

INTEGRALITY OF STICKELBERGER ELEMENTS AND ANNIHILATION OF NATURAL GALOIS MODULES

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ABSTRACT. To each Galois extension L/K of number fields with Galois group G and each integer $r \leq 0$ one can associate Stickelberger elements in the centre of the rational group ring $\mathbb{Q}[G]$ in terms of values of Artin L -series at r . We show that the denominators of their coefficients are bounded by the cardinality of the commutator subgroup G' of G whenever G is nilpotent. Moreover, we show that, after multiplication by $|G'|$ and away from 2-primary parts, they annihilate the class group of L if $r = 0$ and higher Quillen K -groups of the ring of integers in L if $r < 0$. This generalizes recent progress on conjectures of Brumer and of Coates and Sinnott from abelian to nilpotent extensions.

For arbitrary G we show that the denominators remain bounded along the cyclotomic \mathbb{Z}_p -tower of L for every odd prime p . This allows us to give an affirmative answer to a question of Greenberg and of Gross on the behaviour of p -adic Artin L -series at zero.

INTRODUCTION

Let K be a number field and let L be a finite Galois extension of K with Galois group $G = \text{Gal}(L/K)$. Then G acts naturally upon the class group cl_L of L so that we may view the class group as a module over the integral group ring $\mathbb{Z}[G]$. Its structure is very hard to determine in general and even constructing unconditional non-trivial annihilators is very challenging.

The class number obviously annihilates the class group and is related to the leading term of the Laurent series expansion at $s = 1$ (or equivalently at $s = 0$ via the functional equation) of the Dedekind zeta function attached to L by means of the analytic class number formula. It is therefore natural to consider (special) values of Artin L -series $L(s, \chi)$ at $s = 0$ in order to construct interesting annihilators, where χ runs over the irreducible complex-valued characters of G .

More precisely, let S and T be two finite disjoint sets of places of K . We always assume that S contains all archimedean places and all places that ramify in L . The set T is chosen such that no non-trivial root of unity in L is congruent to 1 modulo every place of L lying above one in T . The S -truncated equivariant Artin L -function $\Theta_S(s)$ is a meromorphic function on the whole complex plane with values in $\zeta(\mathbb{C}[G])$, where $\zeta(\Lambda)$ denotes the centre of a ring Λ . There is a natural isomorphism $\zeta(\mathbb{C}[G]) \simeq \prod_{\chi} \mathbb{C}$, where the product runs over all complex valued irreducible characters of G . Then $\Theta_S(s)$ corresponds to the tuple $(L_S(s, \check{\chi}))_{\chi}$, where $\check{\chi}$ denotes the character contragredient to χ , and $L_S(s, \chi)$ denotes the S -truncated Artin L -series associated to χ . The latter is obtained from $L(s, \chi)$ by removing the Euler factors corresponding to the primes in S . The T -modified S -truncated equivariant Artin L -function $\Theta_{S,T}(s)$ is then obtained from $\Theta_S(s)$ by inserting certain modified Euler factors for the primes in T . Let $r \leq 0$ be an

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integer. Then the equivariant L -values $\theta_S^T(r) := \Theta_{S,T}(r)$ are called Stickelberger elements and it follows from a result of Siegel [Sie70] that they indeed have rational coefficients.

If G is abelian, then it was independently shown by Pi. Cassou-Noguès [CN79], Deligne and Ribet [DR80], and by Barsky [Bar78] that one has

$$(0.1) \quad \theta_S^T(r) \in \mathbb{Z}[G]$$

and Brumer's conjecture as discussed by Tate [Tat84] simply asserts that $\theta_S^T(0)$ annihilates the class group cl_L . In the case $K = \mathbb{Q}$ this is equivalent to Stickelberger's theorem from the late 19th century [Sti90].

In the last decades, Brumer's conjecture has been attacked via Iwasawa theory, starting with the seminal work of Wiles [Wil90b]. The method has been refined over the years by several authors [Gre00, GP15, Nic11c, JN19], but in almost all cases the results were dependent on the vanishing of the μ -invariant of a certain Iwasawa module. The latter is conjectured to be true, but this question is still wide open and so this only has led to conditional results. A different approach, which assumes the validity of the relevant special case of the equivariant Tamagawa number conjecture, is due to Greither [Gre07].

Rather recently, ground-breaking work of Dasgupta and Kakde [DK23] has overcome this problem. They have shown that Brumer's conjecture holds unconditionally away from its 2-primary part. They indeed proved refinements thereof, one of which is known as the Brumer–Stark conjecture. Roughly speaking, the latter replaces the class group by a ray class group cl_L^T whose definition also involves the set T . Their method is inspired by Wiles' proof of the main conjecture for totally real fields [Wil90a] and is a very clever variant of 'Ribet's method' using group ring valued Hilbert modular forms. The interest in the Brumer–Stark conjecture comes from its relation to Hilbert's 12th problem, and in [DK24] Dasgupta and Kakde indeed provide an effective method to construct the maximal abelian extension of a totally real field.

Finally, we also want to mention some unconditional and computational results on the 2-primary part of the conjecture for extensions of exponent 2 and 4 due to Sands [San84] and due to Roblot and Tangedal [RT00], respectively. Extensions of degree $2p$ for odd primes p have been considered in [GRT04].

Coates and Sinnott [CS74] have formulated an analogue of Brumer's conjecture when $r < 0$. It asserts that $\theta_S^T(r)$ annihilates the higher Quillen K -group $K_{-2r}(\mathcal{O}_{L,S})$ of the ring of $S(L)$ -integers in L . Here $S(L)$ denotes the set of places of L lying above a place in S . Based on the work of Dasgupta and Kakde, Johnston and the second named author [JN21] have shown that the conjecture of Coates and Sinnott holds away from 2-primary parts. Assuming the vanishing of the relevant μ -invariant, this was already known by work of Burns and Greither [BG03] (see also [GP15] for a different approach). The 2-primary part has been considered by Kolster and Taleb [KT20].

In this article we go a step further and no longer assume that the Galois group G is abelian. Generalizations of Brumer's conjecture have been formulated by the second named author [Nic11b] and, independently and in even greater generality, by Burns [Bur11]. A further, slightly different approach has been developed by Dejou and Roblot [DR14]. Here we mainly follow the treatment in the survey article [Nic19].

