (15) implies that for any $z \in \mathcal{F}(G)$ we have

$$(\alpha)^{z \cdot \delta_T(0)} = \mathfrak{a}^{(1-j)z \cdot \frac{1}{2}x\omega_L\theta_S^T} = (\alpha'_T(\mathfrak{a})^{(1-j)})^{z \cdot \omega_L},$$

where $\alpha'_T(\mathfrak{a}) \in E^T_{S_\alpha}$ and $\alpha_T := \alpha'_T(\mathfrak{a})^{(1-j)}$ is an anti-unit. Hence $\alpha^{z\cdot\delta_T(0)} = u_T \cdot \alpha_T^{z\cdot\omega_L}$, where u_T is a unit and an anti-unit, thus a root of unity. But u_T is also congruent 1 modulo $\mathfrak{M}_T$ and therefore $u_T = 1$.
We are left with the proof of $\frac{1}{2} \in \mathfrak{c}_{S\cup\{\mathfrak{q}\}}(\chi)^{-1}$ (which was only needed for the case $p = 2$). Since $r_S(\chi) = 0$, we have $V_\chi^{G_\mathfrak{P}} = 0$ for any prime $\mathfrak{p} \in S$. Let us fix an infinite $\mathfrak{p} \in S$. Since there is no unramified extension of the rational numbers, there are at least two primes in S such that $\mathfrak{c}_{S\cup\{\mathfrak{q}\}}(\chi)$ is contained in $\mathrm{Fitt}_{\mathfrak{o}_\chi}((T_\chi)_{G_\mathfrak{P}})$ by [3], Rem. 2.3. But $(T_\chi)_{G_\mathfrak{P}} = T_\chi/(1-j)T_\chi = T_\chi/2T_\chi$ such that

$$\mathfrak{c}_{S\cup\{\mathfrak{q}\}}(\chi) \subset \mathrm{Fitt}_{\mathfrak{o}_\chi}((T_\chi)_{G_\mathfrak{P}}) = 2^{\chi(1)}\mathfrak{o}_\chi \subset 2\mathfrak{o}_\chi$$

and we have proven the claim. □

Now we discuss a non-abelian generalization of the strong Brumer-Stark property.

Definition 3.5 $StBS(L/K,S,T,p)$. Let p be a prime and let S and T be two finite sets of places of K such that $Hyp(S,T)$ is satisfied. The CM-extension L/K satisfies the *strong Brumer-Stark property* $StBS(L/K,S,T,p)$ if

$$\begin{array}{lll} \theta_S^T & \in \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L^T(p)) = \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}(\mathrm{cl}_L(p))^- & \text{if} \quad p \neq 2 \\ \frac{1}{2}\theta_S^T & \in \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}(\mathrm{cl}_L(p)) & \text{if} \quad p = 2. \end{array}$$

The above property does not hold in general as follows from the results in [16]. But it is reasonable to state the following conjecture which is the above property over the maximal order.

Conjecture 3.6 $StBS_w(L/K,S,T,p)$. Let p be a prime and let S and T be two finite sets of places of K such that $Hyp(S,T)$ is satisfied. Choose a maximal order Λ_p containing $\mathbb{Z}_pG$. Then

$$\begin{array}{lll} \theta_S^T & \in \mathrm{Fitt}_{(\Lambda_p)_-}((\Lambda_p)_- \otimes_{\mathbb{Z}_pG_-} A_L^T(p)) = \mathrm{Fitt}_{\Lambda_p}(\Lambda_p \otimes_{\mathbb{Z}_pG} \mathrm{cl}_L^T(p))^- & \text{if} \quad p \neq 2 \\ \frac{1}{2}\theta_S^T & \in \mathrm{Fitt}_{\Lambda_p}(\Lambda_p \otimes_{\mathbb{Z}_pG} \mathrm{cl}_L^T(p)) & \text{if} \quad p = 2. \end{array}$$

Moreover, we will say that $StBS(L/K,S,p)$ resp. $StBS_w(L/K,S,p)$ holds if $StBS(L/K,S,T,p)$ resp. $StBS_w(L/K,S,T,p)$ holds for all finite sets of places T such that $Hyp(S,T)$ is satisfied.

Lemma 3.7. *Let p be a prime and let S and T be two finite sets of places of K such that $Hyp(S,T)$ is satisfied.*

i) *If $S \subset S'$, then*

$$StBS(L/K,S,T,p) \Longrightarrow StBS(L/K,S',T,p),$$
$$StBS_w(L/K,S,T,p) \Longrightarrow StBS_w(L/K,S',T,p).$$

ii) *If $T \subset T'$, then*

$$StBS(L/K,S,T,p) \Longrightarrow StBS(L/K,S,T',p),$$
$$StBS_w(L/K,S,T,p) \Longrightarrow StBS_w(L/K,S,T',p).$$

Proof. The first assertion follows from Proposition 1.1 (iv), as we observe that $\theta_{S'}^T = \prod_{\mathfrak{p}\in S'\setminus S} \mathrm{nr}(1-\phi_\mathfrak{P}^{-1}) \cdot \theta_S^T$. For (ii) consider the following special case of sequence (9):

$$E_L^T \rightarrowtail E_L \to (\mathfrak{o}_L/\mathfrak{M}_T)^\times \to \mathrm{cl}_L^T \twoheadrightarrow \mathrm{cl}_L.$$

Since we have a similar sequence with T replaced by T', we find that the kernel of the natural projection $\mathrm{cl}_L^{T'} \twoheadrightarrow \mathrm{cl}_L^T$ equals $\ker((\mathfrak{o}_L/\mathfrak{M}_{T'})^\times \twoheadrightarrow (\mathfrak{o}_L/\mathfrak{M}_T)^\times) = (\mathfrak{o}_L/\mathfrak{M}_{T'\setminus T})^\times$. Now Proposition 1.1 (ii) implies

$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L^{T'}(p)) \supset \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L^T(p)) \cdot \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}((\mathfrak{o}/\mathfrak{M}_{T'\setminus T})^{\times,-}(p))$$

if $p \neq 2$, and the latter contains $\theta_S^T \cdot \prod_{\mathfrak{p} \in T' \setminus T} \mathrm{nr}(1 - N(\mathfrak{p})\phi_{\mathfrak{P}}^{-1}) = \theta_S^{T'}$. A similar argument applies for $p = 2$. Since tensoring with Λ_p is a right exact functor, we also obtain the desired implication in the maximal order case. □

Proposition 3.8. *Let p be a prime and let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Then*

$$StBS(L/K, S, p) \Longrightarrow BS(L/K, S, p),$$
$$StBS_w(L/K, S, p) \Longrightarrow BS_w(L/K, S, p).$$

Proof. Assume that $StBS(L/K, S, p)$ holds and $p \neq 2$. Let $\mathfrak{a}$ be a fractional ideal of L whose class in cl_L has p-power order. As before write $|\mu_L| = (1 - N(\mathfrak{p}_0)) \cdot c$, where $\mathfrak{p}_0 \notin S$ is a totally decomposed prime, coprime to $\mathfrak{a}$ such that $Hyp(S, \{\mathfrak{p}_0\})$ is satisfied, and $v_p(c) = 0$. Then Theorem 1.2 implies that for any $x \in \mathcal{H}(G)$, there is an $\alpha \in L^\times$ such that

$$\mathfrak{a}^{x\omega_L\theta_S} = \mathfrak{a}^{x\mathrm{nr}(c)\theta_S^{\{\mathfrak{p}_0\}}} = (\alpha).$$