A main obstacle is the fact that Stickelberger elements do no longer have integral coefficients. So our first task is to bound their denominators. The theory of noncommutative Fitting invariants [Nic10, JN13] has led to the definition of the so-called integrality ring

$\mathcal{I}(G)$ and the denominator ideal $\mathcal{H}(G)$ of the integral group ring $\mathbb{Z}[G]$. The former is defined to be the smallest subring of $\zeta(\mathbb{Q}[G])$ that contains $\zeta(\mathbb{Z}[G])$ and the reduced norms $\text{nr}(H)$ of all square matrices H with entries in $\mathbb{Z}[G]$. The definition of the denominator ideal is even less explicit. It is an ideal in $\mathcal{I}(G)$ such that

$$\mathcal{H}(G)\mathcal{I}(G) \subseteq \zeta(\mathbb{Z}[G])$$

and it measures the failure of ‘generalized adjoint matrices’ to have coefficients in $\mathbb{Z}[G]$. The Stickelberger elements are now conjectured to lie in the integrality ring so that each integer in $\mathcal{H}(G)$ is conjecturally a bound for the denominators.

It is not hard to see that $|G| \in \mathcal{H}(G)$ and indeed $|G|\theta_S^T(r) \in \zeta(\mathbb{Z}[\frac{1}{2}][G])$ as follows from work of Burns and the second named author (we refer the reader to Theorem 5.3 for details). It is also known that $|G|\theta_S^T(0)$ annihilates the class group away from its 2-primary part as has been shown by Burns and Johnston [BJ11] under certain hypotheses and by the second named author [Nic24] in general (note, however, that the hypotheses in [BJ11] have been tailored to guarantee the validity of the strong Stark conjecture of Chinburg [Chi83] for totally odd characters to the state of knowledge at that time, and it is the latter that has been verified in [Nic24]). For a slightly weaker variant of these results for the non-abelian Brumer–Stark conjecture of Dejou and Roblot see [Rob18].

Already the abelian case makes it clear that this is not the best possible bound. For this reason, we first deal with the purely algebraic question of which integers lie in the denominator ideal. The following result gives a complete answer.

Theorem 1. Let G be a finite group with commutator subgroup G' . Then we have an equality

$$\mathcal{H}(G) \cap \mathbb{Z} = |G'|\mathbb{Z}.$$

The proof is by induction on the cardinality of G and makes heavy use of Clifford theory. The method also allows us to show that the integrality ring $\mathcal{I}(G)$ is strictly larger than $\zeta(\mathbb{Z}[G])$ unless G is abelian (in which case both rings are obviously equal). This observation is not needed in later sections, but we feel that it might be of independent interest.

Given Theorem 1, the integrality conjecture predicts that $|G'|\theta_S^T(r)$ has integral coefficients. We refine the method of proof of Theorem 1 using (0.1) and the results of Dasgupta and Kakde [DK23] and of Johnston and the second named author [JN21] as our base case. The main result of this article is the following.

Theorem 2. Let L/K be a Galois extension of number fields such that $G = \text{Gal}(L/K)$ is nilpotent. Let $r \leq 0$ be an integer. Then we have

$$|G'|\theta_S^T(r) \in \zeta(\mathbb{Z}[G])$$

and moreover the following holds.

- (i) If $r = 0$, then $|G'|\theta_S^T(0)$ annihilates $\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \text{cl}_L^T$.
- (ii) If $r < 0$, then $|G'|\theta_S^T(r)$ annihilates $\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} K_{-2r}(\mathcal{O}_{L,S})$.

We refer the reader to Theorem 4.11 for a more precise statement. We stress that replacing the factor $|G|$ by $|G'|$ produces a considerably stronger result. This is similar in spirit to the fact that the equivariant Tamagawa number conjecture (as formulated by Burns and Flach [BF01]) for the motive $h^0(\text{Spec}(L)(r))$ with coefficients in $\mathbb{Z}[G]$ is stronger than with coefficients in a maximal order containing $\mathbb{Z}[G]$.

We do not know yet how to remove the condition on G to be nilpotent. Roughly speaking, the reason is that Artin L -series behave well under induction and inflation of

characters, but not under Morita equivalence – in contrast to denominator ideals. That is why the condition appears in Theorem 2, but not in Theorem 1.

We finally give an application to Iwasawa theory. Let K be a totally real number field and let p be an odd prime. Let ψ be a totally even Artin character of the absolute Galois group of K . Recall that the S -truncated p -adic Artin L -series $L_{p,S}(s, \psi) : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ attached to ψ is the unique p -adic meromorphic function with the property that for each strictly negative integer r we have

$$(0.2) \quad L_{p,S}(r, \psi) = L_S(r, \psi\omega^{r-1}),$$

where ω denotes the Teichmüller character. If ψ is linear, then (0.2) is also valid when $r = 0$ and the construction of p -adic Artin L -series is due to Pi. Cassou-Noguès [CN79], Deligne and Ribet [DR80], and Barsky [Bar78]. Greenberg [Gre83] then generalized the construction to arbitrary ψ by means of Brauer induction. As the L -values at $r = 0$ may vanish, we cannot deduce that (0.2) still holds at $r = 0$, as we would possibly divide by zero. Greenberg [Gre83, §4] writes that ‘it would be interesting to prove that $L_p(s, \psi)$ is in fact analytic at $s = 0$ and has the correct value’, which is also predicted by a conjecture of Gross [Gro81, Conjecture 2.12b] on leading terms of p -adic L -functions.

It is clear that (0.2) holds with $r = 0$ if the character is monomial, i.e. induced from a linear character. In general, it is known by work of Burns [Bur20] that the left hand side of (0.2) vanishes whenever the right hand side does. This is ultimately a consequence of the main conjecture proved by Wiles [Wil90a]. If Gross’s ‘order of vanishing conjecture’ [Gro81, Conjecture 2.12a] holds for all irreducible characters, then (0.2) with $r = 0$ follows from work of Dasgupta, Kakde and Ventullo [DKV18] as has been observed by Burns [Bur20, Theorem 2.6] (see also [Nic19, Corollary 16.4.18]).

Since every irreducible character of a nilpotent group is monomial by [CR81, Theorem 11.3], we do not want to restrict ourselves to nilpotent extensions. So we cannot apply Theorem 2 directly. However, we can still prove that the denominators of the Stickelberger elements remain bounded along the cyclotomic \mathbb{Z}_p -tower as predicted by the integrality conjecture and our Theorem 1 (we stress that the easy bound $|G| \in \mathcal{H}(G)$ would not be sufficient for this purpose). As a consequence of Proposition 5.5 (ii) we have the following result, which is Theorem 6.1 below. We write $\theta_S^T(L/K, r)$ if the Stickelberger element $\theta_S^T(r)$ is associated with the extension L/K .