But also for any finite set of places T such that $Hyp(S \cup S_\alpha, T)$ is satisfied, there is an $\alpha_T \in E_{S_\alpha}^T$ such that

$$\mathfrak{a}^{x \cdot \theta_S^T} = (\alpha_T).$$

As on earlier occasions we may assume that α and α_T are anti-units such that the equation of ideals

$$(\alpha)^{z\delta_T(0)} = \mathfrak{a}^{z \cdot x\omega_L\theta_S^T} = (\alpha_T)^{z \cdot \omega_L}$$

for all $z \in \mathcal{H}(G)$ actually implies $\alpha^{z\delta_T(0)} = \alpha_T^{z \cdot \omega_L}$. For the modifications for the prime $p = 2$ compare the proof of Theorem 3.4. For the implication of the weaker conjectures, everything remains the same apart from some obvious modifications and the following fact: If M is a finitely presented $\mathbb{Z}_pG$-module, then (cf. [20], Prop. 5.1)

$$\mathcal{F}_p(G) \cdot \mathrm{Fitt}_{\Lambda_p}(\Lambda_p \otimes_{\mathbb{Z}_pG} M) \subset \mathrm{Ann}_{\mathbb{Z}_pG}(M)$$

and similarly on minus parts. □

By a similar argument we can prove a partial converse of Lemma 2.9 which is a non-abelian analogue of [17], Prop. 1.2.

Proposition 3.9. *Let p be an odd prime and let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Assume that $\mu_L(p)$ is c.t. and*

$$\mathcal{H}_p(G)\mathrm{Fitt}_{\mathbb{Z}_pG_-}(\mu_L(p))\theta_S \subset \mathrm{Ann}_{\mathbb{Z}_pG_-}(A_L(p)). \tag{16}$$

Then for each $x \in \mathcal{H}(G)^2$ and each fractional ideal $\mathfrak{a}$ of L whose class in cl_L is of p-power order, there is an anti-unit $\alpha \in L^\times$ such that

$$\mathfrak{a}^{x \cdot \omega_L \cdot \theta_S} = (\alpha)$$

and for each finite set T of primes of K such that $Hyp(S \cup S_\alpha, T)$ holds there is an $\alpha_T \in E_{S_\alpha}^T$ such that

$$\alpha^{z \cdot \delta_T(0)} = \alpha_T^{z \cdot \omega_L}$$

for each $z \in \mathcal{H}(G)$.

Proof. Since the p-part of the roots of unity is c.t. and $\mu_L(p)$ is cyclic as $\mathbb{Z}_pG_-$-module, there is a nonzerodivisor $\lambda \in \mathbb{Z}_pG_-$ such that $\xi := \mathrm{nr}(\lambda)$ generates $\mathrm{Fitt}_{\mathbb{Z}_pG_-}(\mu_L(p))$. Let $\mathfrak{a}$ be a fractional ideal of L whose class $[\mathfrak{a}] \in \mathrm{cl}_L$ has p-power order and let $x' \in \mathcal{H}(G)$. By assumption there is an $\alpha \in L^\times$ such that

$$\mathfrak{a}^{x'\xi\theta_S} = (\alpha).$$

Let T be a finite set of places such that $Hyp(S \cup S_\alpha, T)$ is satisfied. Since p is odd, we may assume that $[\mathfrak{a}] \in A_L(p)$ and we can lift $[\mathfrak{a}]$ to the class $[\mathfrak{a}]_T \in A_L^T(p)$. Then $[\mathfrak{a}]_T^{x'\xi\theta_S}$ lies in the kernel $D(p)$ of the epimorphism $A_L^T(p) \twoheadrightarrow A_L(p)$. But $D(p)$ is c.t. by means of the exact sequence

$$\mu_L(p) \rightarrowtail (\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}(p) \twoheadrightarrow D(p)$$

and the Fitting invariant $\mathrm{Fitt}_{\mathbb{Z}_pG_-}(D(p))$ is generated by $\delta_T(0)\xi^{-1}$. Hence for any $x'' \in \mathcal{H}(G)$, we have

$$1 = [\mathfrak{a}]_T^{x''\delta_T(0)\xi^{-1}x'\xi\theta_S} = [\mathfrak{a}]_T^{x'x''\theta_S^T}.$$

Now we can proceed as in the proof of Proposition 3.8. $\square$

Remark 5.
- i) Since tensoring with the maximal order Λ_p is exact on sequences of finite c.t. modules, we obtain a similar result by replacing $\mathrm{Fitt}_{\mathbb{Z}_pG_-}(\mu_L(p))$ by $\mathrm{Fitt}_{(\Lambda_p)_-}(\Lambda_p \otimes \mu_L)$ and $\mathcal{H}$ by $\mathcal{F}$.
- ii) If $p \nmid |G|$, then $\mu_L(p)$ is c.t. and $\mathcal{H}_p(G)^2 = \mathcal{H}_p(G)$ by Lemma 1.3. Then the above proof shows that we may replace $x \in \mathcal{H}(G)^2$ by $x \in \mathcal{H}(G)$ such that (16) implies $BS(L/K,S,p)$.
- iii) If $\mu_L(p) = 1$, then in fact
$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(\mu_L(p)) = [\mathrm{nr}(\mathbb{Z}_pG_-)]_{\mathrm{nr}(\mathbb{Z}_pG_-)} = [\mathfrak{A}_S(p)]_{\mathrm{nr}(\mathbb{Z}_pG_-)}.$$
In particular, $BS(L/K,S,p)$ is equivalent to $B(L/K,S,p)$ if in addition $p \nmid |G|$.

4. The relation to the strong Stark conjecture

As before let L/K be a finite Galois CM-extension of number fields with Galois group G. We denote the maximal real subfield of L by L^+ and the normal closure of L by L^c. For $n \in \mathbb{N}$ let ζ_n be a primitive complex n^{th} root of unity. The aim of this section is to prove the following result.

Theorem 4.1. *Let p be an odd prime and let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. If the strong Stark conjecture at p holds for all characters χ with $r_S(\chi) = 0$, then $StBS_w(L/K,S,p)$ is true.*

Corollary 4.2. *Let p be an odd prime. Assume that no prime of L^+ above p is split in L/L^+ whenever $L^c \subset (L^c)^+(\zeta_p)$. Then $StBS_w(L/K,S,p)$, $BS_w(L/K,S,p)$ and $B_w(L/K,S,p)$ are true for any set S of places of K containing $S_{\mathrm{ram}} \cup S_\infty$.*

Proof. Since the strong Stark conjecture at p holds in this case by [19], Cor. 2, this follows immediately from Theorem 4.1, Proposition 3.8 and Lemma 2.9. $\square$

Before we start with the proof of Theorem 4.1, we introduce some further notation. We define central idempotents of $\mathbb{Q}_pG_{\mathfrak{P}}$ by

$$\varepsilon'_{\mathfrak{p}} := |I_{\mathfrak{P}}|^{-1}N_{I_{\mathfrak{P}}},\ \varepsilon''_{\mathfrak{p}} = 1 - \varepsilon'_{\mathfrak{p}},$$

and a $\mathbb{Z}_pG_{\mathfrak{P}}$-module $U_{\mathfrak{p}}$ by

$$U_{\mathfrak{p}} := \langle N_{I_{\mathfrak{P}}}, 1 - \varepsilon'_{\mathfrak{p}}\phi_{\mathfrak{P}}^{-1}\rangle_{\mathbb{Z}_pG_{\mathfrak{P}}} \subset \mathbb{Q}_pG_{\mathfrak{P}}.$$

Note that $U_{\mathfrak{p}} = \mathbb{Z}_pG_{\mathfrak{P}}$ if $\mathfrak{p}$ is unramified in L/K. For each irreducible $\mathbb{C}_p$-valued character χ we define a fractional ideal of $\mathfrak{o}_\chi$ by setting

$$U_\chi := \prod_{\mathfrak{p} \in S_{\mathrm{ram}}} \mathrm{nr}_{e_\chi E_\chi G}(e_\chi \mathfrak{M} U_{\mathfrak{p}})\mathfrak{o}_\chi,$$

where as before $\mathfrak{M}$ is a maximal $\mathfrak{o}_\chi$-order in $E_\chi G$ containing $\mathfrak{o}_\chi G$. For any finite set S containing S_∞, there is a Tate sequence (cf. [23])

$$E_S \rightarrowtail A_S \to B_S \twoheadrightarrow \nabla_S, \tag{17}$$

where A_S is G-c.t., B_S is $\mathbb{Z}G$-projective and ∇_S fits into an exact sequence

$$\mathrm{cl}_S \rightarrowtail \nabla_S \twoheadrightarrow \overline{\nabla}_S, \tag{18}$$

where $\overline{\nabla}_S$ is a $\mathbb{Z}G$-lattice. More precisely, $\overline{\nabla}_S$ fits into a short exact sequence

$$\overline{\nabla}_S \rightarrowtail \bigoplus_{\substack{\mathfrak{p}\in S_{\mathrm{ram}}\\ \mathfrak{p}\notin S}} (\mathrm{ind}_{G_{\mathfrak{P}}}^{G}(W_{\mathfrak{P}}^0)) \twoheadrightarrow \mathbb{Z},$$

where $W_{\mathfrak{P}}^0$ can be described as the cokernel of the map (cf. [15], §5)

$$\begin{aligned} \mathbb{Z}\overline{G_{\mathfrak{P}}} &\longrightarrow \mathbb{Z}G_{\mathfrak{P}}/(N_{G_{\mathfrak{P}}}) \times \mathbb{Z}\overline{G_{\mathfrak{P}}} \\ 1 &\mapsto (N_{I_{\mathfrak{P}}}, 1-\phi_{\mathfrak{P}}^{-1}). \end{aligned}$$

Sequence (17) has a uniquely determined extension class $\tau_S \in \mathrm{Ext}_G^2(\nabla_S, E_S)$ which is Tate's canonical class (cf. [27]) if S is sufficiently large. We set $\nabla := \nabla_{S_\infty}$ and $\overline{\nabla} := \overline{\nabla}_{S_\infty}$.

Proof of Theorem 4.1. We seek to compute the Fitting invariant of $A_L^T(p)$ over the maximal order $(\Lambda_p)_-$. By [20], remark 7 this is equivalent to the computation of the Fitting ideals $\mathrm{Fitt}_{\mathfrak{o}_\chi}(A_L^T(p)_\chi)$ for all $\mathbb{C}_p$-valued irreducible odd characters χ. Thus we have to show that for any finite set T of places of K such that $Hyp(S,T)$ holds we have

$$\theta_{S,\chi}^T \in \mathrm{Fitt}_{\mathfrak{o}_\chi}(A_L^T(p)_\chi),$$

where $\theta_S^T = (\theta_{S,\chi}^T)_{\chi\in\mathrm{Irr}_p(G)} \in \zeta(\mathbb{C}_pG)$. Let us fix an odd irreducible character χ; for any finitely generated $\mathbb{Z}G$-module M and $i \in \mathbb{Z}$ we abbreviate the Tate-cohomology groups $H^i(G, M(p)[\chi])$ by $H^i(M)$. For any finite $\mathbb{Z}G$-module M, the homomorphism $t(M(p),\chi)$ induces an equality

$$\mathrm{Fitt}_{\mathfrak{o}_\chi}(M(p)_\chi)\mathrm{Fitt}_{\mathfrak{o}_\chi}(H^0(M)) = \mathrm{Fitt}_{\mathfrak{o}_\chi}(M(p)^\chi)\mathrm{Fitt}_{\mathfrak{o}_\chi}(H^{-1}(M)). \tag{19}$$

Consider the exact sequence of $\mathbb{Z}_pG_-$-modules (cf. sequence (9))

$$\mu_L(p) \rightarrowtail (\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}(p) \to A_L^T(p) \twoheadrightarrow A_L(p). \tag{20}$$

If we denote the kernel of the epimorphism $A_L^T \twoheadrightarrow A_L$ by D, we get two exact sequences of $\mathfrak{o}_\chi$-modules as follows:

$$\mu_L(p)^\chi \rightarrowtail (\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}(p)^\chi \to D(p)^\chi \twoheadrightarrow H^1(\mu_L), \tag{21}$$

$$J_{-2} \rightarrowtail D(p)_\chi \to A_L^T(p)_\chi \twoheadrightarrow A_L(p)_\chi, \tag{22}$$

where J_{-2} denotes the image of the map $H^{-2}(A_L^T) \to H^{-2}(A_L)$. It follows from the proof of the main result in [7] (cf. the end of §12 in loc.cit.) that

$$L_{S_\infty}(0,\check{\chi})U_\chi \mathrm{Fitt}_{\mathfrak{o}_\chi}(\mu_L(p)^\chi)\mathrm{Fitt}_{\mathfrak{o}_\chi}(H^1(\mu_L))^{-1}\mathrm{Fitt}_{\mathfrak{o}_\chi}(H^2(\mu_L)) \subset \mathrm{Fitt}_{\mathfrak{o}_\chi}(A_L(p)^\chi) \tag{23}$$

provided that the strong Stark conjecture at p holds for the character χ. From this one can actually derive annihilation results in spirit of the non-abelian Brumer conjecture (cf. loc.cit. Th. 1.2), but this inclusion is not sufficient for our purposes such that we have to take care of the difference of the above $\mathfrak{o}_\chi$-ideals. The only inclusion of the proof of (23) derives from the two short exact sequences of finite $\mathbb{Z}[\frac{1}{2}]G_-$-modules

$$A_L \rightarrowtail \frac{\nabla^-}{\delta(C)^-} \twoheadrightarrow \frac{\overline{\nabla}^-}{\delta(C)^-}, \quad \frac{\overline{\nabla}^-}{\delta(C)^-} \rightarrowtail \frac{x^{-1}\delta(C)^-}{\delta(C)^-} \twoheadrightarrow \frac{x^{-1}\delta(C)^-}{\overline{\nabla}^-}, \tag{24}$$