Theorem 3. Let p be an odd prime and let $r \leq 0$ be an integer. Let L/K be a Galois CM extension of number fields and denote the n -th layer of the cyclotomic \mathbb{Z}_p -extension of L by L_n . Then the denominators of the Stickelberger elements $\theta_S^T(L_n/K, r) \in \mathbb{Q}_p[\text{Gal}(L_n/K)]$ are bounded by a constant that does not depend on either n or r .

In this way, we obtain a new direct construction of p -adic Artin L -series with the additional property that the value at zero is as predicted by Greenberg and by Gross. This essentially follows by construction so that we obtain the following result, which is Corollary 6.4 below.

Theorem 4. For each totally even character ψ and each odd prime p one has an equality

$$L_{p,S}(0, \psi) = L_S(0, \psi\omega^{-1}).$$

As a further application, we give a new proof of the p -adic Artin conjecture [Gre83, p. 82], which does not rely on the validity of the main conjecture.

Notation and conventions. All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. We denote the set of all $m \times n$ matrices with entries in a ring R by $M_{m \times n}(R)$ and in the case $m = n$ the group of all invertible elements of $M_n(R) := M_{n \times n}(R)$ by $\mathrm{GL}_n(R)$. We let $\mathbf{1}_{n \times n}$ be the $n \times n$ identity matrix. We write $\zeta(R)$ for the centre of a ring R and $\mathrm{Ann}_R(M)$ for the annihilator ideal of an R -module M . We shall sometimes abuse notation by using the symbol \oplus to denote the direct product of rings or orders.

If q is a prime power, we denote the finite field with q elements by \mathbb{F}_q . If K is a field, we choose a separable closure K^c of K and set $G_K := \mathrm{Gal}(K^c/K)$.

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1. THE INTEGRALITY RING AND THE DENOMINATOR IDEAL

1.1. Reduced norms and the integrality ring. Let \mathcal{O} be a noetherian integrally closed domain with field of quotients F and let A be a finite dimensional separable F -algebra. If e_1, \dots, e_t are the central primitive idempotents of A then

$$A = A_1 \oplus \dots \oplus A_t$$

where $A_i := Ae_i = e_iA$. Each A_i is isomorphic to an algebra of $n_i \times n_i$ matrices over a skewfield D_i , and $F_i := \zeta(A_i) = \zeta(D_i)$ is a finite separable field extension of F ; in particular, each A_i is a central simple F_i -algebra. We denote the Schur index of D_i by s_i so that $[D_i : F_i] = s_i^2$. The reduced norm map

$$\mathrm{nr} = \mathrm{nr}_A : A \longrightarrow \zeta(A) = F_1 \oplus \dots \oplus F_t$$

is defined componentwise (see [Rei03, §9]) and extends to matrix rings over A in the obvious way.

Let Λ be an \mathcal{O} -order in A . Then Λ is noetherian and by [Rei03, Corollary 10.4] we may choose a maximal \mathcal{O} -order Λ' in A that contains Λ . Moreover, Λ' decomposes into a finite product

$$\Lambda' = \Lambda'_1 \oplus \dots \oplus \Lambda'_t$$

where $\Lambda'_i := \Lambda'e_i$. Denote the integral closure of \mathcal{O} in F_i by \mathcal{O}_i . By [Rei03, Theorem 10.5] each Λ'_i is a maximal \mathcal{O}_i -order in the F_i -algebra A_i and one has $\zeta(\Lambda'_i) = \mathcal{O}_i$. We point out that the reduced norm maps an element in Λ or, more generally, in $M_n(\Lambda)$ into $\zeta(\Lambda') = \mathcal{O}_1 \oplus \dots \oplus \mathcal{O}_t$ by [Rei03, Theorem 10.1], but not necessarily into $\zeta(\Lambda)$. Following [JN13, §3.4], we define a $\zeta(\Lambda)$ -submodule of $\zeta(A)$ by

$$\mathcal{I}(\Lambda) := \langle \mathrm{nr}(H) \mid H \in M_n(\Lambda), n \in \mathbb{N} \rangle_{\zeta(\Lambda)}.$$

As the reduced norm is multiplicative, this is in fact an \mathcal{O} -order in $\zeta(A)$ contained in $\zeta(\Lambda')$. We call $\mathcal{I}(\Lambda)$ the *integrality ring* of Λ . It is clear that $\mathcal{I}(\Lambda)$ always contains the

centre $\zeta(\Lambda)$, and that $\mathcal{I}(\Lambda) = \zeta(\Lambda)$ whenever Λ is commutative. In general, however, the inclusion $\zeta(\Lambda) \subseteq \mathcal{I}(\Lambda)$ might be strict. We consider some examples below.

We are mainly interested in the case $\mathcal{O} = \mathbb{Z}$ and $\Lambda = \mathbb{Z}[G]$, where G is a finite group. Then Λ is a \mathbb{Z} -order in the \mathbb{Q} -algebra $A = \mathbb{Q}[G]$ which is not maximal unless $G = 1$. We will also consider certain variations thereof. In particular, if p is a prime, we shall be interested in the case $\mathcal{O} = \mathbb{Z}_p$, $\Lambda = \mathbb{Z}_p[G]$ and $A = \mathbb{Q}_p[G]$. For brevity, we set $\mathcal{I}(G) := \mathcal{I}(\mathbb{Z}[G])$ and $\mathcal{I}_p(G) := \mathcal{I}(\mathbb{Z}_p[G])$.

For any commutative ring \mathcal{O} the group ring $\mathcal{O}[G]$ is free of rank $|G|$ as an \mathcal{O} -module. Its centre $\zeta(\mathcal{O}[G])$ is also a free \mathcal{O} -module of rank $c(G)$, where $c(G)$ denotes the number of conjugacy classes of G . More precisely, if \mathcal{C}_i , $1 \leq i \leq c(G)$, are the distinct conjugacy classes, then $C_i := \sum_{g \in \mathcal{C}_i} g$ constitute an \mathcal{O} -basis of $\zeta(\mathcal{O}[G])$. We denote the commutator subgroup of G by G' .