where C is a free $\mathbb{Z}G$-module of rank $|S_{\mathrm{ram}}|$ and the map $\delta : C \to \nabla$ is injective. By abuse of notation we also write δ for the induced map $C \to \overline{\nabla}$ and note that this map is still injective. Moreover, x is a natural number such that $x\nabla^- \subset \delta(C)^-$. Following the notation of loc.cit. we put

$$M_1 := \frac{\nabla^-}{\delta(C)^-}, \quad M_2 := \frac{x^{-1}\delta(C)^-}{\delta(C)^-}, \quad M_3 := \frac{x^{-1}\delta(C)^-}{\overline{\nabla}^-}$$

and in addition

$$\overline{M}_1 := \frac{\overline{\nabla}^-}{\delta(C)^-}.$$

Since M_2 is c.t. and $H^i(M_3) \simeq H^{i+1}(\overline{\nabla}^-)$, we obtain from (24) the following exact sequences of $\mathfrak{o}_\chi$-modules:

$$A_L(p)^\chi \rightarrowtail M_1(p)^\chi \to \overline{M}_1(p)^\chi \twoheadrightarrow J_1, \tag{25}$$

$$H^{-1}(\overline{\nabla}^-) \rightarrowtail \overline{M}_1(p)_\chi \to M_2(p)_\chi \twoheadrightarrow M_3(p)_\chi, \tag{26}$$

where J_1 denotes the kernel of the map $H^1(A_L) \to H^1(M_1)$. Now we observe that we have isomorphisms

$$H^i(M_1) \simeq H^i(\nabla^-) \simeq H^{i+2}(\mu_L) \simeq H^{i+1}(D),$$

where the second isomorphism derives from the Tate sequence (17) for the set S_∞ whereas the last isomorphism is induced by the exact sequence

$$\mathbb{Z}[\tfrac{1}{2}] \otimes \mu_L \rightarrowtail (\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-} \twoheadrightarrow D.$$

Now choose a finite set S' of totally decomposed primes which generate the ray class group cl_L^T. The two exact sequences (cf. sequence (8) and (9))

$$(E_{S'}^T)^- \rightarrowtail (\mathbb{Z}S')^- \twoheadrightarrow A_L^T, \quad (E_{S'}^T)^- \rightarrowtail E_{S'}^- \twoheadrightarrow (\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}$$

imply the first two isomorphisms of

$$H^{i+1}(A_L^T) \simeq H^{i+2}(E_{S'}^{T,-}) \simeq H^{i+2}(E_{S'}^-) \simeq H^i(\overline{\nabla}^-) \simeq H^i(\overline{M}_1).$$

The last isomorphism is clear and the third is induced by a Tate sequence for S', since $\nabla_{S'} \simeq \overline{\nabla} \oplus \mathbb{Z}S'$. The natural behavior of Tate sequences yields a commutative diagram

$$\begin{tikzcd}
E_L \arrow[r, hook] \arrow[d, hook, "\iota_{S'}"] & A_{S_\infty} \arrow[r] \arrow[d, hook] & B_{S_\infty} \arrow[r, two heads] \arrow[d, hook] & \nabla \arrow[d] \arrow[dr, two heads] & \\
E_{S'} \arrow[r, hook] & A_{S'} \arrow[r] & B_{S'} \arrow[r, two heads] & \nabla_{S'} & \overline{\nabla} \arrow[l, hook]
\end{tikzcd}$$

which implies that the squares

$$\begin{tikzcd}
H^i(M_1) \arrow[r, "\sim"] \arrow[d, "H^i(\pi_1)"] & H^{i+2}(\mu_L) \arrow[d, "H^{i+2}(\iota_{S'})"] \\
H^i(\overline{M}_1) \arrow[r, "\sim"] & H^{i+2}(E_{S'}^-)
\end{tikzcd}$$

commute for all $i \in \mathbb{Z}$, where π_1 denotes the surjection $M_1 \twoheadrightarrow \overline{M}_1$. Moreover, the commutative diagram

$$\begin{tikzcd}
 & E_{S'}^{T,-} \arrow[r, hook] \arrow[d, hook] & (\mathbb{Z}S')^- \arrow[r, two heads] \arrow[d, equal] & A_L^T \arrow[d] \\
\mathbb{Z}[\frac{1}{2}] \otimes \mu_L \arrow[r, hook, "\iota_{S'}^-"] \arrow[d, equal] & E_{S'}^- \arrow[r] \arrow[d] & (\mathbb{Z}S')^- \arrow[r, two heads] & A_L \\
\mathbb{Z}[\frac{1}{2}] \otimes \mu_L \arrow[r, hook] & (\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-} \arrow[r, two heads] & D \arrow[uur, hook] &
\end{tikzcd}$$

implies that indeed the diagram

$$\begin{tikzcd}
H^i(M_1) \arrow[r, "\sim"] \arrow[d, "H^i(\pi_1)"] & H^{i+2}(\mu_L) \arrow[r, "\sim"] \arrow[d, "H^{i+2}(\iota_{S'})"] & H^{i+1}(D) \arrow[d, "H^{i+1}(\iota_D)"] \\
H^i(\overline{M}_1) \arrow[r, "\sim"] & H^{i+2}(E_{S'}^-) \arrow[r, "\sim"] & H^{i+1}(A_L^T)
\end{tikzcd}$$

commutes for all $i \in \mathbb{Z}$, where ι_D denotes the inclusion $D \rightarrowtail A_L^T$. In particular, we have $J_1 \simeq \operatorname{cok}(H^0(\pi_1)) \simeq \operatorname{cok}(H^1(\iota_D))$ and thus there is an exact sequence

$$\begin{array}{ccccccccc} & J_{-2} & \rightarrowtail & H^{-1}(D) & \to & H^{-1}(A_L^T) & \to & H^{-1}(A_L) & \to & H^0(D) \\ \to & H^0(A_L^T) & \to & H^0(A_L) & \to & H^1(D) & \to & H^1(A_L^T) & \twoheadrightarrow & J_1. \end{array}$$

Taking this into account, we can use the sequences (21), (22), (25), (26) and the equality (19) to calculate the desired Fitting ideal and end up with

$$\begin{aligned} \mathrm{Fitt}_{\mathfrak{o}_\chi}(A_L(p)_\chi) = {} & \mathrm{Fitt}_{\mathfrak{o}_\chi}((\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}(p)^\chi)\mathrm{Fitt}_{\mathfrak{o}_\chi}(M_1(p)^\chi)\mathrm{Fitt}_{\mathfrak{o}_\chi}(M_2(p)_\chi)^{-1}\mathrm{Fitt}_{\mathfrak{o}_\chi}((M_3(p)_\chi)) \cdot \\ & \cdot \mathrm{Fitt}_{\mathfrak{o}_\chi}(\mu_L(p)^\chi)^{-1}\mathrm{Fitt}_{\mathfrak{o}_\chi}(H^1(\mu_L))\mathrm{Fitt}_{\mathfrak{o}_\chi}(H^2(\mu_L))^{-1} \end{aligned}$$