Example 1.1. Let $G = D_8 = \langle a, x \mid a^8 = x^2 = 1, xax = a^{-1} \rangle$ be the dihedral group of order 16. Then the commutator subgroup $D'_8 = \langle a^2 \rangle$ is cyclic of order 4. Furthermore, the conjugacy classes of D_8 are given by

$$\begin{aligned} \mathcal{C}_1 &= \{1\}, & \mathcal{C}_2 &= \{a^4\}, & \mathcal{C}_3 &= \{a^2, a^6\}, & \mathcal{C}_4 &= \{a, a^7\}, \\ \mathcal{C}_5 &= \{a^3, a^5\}, & \mathcal{C}_6 &= \{x, a^2x, a^4x, a^6x\}, & \mathcal{C}_7 &= \{ax, a^3x, a^5x, a^7x\}. \end{aligned}$$

An easy calculation gives that

$$\text{nr}(a) = \frac{1}{4}(3C_1 - C_2 - C_3 + C_4 + C_5) \in \mathcal{I}(D_8).$$

On the one hand, we see that $\zeta(\mathbb{Z}[D_8])$ is strictly contained in $\mathcal{I}(D_8)$. On the other hand, we have that $|D'_8| \text{nr}(a) = 4\text{nr}(a) \in \zeta(\mathbb{Z}[D_8])$.

Example 1.2. Let $G = \text{SL}_2(\mathbb{F}_3)$ be the special linear group of degree 2 over \mathbb{F}_3 . This group is generated by the matrices

$$\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and indeed has the presentation

$$\text{SL}_2(\mathbb{F}_3) = \langle \alpha, \beta, \gamma \mid \alpha^4 = \beta^4 = \gamma^3 = 1, \beta\gamma = \gamma\alpha, \gamma\beta = \alpha\beta\gamma \rangle.$$

There are seven conjugacy classes. We let $\mathcal{C}_1 = \{\mathbf{1}_{2 \times 2}\}$ and $\mathcal{C}_2 = \{-\mathbf{1}_{2 \times 2}\}$. Moreover, we denote the conjugacy classes containing γ and γ^2 by \mathcal{C}_3 and \mathcal{C}_4 , and those containing $\alpha^2\gamma$ and $\alpha^2\gamma^2$ by \mathcal{C}_5 and \mathcal{C}_6 , respectively. Finally, \mathcal{C}_7 denotes the class that contains α and hence $\beta = \gamma\alpha\gamma^{-1}$. The commutator subgroup equals $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_7$ and has cardinality 8. Then clearly $\text{nr}(\alpha) = \text{nr}(\beta) = 1$. However, a lengthy computation shows that

$$\text{nr}(\gamma) = \frac{1}{8}(3C_1 + 3C_2 + C_3 - 2C_4 + 2C_5 + C_6 - C_7) \in \mathcal{I}(\text{SL}_2(\mathbb{F}_3)).$$

We see that the integrality ring $\mathcal{I}(\text{SL}_2(\mathbb{F}_3))$ is again strictly larger than $\zeta(\mathbb{Z}[\text{SL}_2(\mathbb{F}_3)])$ and that $|\text{SL}_2(\mathbb{F}_3)'| \text{nr}(\gamma) = 8\text{nr}(\gamma) \in \zeta(\mathbb{Z}[\text{SL}_2(\mathbb{F}_3)])$.

Lemma 1.3. *Let G be a finite group and let N be a normal subgroup of G . Let \mathcal{O} be a noetherian integrally closed domain with field of quotients F whose characteristic does not divide the cardinality of G . Then the canonical projection $\pi : G \rightarrow \overline{G} := G/N$ induces a map $\pi : F[G] \rightarrow F[\overline{G}]$ such that the following hold.*

- (i) *We have inclusions $\pi(\zeta(\mathcal{O}[G])) \subseteq \zeta(\mathcal{O}[\overline{G}])$ and $\pi(\mathcal{I}(\mathcal{O}[G])) \subseteq \mathcal{I}(\mathcal{O}[\overline{G}])$.*

- (ii) If $\pi : \zeta(\mathcal{O}[G]) \rightarrow \zeta(\mathcal{O}[\overline{G}])$ is surjective, then $\pi : \mathcal{I}(\mathcal{O}[G]) \rightarrow \mathcal{I}(\mathcal{O}[\overline{G}])$ is also surjective.
- (iii) The map $\pi : \mathcal{I}(\mathcal{O}[G]) \rightarrow \mathcal{I}(\mathcal{O}[\overline{G}])$ is surjective whenever \overline{G} is abelian.

Proof. We first observe that $F[G]$ is a separable F -algebra by Maschke's theorem [CR81, Theorem 3.14]. If x is central in $\mathcal{O}[G]$, then clearly $\pi(x)$ is central in $\mathcal{O}[\overline{G}]$, which is the first inclusion in (i). Let us write nr_G for the reduced norm on (matrix rings over) $F[G]$ and similarly for \overline{G} . Then π induces a surjective map $\pi_n : M_n(\mathcal{O}[G]) \rightarrow M_n(\mathcal{O}[\overline{G}])$ for each $n \in \mathbb{N}$ and one has

$$\pi(\text{nr}_G(H)) = \text{nr}_{\overline{G}}(\pi_n(H))$$

for each $H \in M_n(\mathcal{O}[G])$. Since every $x \in \mathcal{I}(\mathcal{O}[G])$ can be written as a finite sum $x = \sum_j x_j \text{nr}_G(H_j)$ with x_j central in $\mathcal{O}[G]$ and square matrices H_j with entries in $\mathcal{O}[G]$, this proves (i). For (ii) we let $y \in \mathcal{I}(\mathcal{O}[\overline{G}])$ be arbitrary. Then likewise y may be written as a finite sum $y = \sum_j y_j \text{nr}_{\overline{G}}(Z_j)$ with y_j central in $\mathcal{O}[\overline{G}]$ and matrices $Z_j \in M_{n_j}(\mathcal{O}[\overline{G}])$ with $n_j \in \mathbb{N}$. Since π_n is surjective for every n , there are matrices $H_j \in M_{n_j}(\mathcal{O}[G])$ such that $\pi_{n_j}(H_j) = Z_j$ for all j . If $\pi : \zeta(\mathcal{O}[G]) \rightarrow \zeta(\mathcal{O}[\overline{G}])$ is surjective, then there likewise are $x_j \in \zeta(\mathcal{O}[G])$ such that $\pi(x_j) = y_j$ and hence $x := \sum_j x_j \text{nr}_G(H_j)$ is a preimage of y such that $x \in \mathcal{I}(\mathcal{O}[G])$. For (iii) we assume that \overline{G} is abelian. Then we have that $\mathcal{I}(\mathcal{O}[\overline{G}]) = \mathcal{O}[\overline{G}]$ and we can simply write $y = \text{nr}_{\overline{G}}(y) = \pi(\text{nr}_G(x))$ for any $x \in \mathcal{O}[G]$ with $\pi(x) = y$. \square

Example 1.4. Let $G = S_3$ be the symmetric group on three letters and let $N = A_3$ be the unique non-trivial normal subgroup of S_3 . Then S_3/A_3 is cyclic of order 2 so that the canonical map $\mathcal{I}(\mathcal{O}[S_3]) \rightarrow \mathcal{I}(\mathcal{O}[S_3/A_3]) = \mathcal{O}[S_3/A_3]$ is surjective by Lemma 1.3 (iii). However, we show that $\pi : \zeta(\mathcal{O}[S_3]) \rightarrow \zeta(\mathcal{O}[S_3/A_3]) = \mathcal{O}[S_3/A_3]$ is in general not surjective. There are three conjugacy classes \mathcal{C}_i of S_3 , where each \mathcal{C}_i consists of the i -cycles, $1 \leq i \leq 3$. Let us denote the non-trivial element of S_3/A_3 by τ . Then $\pi(\mathcal{C}_1) = 1$, $\pi(\mathcal{C}_2) = 3\tau$ and $\pi(\mathcal{C}_3) = 2$ so that the image of π equals $\mathcal{O} + 3\mathcal{O}\tau$. Hence π is surjective if and only if 3 is a unit in \mathcal{O} .

Remark 1.5. In [BS20, §3.1] Burns and Sano introduce a variant of the integrality ring when \mathcal{O} is a Dedekind domain. Assume for simplicity that \mathcal{O} is a discrete valuation ring. Then they let $\xi(\Lambda)$ be the \mathcal{O} -submodule of $\zeta(\Lambda')$ generated by the elements $\text{nr}(H)$, where H runs over all square matrices with entries in Λ , and call it the Whitehead order of Λ . Then $\xi(\Lambda)$ is clearly contained in $\mathcal{I}(\Lambda)$, and this inclusion can be strict in general. In the setting of Lemma 1.3 the Whitehead order indeed behaves better under projections. The proof of Lemma 1.3 (ii) shows that one always has $\pi(\xi(\mathcal{O}[G])) = \xi(\mathcal{O}[\overline{G}])$. See also [BS20, Lemma 3.2(v)].

1.2. Generalised adjoints. We keep the notation of §1.1. Let $H \in M_n(A)$ and decompose $H = (H_1, \dots, H_t)$, where $H_i = He_i \in M_n(A_i)$. Following [Rei03, §9], we let $f_{H_i}(X) \in F_i[X]$ be the reduced characteristic polynomial of H_i and set $f_H(X) := (f_{H_i}(X))_i \in \bigoplus_{i=1}^t F_i[X]$. The latter may itself be written as reduced norm of the separable $F(X)$ -algebra $A \otimes_F F(X)$. Indeed a straightforward exercise (see also [Wat18, Lemma 1.6.6]) shows that

$$(1.1) \quad f_H(X) = \text{nr}_{A \otimes_F F(X)}(1 \otimes X - H \otimes 1) \in \zeta(A)[X] = \bigoplus_{i=1}^t F_i[X].$$

Set $m_i := n_i \cdot s_i \cdot n$. Then we may write $f_{H_i}(X) = \sum_{j=0}^{m_i} \alpha_{ij} X^j$ with $\alpha_{ij} \in F_i$ for all $1 \leq j \leq m_i$. Note that the constant term α_{i0} equals $(-1)^{m_i} \cdot \text{nr}(H_i)$. We define

$$H_i^* := (-1)^{m_i+1} \sum_{j=1}^{m_i} \alpha_{ij} H_i^{j-1} \in M_n(A_i)$$

and call $H^* := (H_1^*, \dots, H_t^*) \in M_n(A)$ the *generalised adjoint* of H . Since $f_{H_i}(H_i)$ vanishes for each i , one has

$$HH^* = H^*H = \text{nr}(H) \cdot \mathbf{1}_{n \times n}.$$

Note that we follow the conventions in [JN13, §3.6] which slightly differ from those given in [Nic10, §4].

Now suppose that $H \in M_n(\Lambda)$. Then by another application of [Rei03, Theorem 10.1] the generalised adjoint H^* has entries in Λ' , but not necessarily in Λ . This motivates the following definition. We set

$$\mathcal{H}(\Lambda) := \{x \in \zeta(\Lambda) \mid xH^* \in M_n(\Lambda) \forall H \in M_n(\Lambda) \forall n \in \mathbb{N}\}$$

and call $\mathcal{H}(\Lambda)$ the *denominator ideal* of Λ . It is easy to see that $\mathcal{H}(\Lambda)$ is an ideal in $\zeta(\Lambda)$. With a little more work [Wat18, Lemma 1.10.9] one can actually show that it is an ideal in $\mathcal{I}(\Lambda)$ such that

$$(1.2) \quad \mathcal{H}(\Lambda) \cdot \mathcal{I}(\Lambda) \subseteq \zeta(\Lambda).$$

If G is a finite group, we set $\mathcal{H}(G) := \mathcal{H}(\mathbb{Z}[G])$ and $\mathcal{H}_p(G) := \mathcal{H}(\mathbb{Z}_p[G])$ for each prime p .

General results on denominator ideals are rather sparse. Watson [Wat18] has computed $\mathcal{H}(\mathcal{O}[G])$ whenever \mathcal{O} has characteristic 0 and G is a finite p -group such that $|G'| = p$. Moreover, we have the following result.

Proposition 1.6. *Let G be a finite group and let p be a prime. Then we have $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_p[G])$ if and only if p does not divide the order of the commutator subgroup G' of G . In this case, we also have that $\mathcal{I}_p(G) = \zeta(\mathbb{Z}_p[G])$.*

Proof. The first claim is [JN13, Proposition 4.4], but note that this can also be easily deduced from Theorem 1 (whose proof does not depend on Proposition 1.6). The second claim then follows immediately from (1.2). \square

We are mainly interested in the ideal $\mathbb{Z} \cap \mathcal{H}(G)$. The following important example, which is essentially taken from the proof of Proposition 1.6 given in [JN13], provides an upper bound for that ideal.

Example 1.7. Assume that \mathcal{O} has characteristic 0. We consider $H = 0 \in M_1(\mathcal{O}[G])$. Decompose $H = (H_1, \dots, H_t)$ as above. Then the reduced characteristic polynomial of H_i is given by $f_{H_i}(X) = X^{m_i}$, where $m_i = n_i \cdot s_i$, so that $H_i^* = h_i(0)$ with $h_i(X) = X^{m_i-1}$. In other words, we have $H_i^* = 1$ if $m_i = 1$, and $H_i^* = 0$ otherwise. This implies that

$$0^* = \frac{1}{|G'|} \text{Tr}_{G'},$$

where $\text{Tr}_{G'} := \sum_{g \in G'} g$. In particular, every integer in $\mathbb{Z} \cap \mathcal{H}(G)$ must be divisible by $|G'|$.