Now it follows from the proof of loc.cit., Prop. 9.1 and Th. 1.2 that the left hand side of the inclusion (23) equals $\mathrm{Fitt}_{\mathfrak{o}_\chi}(M_1(p)^\chi)\mathrm{Fitt}_{\mathfrak{o}_\chi}(M_2(p)_\chi)^{-1}\mathrm{Fitt}_{\mathfrak{o}_\chi}((M_3(p)_\chi))$. Hence we obtain

$$\begin{aligned} \mathrm{Fitt}_{\mathfrak{o}_\chi}(A_L(p)_\chi) &= L_{S_\infty}(0, \check{\chi}) \cdot \mathrm{Fitt}_{\mathfrak{o}_\chi}((\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}(p)^\chi)U_\chi \\ &= L_{S_\infty}(0, \check{\chi}) \cdot \delta_T(0, \chi)U_\chi \end{aligned}$$

which in particular contains $(\theta_S^T)_\chi$. □

5. The relation to the equivariant Tamagawa number conjecture

In [2] the author defines the following element of $K_0(\mathbb{Z}G, \mathbb{R})$:

$$T\Omega(L/K, 0) := \psi_G^*(\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1}) + \hat{\partial}_G(L_S^*(0)^\sharp)).$$

Here, ψ_G^* is a certain involution on $K_0(\mathbb{Z}G, \mathbb{R})$ which is not important for our purposes, since we will be only interested in the nullity of $T\Omega(L/K, 0)$. Furthermore, $\tau_S \in \mathrm{Ext}_G^2(E_S, X_S)$ is Tate's canonical class (cf. [27]). Finally, $\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1})$ is the refined Euler characteristic associated to the perfect 2-extension $A_S \to B_S$ whose extension class is τ_S, metrised by λ_S^{-1}. For more precise definitions we refer the reader to [2]. By loc.cit., Th. 2.4.1 the ETNC for the motive $h^0(L)$ with coefficients in $\mathbb{Z}G$ in this context asserts that the element $T\Omega(L/K, 0)$ is zero. Note that this statement is also equivalent to the Lifted Root Number Conjecture formulated by Gruenberg, Ritter and Weiss [18] by [2], Th. 2.3.3.

It is also proven in [2] that $T\Omega(L/K, 0)$ lies in $K_0(\mathbb{Z}G, \mathbb{Q})$ if and only if Stark's conjecture holds. In this case the ETNC decomposes into local conjectures at each prime p by means of the isomorphism

$$K_0(\mathbb{Z}G,\mathbb{Q}) \simeq \bigoplus_{p\nmid\infty} K_0(\mathbb{Z}_pG,\mathbb{Q}_p).$$

Since Stark's conjecture holds for odd characters, $T\Omega(L/K,0)$ has a well defined image $T\Omega(L/K,0)_p^-$ in $K_0(\mathbb{Z}_pG_-,\mathbb{Q}_p)$.

Theorem 5.1. *Let p be an odd prime and assume that $T\Omega(L/K,0)_p^- = 0$. If $\mu_L(p)$ is a c.t. G-module, then $BS(L/K,S,p)$ holds for all finite sets S of places of K containing $S_{\mathrm{ram}} \cup S_\infty$.*

Proof. If $T\Omega(L/K,0)_p^- = 0$ and $\mu_L(p)$ is c.t., Proposition 1.1 (iv) and [20], Th. 7.1 imply that

$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(\mu_L(p)) \cdot [L(0)^\sharp \prod_{\mathfrak{p}\in S_{\mathrm{ram}}} \mathrm{nr}(U_\mathfrak{p})]_{\mathrm{nr}(\mathbb{Z}_pG_-)} \subset \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L(p)^\vee)^\sharp. \tag{27}$$

As in the last section, we consider sequence (20), where the kernel $D(p)$ of the surjection $A_L^T(p) \twoheadrightarrow A_L(p)$ now is c.t. As the Pontryagin dual of $\mu_L(p)$ is again $\mu_L(p)$, we obtain the following dual sequences:

$$D(p)^\vee \rightarrowtail ((\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}(p))^\vee \twoheadrightarrow \mu_L(p),$$
$$A_L(p)^\vee \rightarrowtail A_L^T(p)^\vee \twoheadrightarrow D(p)^\vee.$$

Since the Fitting invariant of $((\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}(p))^\vee$ is generated by $\delta_T(0)^\sharp$, Proposition 1.1 implies that

$$[\delta_T(0)\cdot L(0)^\sharp \prod_{\mathfrak{p}\in S_{\mathrm{ram}}} \mathrm{nr}(U_\mathfrak{p})]_{\mathrm{nr}(\mathbb{Z}_pG_-)} \subset \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L^T(p)^\vee)^\sharp.$$

Since the left hand side contains θ_S^T if $Hyp(S,T)$ is satisfied, the group ring elements $x\cdot\theta_S^T$, $x \in \mathcal{H}_p(G)$ annihilate $A_L^T(p)$ by (4). Now we can proceed as in the proof of Proposition 3.8. $\square$

In particular, the inclusion (27) shows the following result (cf. [20], Cor. 7.2).

Corollary 5.2. *Let p be an odd prime and assume that $T\Omega(L/K,0)_p^- = 0$. If $\mu_L(p)$ is a c.t. G-module and S is a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$, then*

$$\mathcal{H}_p(G)\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(\mu_L(p))\theta_S \subset \mathrm{Ann}_{\mathbb{Z}_pG_-}(A_L(p)).$$

We also can derive the strong Brumer-Stark property from the ETNC if the ramification above p is (almost) tame:

Theorem 5.3. *Let p be an odd prime and let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. Assume that $T\Omega(L/K,0)_p^- = 0$. Then $StBS(L/K,S,p)$ holds, whenever all primes $\mathfrak{p}$ of K above p are at most tamely ramified in L/K or $j \in G_\mathfrak{P}$. In particular, $BS(L/K,S,p)$ and $B(L/K,S,p)$ are true in this case.*

Remark 6. Assume that all primes $\mathfrak{p}$ of K above p are at most tamely ramified in L/K or $j \in G_\mathfrak{P}$. In [21] the author was meanwhile able to deduce the vanishing of $T\Omega(L/K,0)_p^-$ under some further restrictions from the validity of the equivariant Iwasawa main conjecture which has been proven by Ritter and Weiss [24] provided that Iwasawa's μ-invariant vanishes. For further connections of the work of Ritter and Weiss to the ETNC we refer the reader to [5].

Proof of Theorem 5.3. Let S and T be finite sets of places of K such that $Hyp(S,T)$ is satisfied. We denote the set of places of K above p by S_p and put

$$T' := T \cup (S_{\mathrm{ram}} \setminus (S_{\mathrm{ram}} \cap S_p)).$$

If the set T consists of only one prime, then $A_L^{T'}(p)$ is G-c.t. by [19], Th. 1. But if $T \subset T_0$, the exact sequence

$$(\mathfrak{o}_L/\mathfrak{M}_{T_0\setminus T})^{\times,-}(p) \rightarrowtail A_L^{T_0'}(p) \twoheadrightarrow A_L^{T'}(p)$$

implies that we may add or remove primes without changing the cohomology of $A_L^{T'}(p)$ and this module is hence c.t. for all admissible sets T. Since the Fitting invariant of $(\mathfrak{o}_L/\mathfrak{M}_{T_0\setminus T})^{\times,-}(p)$ is generated by $\delta_{T_0\setminus T}(0)$, loc.cit., Th. 2 implies that

$$T\Omega(L/K,0)_p^- = 0 \iff \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^{T'}(p)) = [\langle \theta_{S_1}^{T'}\rangle]_{\mathrm{nr}(\mathbb{Z}_pG_-)}, \tag{28}$$

where S_1 denotes the set of all primes of K which are wildly ramified in L/K. Moreover, we have an exact sequence

$$(\mathfrak{o}_L/\mathfrak{M}_{T'\setminus T})^{\times,-}(p) \rightarrowtail A_L^{T'}(p) \twoheadrightarrow A_L^{T}(p). \tag{29}$$

Let $\mathfrak{p}$ be a finite prime of K and choose a prime $\mathfrak{P}$ in L above $\mathfrak{p}$. We denote the kernel of the augmentation map $\mathbb{Z}G_{\mathfrak{P}} \twoheadrightarrow \mathbb{Z}$ which sends each $g \in G_{\mathfrak{P}}$ to 1 by $\Delta G_{\mathfrak{P}}$. Take an exact sequence

$$L_{\mathfrak{P}}^{\times} \rightarrowtail V_{\mathfrak{P}} \twoheadrightarrow \Delta G_{\mathfrak{P}}$$

whose extension class in $\mathrm{Ext}^1_{G_{\mathfrak{P}}}(\Delta G_{\mathfrak{P}}, L_{\mathfrak{P}}^{\times}) \simeq H^2(G_{\mathfrak{P}}, L_{\mathfrak{P}}^{\times})$ is the local fundamental class of the extension $L_{\mathfrak{P}}/K_{\mathfrak{p}}$. By [29], Th. 4 the inertial lattice

$$W_{\mathfrak{P}} := \left\{(x,y) \in \Delta G_{\mathfrak{P}} \times \mathbb{Z}\overline{G_{\mathfrak{P}}} | \overline{x} = (\phi_{\mathfrak{P}} - 1)y\right\}$$

is the push-out of this sequence along the normalized valuation $v_{\mathfrak{P}} : L_{\mathfrak{P}}^{\times} \twoheadrightarrow \mathbb{Z}$. We have two exact sequences

$$E_{\mathfrak{P}} \rightarrowtail V_{\mathfrak{P}} \twoheadrightarrow W_{\mathfrak{P}}, \quad E_{\mathfrak{P}}^1 \rightarrowtail E_{\mathfrak{P}} \twoheadrightarrow (\mathfrak{o}_L/\mathfrak{P})^{\times},$$

where $E_{\mathfrak{P}}$ is the group of local units and $E_{\mathfrak{P}}^1$ denotes the local units which are congruent 1 modulo $\mathfrak{P}$. We define $T_{\mathfrak{P}}$ to be the push-out of the first sequence along the projection of the second such that we obtain an exact sequence

$$(\mathfrak{o}_L/\mathfrak{P})^{\times} \rightarrowtail T_{\mathfrak{P}} \twoheadrightarrow W_{\mathfrak{P}}. \tag{30}$$

The following result is [19], Lemma 3 (i).

Lemma 5.4. *The G-module $(\mathrm{ind}_{G_{\mathfrak{P}}}^G T_{\mathfrak{P}})(p)$ is c.t. for each finite prime $\mathfrak{p} \nmid p$ of K and for each finite prime $\mathfrak{p}$ of K which is at most tamely ramified in L/K.*

We write $e_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ for the ramification index and the degree of the residue field extension at $\mathfrak{p}$, respectively. We observe that there is an isomorphism $\mathbb{Q}_p \otimes W_{\mathfrak{P}} \simeq \mathbb{Q}_pG_{\mathfrak{P}}$ and we specify a generator $c'_{\mathfrak{P}} \in W_{\mathfrak{P}}(p)$ as follows:

$$c'_{\mathfrak{P}} := (|G_{\mathfrak{P}}| - N_{G_{\mathfrak{P}}}, N_{\overline{G_{\mathfrak{P}}}} + e_{\mathfrak{p}}(\phi_{\mathfrak{P}} - 1)^{-1}(f_{\mathfrak{p}} - N_{\overline{G_{\mathfrak{P}}}})),$$

where we write $(\phi_{\mathfrak{P}} - 1)^{-1} = f_{\mathfrak{p}}^{-1}\sum_{i=0}^{f_{\mathfrak{p}}-1} i\phi_{\mathfrak{P}}^i$ in an intuitive notation. Note that

$$(\phi_{\mathfrak{P}} - 1)^{-1}(f_{\mathfrak{p}} - N_{\overline{G_{\mathfrak{P}}}}) = \sum_{i=0}^{f_{\mathfrak{p}}-1} i\phi_{\mathfrak{P}}^i - \frac{f_{\mathfrak{p}} - 1}{2}N_{\overline{G_{\mathfrak{P}}}}$$

lies in $\mathbb{Z}_p\overline{G_{\mathfrak{P}}}$ as $p \neq 2$. We pick a preimage $t'_{\mathfrak{P}} \in T_{\mathfrak{P}}(p)$ of $c'_{\mathfrak{P}}$. The maps $\mathbb{Z}_pG_{\mathfrak{P}} \to W_{\mathfrak{P}}(p),\ 1 \mapsto c'_{\mathfrak{P}}$ and $\mathbb{Z}_pG_{\mathfrak{P}} \to T_{\mathfrak{P}}(p),\ 1 \mapsto t'_{\mathfrak{P}}$ are injective and become isomorphisms after tensoring with $\mathbb{Q}_p$. Hence, the direct sum

$$\mathcal{T} := \bigoplus_{\mathfrak{p} \in T'\setminus T} \mathrm{ind}_{G_{\mathfrak{P}}}^G (T_{\mathfrak{P}}(p)/t'_{\mathfrak{P}})$$

is finite and c.t. by Lemma 5.4. Let $\mathcal{W}$ be the direct sum of the modules $\mathrm{ind}_{G_{\mathfrak{P}}}^{G}(W_{\mathfrak{P}}(p)/c'_{\mathfrak{P}})$, $\mathfrak{p} \in T' \backslash T$. Then the exact sequences (30) for these primes induce an exact sequence of G-modules

$$(\mathfrak{o}_L/\mathfrak{M}_{T'\backslash T})^{\times,-}(p) \rightarrowtail \mathcal{T}^- \twoheadrightarrow \mathcal{W}^-.$$