2. CLIFFORD THEORY

2.1. General notation and Clifford's Theorem. Let K be a field of characteristic 0 and let G be a finite group. We denote the set of K -irreducible characters of G by $\text{Irr}_K(G)$. If K is clear from context, we simply write $\text{Irr}(G)$ for $\text{Irr}_{K^c}(G)$.

We will always assume in this section that K is sufficiently large in the sense that it is a splitting field for G and all of its subgroups. In particular, we have $\text{Irr}_K(G) = \text{Irr}(G)$ and the Wedderburn decomposition of $K[G]$ is given by

$$K[G] = \bigoplus_{\chi \in \text{Irr}(G)} K[G]e_\chi \simeq \bigoplus_{\chi \in \text{Irr}(G)} M_{\chi(1)}(K),$$

where $e_\chi := \chi(1)|G|^{-1} \sum_{g \in G} \chi(g^{-1})g$ are the primitive central idempotents of $K[G]$. We write $R^+(G)$ for the set of characters associated to finite-dimensional K^c -valued representations of G , and $R(G)$ for the ring of virtual characters generated by $R^+(G)$. For each character $\chi \in R^+(G)$ we choose a $K[G]$ -module V_χ with character χ . Conversely, if V is a finitely generated $K[G]$ -module, we denote the associated character by χ_V . If U is a subgroup of G , we write $\text{res}_U^G \chi$ for the character in $R^+(U)$ obtained by restriction. Conversely, for $\psi \in R^+(U)$ we let $\text{ind}_U^G \psi \in R^+(G)$ be the induced character. The associated $K[G]$ -module is given by $K[G] \otimes_{K[U]} V_\psi$, from which one can easily deduce the well-known formula (see [CR81, (10.2)])

$$(2.1) \quad (\text{ind}_U^G \psi)(g) = \frac{1}{|U|} \sum_{h \in G: h^{-1}gh \in U} \psi(h^{-1}gh).$$

We will use (2.1) frequently. Finally, if N is a normal subgroup of G and $\chi \in R^+(G/N)$, we write $\text{infl}_{G/N}^G \chi \in R^+(G)$ for the inflated character. These notions extend to virtual characters by linearity. For $\chi, \psi \in R(G)$ we write $\langle \chi, \psi \rangle_G$ for the usual inner product of virtual characters. We recall the following well-known fact [CR81, Theorem 10.9].

Proposition 2.1 (Frobenius reciprocity). *Let U be a subgroup of G . Then for each $\chi \in R(G)$ and $\psi \in R(U)$ one has*

$$\langle \chi, \text{ind}_U^G \psi \rangle_G = \langle \text{res}_U^G \chi, \psi \rangle_U.$$

Now let N be a normal subgroup of G . Then G acts upon $\eta \in R(N)$ via ${}^g\eta(x) = \eta(g^{-1}xg)$, $g \in G$, $x \in N$. We denote its stabilizer $\{g \in G \mid {}^g\eta = \eta\}$ by G_η and note that N is always contained in G_η . The following result is due to Clifford [Cli37] (see also [CR81, Proposition 11.4]) and will be used extensively in the following.

Theorem 2.2 (Clifford). *Let N be a normal subgroup of a finite group G . Let $\chi \in \text{Irr}(G)$ and chose an irreducible constituent η of $\text{res}_N^G \chi$. Then there is a positive integer m such that*

$$\text{res}_N^G \chi = m \sum_{g \in G/G_\eta} {}^g\eta.$$

In the situation of Theorem 2.2 we write

$$(2.2) \quad \text{ind}_N^G \eta = \sum_{i=1}^s m_i \psi_i,$$

where s and m_1, \dots, m_s are positive integers and $\psi_i \in \text{Irr}(G_\eta)$ are distinct irreducible characters. By [Hup67, Kapitel V, Satz 17.11] the characters $\chi_i := \text{ind}_{G_\eta}^G \psi_i$ are irreducible

and each irreducible character of G whose restriction to N is divisible by η is a χ_i . So we may and do assume that $\chi = \chi_1$ and set $\psi := \psi_1$. It follows that

$$(2.3) \quad \text{ind}_N^G \eta = \sum_{i=1}^s m_i \chi_i$$

and we have $m_1 = m$ by Frobenius reciprocity. Likewise we have that

$$m_i = \langle \psi_i, \text{ind}_N^{G_\eta} \eta \rangle_{G_\eta} = \langle \text{res}_N^{G_\eta} \psi_i, \eta \rangle_N.$$

Since G_η acts trivially upon η , Clifford's Theorem 2.2 implies that we have

$$(2.4) \quad \text{res}_N^{G_\eta} \psi_i = m_i \eta$$

for $1 \leq i \leq s$. See also [CR81, Proposition 11.4].

2.2. Clifford theory for subgroups containing the commutator subgroup. We keep the notation of §2.1 and assume in addition that N contains the commutator subgroup G' of G . Then N is normal in G_η and the quotient G_η/N is abelian.

Proposition 2.3. *Let G be a finite group and let N be a (normal) subgroup containing the commutator subgroup G' of G . Let $\chi \in \text{Irr}(G)$ be an irreducible character and let η be an irreducible constituent of $\text{res}_N^G \chi$. Write $\text{ind}_N^{G_\eta} \eta = \sum_{i=1}^s m_i \psi_i$ as in (2.2) and recall that $\psi = \psi_1$ and $m = m_1$. Then the following holds.*

- (i) *Each ψ_i is of the form $\psi_i = \psi \otimes \omega_i$ for some $\omega_i \in \text{Irr}(G_\eta/N)$.*
- (ii) *Each $\psi \otimes \omega$, $\omega \in \text{Irr}(G_\eta/N)$ is a ψ_i .*
- (iii) *We have that $m_i = m$ for all $1 \leq i \leq s$ and $[G_\eta : N] = sm^2$.*

Proof. We first note that $\psi \otimes \omega$ is irreducible for every $\omega \in \text{Irr}(G_\eta/N)$, since the quotient G_η/N is abelian. We claim that we have an equality

$$m \text{ind}_N^{G_\eta} \eta = \sum_{\omega \in \text{Irr}(G_\eta/N)} \psi \otimes \omega.$$