Now take any finite c.t. $\mathbb{Z}_pG_-$-module $\mathcal{P}$ which maps onto $(\mathfrak{o}_L/\mathfrak{M}_{T'\backslash T})^{\times,-}(p)$ (for example, choose $\mathcal{P}$ to be the direct sum of the modules $(\mathrm{ind}_{G_{\mathfrak{P}}}^{G}\mathbb{Z}_pG_{\mathfrak{P}}/(N(\mathfrak{P})-1))^-$, $\mathfrak{p} \in T' \setminus T$) and denote the kernel by $\mathcal{K}$. Then we obtain two exact sequences

$$\mathcal{K} \rightarrowtail \mathcal{P} \to \mathcal{T}^- \twoheadrightarrow \mathcal{W}^-, \quad \mathcal{K} \rightarrowtail \mathcal{P} \to A_L^{T'}(p) \twoheadrightarrow A_L^T(p),$$

where the second sequence derives from (29). Hence Proposition 1.1 (v) implies that

$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L^T(p)) = \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^{T'}(p))\mathrm{Fitt}_{\mathbb{Z}_pG_-}(\mathcal{T}^-)^{-1}\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(\mathcal{W}^-). \tag{31}$$

The first Fitting invariant on the right hand side is given by the ETNC and we have to compute the other two. In fact [19], Prop. 6 (4) gives

$$\mathrm{Fitt}_{\mathbb{Z}_pG_-}(\mathcal{T}^-) = [\prod_{\mathfrak{p}\in T'\backslash T} \mathrm{nr}(\tau_{\mathfrak{p}})]_{\mathrm{nr}(\mathbb{Z}_pG_-)}, \text{ where} \tag{32}$$

$$\tau_{\mathfrak{p}} := e_{\mathfrak{p}}^{-1}(1-N(\mathfrak{p}))N_{G_{\mathfrak{P}}} + \left(\frac{N(\mathfrak{p})-\phi_{\mathfrak{P}}}{1-\phi_{\mathfrak{P}}}\varepsilon'_{\mathfrak{p}} + \varepsilon''_{\mathfrak{p}}\right)(|G_{\mathfrak{P}}| - N_{G_{\mathfrak{P}}}).$$

For the last Fitting invariant we prove the following lemma which is the non-commutative analogue of [19], Lemma 8:

Lemma 5.5. *Let $\mathfrak{p} \notin S_p$ be a finite prime of K. Then*

$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_{\mathfrak{P}}}(W_{\mathfrak{P}}(p)/c'_{\mathfrak{P}}) \supset [\mathrm{nr}\,(X_{\mathfrak{p}})]_{\mathrm{nr}(\mathbb{Z}_pG_-)},$$

$$X_{\mathfrak{p}} := \langle N_{G_{\mathfrak{P}}} - |G_{\mathfrak{P}}|, N_{G_{\mathfrak{P}}} + e_{\mathfrak{p}}(f_{\mathfrak{p}}N_{I_{\mathfrak{P}}} - N_{G_{\mathfrak{P}}})(\phi_{\mathfrak{P}}-1)^{-1}\rangle_{\mathbb{Z}_pG_{\mathfrak{P}}}.$$

Before we prove the lemma, we observe that this lemma, (31), (32) and (28) imply the following result:

Corollary 5.6. *Let p be an odd prime and assume that $T\Omega(L/K,0)_p^- = 0$. Moreover, assume that all primes $\mathfrak{p}$ of K above p are at most tamely ramified in L/K or $j \in G_{\mathfrak{P}}$. Then for any finite set T of primes of K such that $Hyp(S_{\mathrm{ram}}, T)$ is satisfied, we have*

$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L^T(p)) \supset [\delta_T(0)L(0)^{\sharp} \prod_{\mathfrak{p}\in S_{\mathrm{ram}}} \mathrm{nr}(U_{\mathfrak{p}})]_{\mathrm{nr}(\mathbb{Z}_pG_-)}.$$

In particular, $\theta^T_{S_{\mathrm{ram}}}$ and hence θ^T_S is contained in $\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG_-}(A_L^T(p))$. This finishes the proof of the theorem. □

We are left with

Proof of Lemma 5.5. Let l be the rational prime below $\mathfrak{p}$ and let $R_{\mathfrak{P}}$ denote the l-Sylow subgroup of $I_{\mathfrak{P}}$. Since $R_{\mathfrak{P}}$ is normal in $G_{\mathfrak{P}}$ and $l \neq p$, the central idempotents

$$r'_{\mathfrak{p}} := |R_{\mathfrak{P}}|^{-1}N_{R_{\mathfrak{P}}}, \quad r''_{\mathfrak{p}} := 1 - r'_{\mathfrak{p}}$$

belong to the group ring $\mathbb{Z}_pG_{\mathfrak{P}}$ and there is an isomorphism $r''_{\mathfrak{p}}(W_{\mathfrak{P}}(p)) \simeq r''_{\mathfrak{p}}\mathbb{Z}_pG_{\mathfrak{P}}$ which maps $r''_{\mathfrak{p}}c'_{\mathfrak{P}}$ to $r''_{\mathfrak{p}}(|G_{\mathfrak{P}}| - N_{G_{\mathfrak{P}}})$. Hence we may assume that $\mathfrak{p}$ is tamely ramified in L/K.
Let us drop the subscripts $\mathfrak{p}$ from the notation and simply write e for $e_{\mathfrak{p}}$, and f for $f_{\mathfrak{p}}$. We keep the notation of [10], Lemma 6.2. So choose a generator a of $I_{\mathfrak{P}}$ and let $b^{-1} \in G_{\mathfrak{P}}$ be a lift of the

Frobenius automorphism which is of maximal order $|b|$ among all such elements. Then $b^{-f} = a^c$ for a divisor c of e and $b^{-1}ab = a^q$, where $q = N(\mathfrak{p})$. Define a map

$$\pi : \mathbb{Z}_p G_{\mathfrak{P}} e_1 \oplus \mathbb{Z}_p G_{\mathfrak{P}} e_2 \to W_{\mathfrak{P}}(p)$$

by $\pi(e_1) = (b^{-1}-1, 1)$ and $\pi(e_2) = (a-1, 0)$. Now let $(x,y) \in W_{\mathfrak{P}}(p)$ be arbitrary. Since $\Delta G_{\mathfrak{P}}$ is generated by $a-1$ and $b-1$, there is an $y' \in \mathbb{Z}_p\overline{G_{\mathfrak{P}}}$ such that $(x,y') \in \operatorname{im}(\pi)$. Hence $\phi_{\mathfrak{P}} - 1$ annihilates $y - y'$ and there is $z \in \mathbb{Z}_p$ such that $y - y' = zN_{\overline{G_{\mathfrak{P}}}}$. But

$$\begin{aligned}\pi(\sum_{i=0}^{f-1} b^{-i} e_1 - \sum_{i=0}^{c-1} a^i e_2) &= (b^{-f} - 1 - (a^c - 1), N_{\overline{G_{\mathfrak{P}}}}) \\ &= (0, N_{\overline{G_{\mathfrak{P}}}})\end{aligned}$$

such that π is an epimorphism. We claim that the kernel is generated by $N_{I_{\mathfrak{P}}} e_2$ and $(a^q - 1)e_1 + (1 - b^{-1})e_2$. For this, assume that