Then (i) and (ii) follow from (2.2). For the claim we evaluate both sides of the equation at $g \in G_\eta$. Since N is normal in G_η , the left-hand side vanishes for every $g \notin N$; likewise the right-hand side vanishes, as $\sum_{\omega} \omega$ is the regular character of G_η/N . Now let $g \in N$. Then the formula (2.1) for induced characters implies that the left-hand side equals $m[G_\eta : N]\eta(g)$. Likewise we have that $(\sum_{\omega} \psi \otimes \omega)(g) = [G_\eta : N]\psi(g) = m[G_\eta : N]\eta(g)$, where the last equality is (2.4). This shows the claim. Again by (2.4) we see that for each $1 \leq i \leq s$ one has

$$m_i \eta(1) = \psi_i(1) = \psi(1) \omega_i(1) = \psi(1) = m \eta(1).$$

As $\eta(1)$ is non-zero, we must have $m_i = m$ for all i . Finally, we compute

$$[G_\eta : N]\eta(1) = \text{ind}_N^{G_\eta} \eta(1) = m \sum_{i=1}^s \psi_i(1) = sm^2 \eta(1).$$

This finishes the proof of (iii). □

Corollary 2.4. *Keep the notation of Proposition 2.3 and recall that $\chi_i = \text{ind}_{G_\eta}^G \psi_i$ is an irreducible character of G . Then the following holds.*

(i) The sum $\sum_{i=1}^s e_{\chi_i}$ is an element of $K[N]$ and indeed

$$\sum_{i=1}^s e_{\chi_i} = \sum_{x \in G/G_\eta} e_{x\eta}.$$

(ii) The sum $\sum_{i=1}^s e_{\psi_i}$ is an element of $K[N]$ and indeed

$$\sum_{i=1}^s e_{\psi_i} = e_\eta.$$

(iii) For each $1 \leq i \leq s$ the idempotent e_{χ_i} is an element of $K[G_\eta]$ and indeed

$$e_{\chi_i} = \sum_{x \in G/G_\eta} e_{x\psi_i}.$$

Proof. We first note that for each $1 \leq i \leq s$ one has

$$\chi_i(1) = [G : G_\eta] \psi_i(1) = [G : G_\eta] m \eta(1),$$

where we have used Proposition 2.3 (iii) for the last equality. Therefore we have that

$$\sum_{i=1}^s e_{\chi_i} = \frac{m\eta(1)}{|G_\eta|} \sum_{i=1}^s \sum_{g \in G} \chi_i(g^{-1})g = \frac{\eta(1)}{|G_\eta|} \sum_{g \in G} (\text{ind}_N^G \eta)(g^{-1})g,$$

where the second equality is (2.3). Since $(\text{ind}_N^G \eta)(g^{-1})$ vanishes whenever $g \in G \setminus N$, and equals $[G_\eta : N] \sum_{x \in G/G_\eta} x\eta(g^{-1})$ otherwise, claim (i) follows. (ii) and (iii) are special cases of (i). For (iii) note that by (2.4) we have an inclusion $G_{\psi_i} \subseteq G_\eta$ which is indeed an equality. To see this, we first observe that $\text{res}_{G_\eta}^G \chi_i = \sum_{x \in G/G_{\psi_i}} x\psi_i$ by Clifford's Theorem and Frobenius reciprocity. This yields

$$[G : G_{\psi_i}] \psi_i(1) = \chi_i(1) = \text{ind}_{G_\eta}^G \psi_i(1) = [G : G_\eta] \psi_i(1)$$

as desired. \square

The characters $\psi \otimes \omega$ for varying $\omega \in \text{Irr}(G_\eta/N)$ may not be distinct. To remedy this, we introduce the following notion. For each $1 \leq i \leq s$ we let U_{ψ_i} be the smallest subgroup of G_η that contains N and all $g \in G_\eta$ such that $\psi_i(g) \neq 0$. Since each ψ_i is of the form $\psi \otimes \omega_i$ for some linear character ω_i by Proposition 2.3 (i), this definition actually does not depend on i . So we simply write U_ψ in the following.

Lemma 2.5. *Let $\omega, \tilde{\omega} \in \text{Irr}(G_\eta/N)$. Then $\psi \otimes \omega = \psi \otimes \tilde{\omega}$ if and only if ω and $\tilde{\omega}$ coincide on U_ψ .*

Proof. The set of all $g \in G_\eta$ such that $\omega(g) = \tilde{\omega}(g)$ is equal to the kernel of $\omega^{-1}\tilde{\omega}$ and thus a normal subgroup of G_η . It is clear that $\psi(g)\omega(g) = \psi(g)\tilde{\omega}(g)$ for all $g \in G_\eta$ if and only if $\omega(g) = \tilde{\omega}(g)$ whenever $\psi(g) \neq 0$. Since both ω and $\tilde{\omega}$ are trivial on N , the result follows. \square

By definition the characters ψ_i vanish outside U_ψ . In particular, the associated idempotents e_{ψ_i} belong to the group ring $K[U_\psi]$. We actually need to consider a slightly more general situation. Let U be a subgroup of G_η containing U_ψ . Then U in particular contains N and thus the commutator subgroups of G and a fortiori that of G_η . Hence we can play the same game as above with (G, N, χ) replaced by (G_η, U, ψ) . Choose an

irreducible constituent ρ of $\text{res}_U^{G_\eta} \psi$ and denote the stabiliser of ρ in G_η by $G_{\eta, \rho}$. Then by Clifford's Theorem there is a positive integer f such that

$$\text{res}_U^{G_\eta} \psi = f \sum_{x \in G_\eta / G_{\eta, \rho}} {}^x \rho.$$

Since ${}^x \omega = \omega$ for every $\omega \in \text{Irr}(G_\eta / N)$, $x \in G_\eta$, it is straightforward to show that likewise

$$(2.5) \quad \text{res}_U^{G_\eta} \psi_i = f \sum_{x \in G_\eta / G_{\eta, \rho}} {}^x \rho_i$$

with the same f and where we set

$$(2.6) \quad \rho_i := \rho \otimes (\text{res}_U^{G_\eta} \omega_i)$$

for all $1 \leq i \leq s$.

Lemma 2.6. *With the above notation we have $\text{ind}_U^{G_\eta} \rho_i = f \psi_i$ and $[G_{\eta, \rho_i} : U] = f^2$ for all $1 \leq i \leq s$.*

Proof. The characters ψ_i and $\text{ind}_U^{G_\eta} \rho_i$ both vanish outside U . On U the formula for the character $\text{ind}_U^{G_\eta} \rho_i$ almost coincides with that of $\text{res}_U^{G_\eta} \psi_i$ given in (2.5), only f is replaced by $[G_{\eta, \rho_i} : U]$. Hence we have that

$$\text{ind}_U^{G_\eta} \rho_i = \frac{[G_{\eta, \rho_i} : U]}{f} \psi_i.$$

However, by Frobenius reciprocity the multiplicity of ψ_i in $\text{ind}_U^{G_\eta} \rho_i$ is given by

$$\langle \psi_i, \text{ind}_U^{G_\eta} \rho_i \rangle_{G_\eta} = \langle \text{res}_U^{G_\eta} \psi_i, \rho_i \rangle_U = f.$$

Both claims follow. \square

We record the following consequence of Lemma 2.6 and Corollary 2.4.