$$\pi(x_1 e_1 + x_2 e_2) = (x_1(b^{-1} - 1) + x_2(a-1), \overline{x}_1) = 0.$$

Since a^q is also a generator of $I_{\mathfrak{P}}$, we have $x_1 = x_1'(a^q - 1)$ for an appropriate $x_1' \in \mathbb{Z}_p G_{\mathfrak{P}}$ by [10], Lemma 6.6. By the same Lemma we get $x_1'(b^{-1} - 1) + x_2 = y \cdot N_{I_{\mathfrak{P}}}$ with $y \in \mathbb{Z}_p G_{\mathfrak{P}}$, since the left-hand side is annihilated by $(a-1)$. Hence

$$\begin{aligned}x_1 e_1 + x_2 e_2 &= x_1'(a^q - 1)e_1 + x_1'(1 - b^{-1})e_2 + (x_1'(b^{-1} - 1) + x_2)e_2 \\ &= x_1'((a^q - 1)e_1 + (1 - b^{-1})e_2) + yN_{I_{\mathfrak{P}}} e_2\end{aligned}$$

which proves the claim. Define two group ring elements

$$\delta_1 := \sum_{i=0}^{f-1} b^{-i} + (fN_{I_{\mathfrak{P}}} - N_{G_{\mathfrak{P}}})(b^{-1} - 1)^{-1} \in \mathbb{Z}_p G_{\mathfrak{P}},$$

$$\delta_2 := \sum_{i=0}^{c-1} a^i + f \cdot \sum_{i=1}^{e-1} \sum_{j=0}^{i-1} a^j \in \mathbb{Z}_p G_{\mathfrak{P}}.$$

Now we compute

$$\begin{aligned}\delta_1(b^{-1} - 1) - \delta_2(a-1) &= b^{-f} - 1 + fN_{I_{\mathfrak{P}}} - N_{G_{\mathfrak{P}}} - (a^c - 1 + f(N_{I_{\mathfrak{P}}} - e)) \\ &= |G_{\mathfrak{P}}| - N_{G_{\mathfrak{P}}}\end{aligned}$$

and thus $\pi(\delta_1 e_1 - \delta_2 e_2) = c_{\mathfrak{P}}'$. Hence, the kernel of the epimorphism

$$\mathbb{Z}_p G_{\mathfrak{P}} e_1 \oplus \mathbb{Z}_p G_{\mathfrak{P}} e_2 \twoheadrightarrow W_{\mathfrak{P}}(p)/c_{\mathfrak{P}}'$$

induced by π is generated by the kernel of π and $\delta_1 e_1 - \delta_2 e_2$. The reduced norms of the following three matrices generate a Fitting invariant of $W_{\mathfrak{P}}(p)/c_{\mathfrak{P}}'$:

$$A := \begin{pmatrix} 0 & a^q - 1 \\ N_{I_{\mathfrak{P}}} & 1 - b^{-1} \end{pmatrix}, \quad B := \begin{pmatrix} 0 & \delta_1 \\ N_{I_{\mathfrak{P}}} & -\delta_2 \end{pmatrix}, \quad C := \begin{pmatrix} a^q - 1 & \delta_1 \\ 1 - b^{-1} & -\delta_2 \end{pmatrix}.$$

Since $N_{I_{\mathfrak{P}}}(a^q - 1) = 0$, we have $\operatorname{nr}(A) = 0$. For the matrix B we have $\operatorname{nr}(B) = \operatorname{nr}(-N_{I_{\mathfrak{P}}}\delta_1)$ and

$$N_{I_{\mathfrak{P}}}\delta_1 = N_{G_{\mathfrak{P}}} + e(fN_{I_{\mathfrak{P}}} - N_{G_{\mathfrak{P}}})(\phi_{\mathfrak{P}} - 1)^{-1}.$$

The reduced norm is defined component wise and we compute $\operatorname{nr}(C)$ in two steps. Recall that $\varepsilon_{\mathfrak{p}}' = e^{-1}N_{I_{\mathfrak{P}}}$ and $\varepsilon_{\mathfrak{p}}'' = 1 - \varepsilon_{\mathfrak{p}}'$. Since $\varepsilon_{\mathfrak{p}}'(a^q - 1) = 0$, we have on the one hand

$$\begin{aligned}\operatorname{nr}(C\varepsilon_{\mathfrak{p}}') &= \operatorname{nr}((b^{-1} - 1)\delta_1 \varepsilon_{\mathfrak{p}}') \\ &= \operatorname{nr}((b^{-f} - 1)\varepsilon_{\mathfrak{p}}' + (fe - N_{G_{\mathfrak{P}}})\varepsilon_{\mathfrak{p}}') \\ &= \operatorname{nr}((|G_{\mathfrak{P}}| - N_{G_{\mathfrak{P}}})\varepsilon_{\mathfrak{p}}').\end{aligned}$$

On the other hand, $(a^q-1)\varepsilon''_{\mathfrak{p}}$ and likewise $(a-1)\varepsilon''_{\mathfrak{p}}$ are invertible and we compute

$$\begin{aligned}
\mathrm{nr}(C\varepsilon''_{\mathfrak{p}}) &= \mathrm{nr}\left(\begin{pmatrix} a^q-1 & \delta_1 \\ 0 & \delta_1(a^q-1)^{-1}(b^{-1}-1)-\delta_2 \end{pmatrix}\varepsilon''_{\mathfrak{p}}\right) \\
&= \mathrm{nr}((a^q-1)(\delta_1(b^{-1}-1)(a-1)^{-1}-\delta_2)\varepsilon''_{\mathfrak{p}}) \\
&= \mathrm{nr}((a^q-1)((\delta_2(a-1)+|G_{\mathfrak{P}}|-N_{G_{\mathfrak{P}}})(a-1)^{-1}-\delta_2)\varepsilon''_{\mathfrak{p}}) \\
&= \mathrm{nr}((a^q-1)(|G_{\mathfrak{P}}|-N_{G_{\mathfrak{P}}})(a-1)^{-1}\varepsilon''_{\mathfrak{p}}) \\
&= \mathrm{nr}((|G_{\mathfrak{P}}|-N_{G_{\mathfrak{P}}})\varepsilon''_{\mathfrak{p}}),
\end{aligned}$$

where the last equation holds, since $b^{-1}ab=a^q$ and the reduced norm is invariant under conjugation. We have shown that $\mathrm{nr}(C)=\mathrm{nr}(|G_{\mathfrak{P}}|-N_{G_{\mathfrak{P}}})$. Now let $x_1,x_2\in\mathbb{Z}_pG_{\mathfrak{P}}$ be arbitrary. Then also

$$\mathrm{nr}\begin{pmatrix} x_2(a^q-1) & \delta_1 \\ x_1N_{I_{\mathfrak{P}}}+x_2(1-b^{-1}) & -\delta_2 \end{pmatrix} = \mathrm{nr}(-x_1(N_{G_{\mathfrak{P}}}+e(fN_{I_{\mathfrak{P}}}-N_{G_{\mathfrak{P}}})(\phi_{\mathfrak{P}}-1)^{-1})+x_2(|G_{\mathfrak{P}}|-N_{G_{\mathfrak{P}}}))$$

belongs to $\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}(W_{\mathfrak{P}}(p)/c'_{\mathfrak{P}})$. $\square$

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