Corollary 2.7. *For each $1 \leq i \leq s$ the idempotent e_{ψ_i} is an element of $K[U]$ and indeed*

$$e_{\psi_i} = \sum_{x \in G_\eta / G_{\eta, \rho}} e_{x \rho_i}.$$

2.3. More Clifford theory. In this final subsection we provide a few technical lemmas which will be used in §4.

Lemma 2.8. *Let G be a finite group and let $\psi \in \text{Irr}(G)$ be an irreducible character. Let H be a normal subgroup of G such that $\psi = \text{ind}_H^G \lambda$ for some (necessarily irreducible) character λ of H . Then $G_\lambda = H$ and $\text{res}_H^G \psi = \sum_{c \in G/H} {}^c \lambda$.*

Proof. By Frobenius reciprocity we have

$$\langle \text{res}_H^G \psi, \lambda \rangle_H = \langle \psi, \text{ind}_H^G \lambda \rangle_G = \langle \psi, \psi \rangle_G = 1$$

so that Clifford's Theorem implies $\text{res}_H^G \psi = \sum_{c \in G/G_\lambda} {}^c \lambda$. In particular, we have $\psi(1) = [G : G_\lambda] \lambda(1)$. As $\psi = \text{ind}_H^G \lambda$, we also have $\psi(1) = [G : H] \lambda(1)$. Hence the inclusion $H \subseteq G_\lambda$ must be an equality. \square

Lemma 2.9. *We keep the notation and assumptions of Lemma 2.8. Let N be a normal subgroup of G which contains the commutator subgroup G' . Assume that $\eta := \text{res}_N^G \psi$ is irreducible (so that in particular $G_\eta = G$) and that $U_\psi = G$. Then the inclusion $H \hookrightarrow G$ induces an isomorphism $H/(H \cap N) \simeq G/N$ by which we identify $\text{Irr}(H/(H \cap N))$ and $\text{Irr}(G/N)$. Then for all $\omega \in \text{Irr}(G/N)$*

- (i) we have $\psi \otimes \omega = \text{ind}_H^G(\lambda \otimes \omega)$;
- (ii) the idempotent $e_{\psi \otimes \omega}$ is an element of $K[H]$ and indeed

$$e_{\psi \otimes \omega} = \sum_{c \in G/H} e_{c\lambda \otimes \omega}.$$

Proof. We first note that indeed $G_\eta = G$ by Clifford's Theorem. Since ψ is induced from a character of the normal subgroup H , it vanishes outside H and hence U_ψ is contained in NH . By assumption we have $U_\psi = G$ so that also $NH = G$ which shows that the natural inclusion $H/(H \cap N) \rightarrow G/N$ is an isomorphism. We now prove (i). It is clear that both characters vanish outside H . For $h \in H$ we compute

$$\begin{aligned} \text{ind}_H^G(\lambda \otimes \omega)(h) &= \frac{1}{|H|} \sum_{g \in G} (\lambda \otimes \omega)(g^{-1}hg) = \frac{1}{|H|} \sum_{g \in G} \lambda(g^{-1}hg)\omega(h) \\ &= \text{ind}_H^G(\lambda)(h)\omega(h) = (\psi \otimes \omega)(h) \end{aligned}$$

as desired. For (ii) we have

$$\begin{aligned} e_{\psi \otimes \omega} &= \frac{\psi(1)}{|G|} \sum_{g \in G} (\psi \otimes \omega)(g^{-1})g \\ &= \frac{[G : H]\lambda(1)}{|G|} \sum_{h \in H} \sum_{c \in G/H} ({}^c\lambda \otimes \omega)(h^{-1})h \\ &= \sum_{c \in G/H} \frac{\lambda(1)}{|H|} \sum_{h \in H} ({}^c\lambda \otimes \omega)(h^{-1})h \\ &= \sum_{c \in G/H} e_{c\lambda \otimes \omega}, \end{aligned}$$

where the second equality follows from Lemma 2.8. \square

Lemma 2.10. *We keep the notation and assumptions of the two preceding lemmas. Then the following holds.*

- (i) For each $c \in G/H$ the character

$$\eta_c := \text{res}_{H \cap N}^H {}^c\lambda = \text{res}_{H \cap N}^H {}^c\lambda \otimes \omega$$

is irreducible and we have an equality

$$\text{ind}_{H \cap N}^H \eta_c = \sum_{\omega \in \text{Irr}(G/N)} {}^c\lambda \otimes \omega.$$

- (ii) For $c \in G/H$ we set $e(\eta_c) := \sum_{\omega \in \text{Irr}(G/N)} e_{c\lambda \otimes \omega} \in K[H \cap N]$. Then we have an equality

$$e_\eta = \sum_{c \in G/H} e(\eta_c).$$

Proof. Let $\tilde{\eta}_c \in \text{Irr}(H \cap N)$ be an irreducible constituent of η_c . By Frobenius reciprocity we have

$$(2.7) \quad \langle {}^c\lambda \otimes \omega, \text{ind}_{H \cap N}^H \tilde{\eta}_c \rangle_H = \langle \eta_c, \tilde{\eta}_c \rangle_{H \cap N} > 0.$$

Since the characters ${}^c\lambda \otimes \omega$, $\omega \in \text{Irr}(G/N)$ are pairwise distinct by Lemmas 2.9 (i) and 2.5, this shows the inequality

$$[G : N]\lambda(1) \leq [H : H \cap N]\tilde{\eta}_c(1) = [G : N]\tilde{\eta}_c(1).$$

However, we clearly have $\tilde{\eta}_c(1) \leq \lambda(1)$ as $\tilde{\eta}_c$ divides η_c . It follows that $\tilde{\eta}_c(1) = \lambda(1) = \eta_c(1)$ and thus $\eta_c = \tilde{\eta}_c$ is irreducible. By (2.7) it suffices to compare degrees for the final claim in (i). We compute

$$\begin{aligned} \text{ind}_{H \cap N}^H \eta_c(1) &= [H : H \cap N] \eta_c(1) \\ &= [G : N]^c \lambda(1) \\ &= \sum_{\omega \in \text{Irr}(G/N)} ({}^c \lambda \otimes \omega)(1). \end{aligned}$$

For (ii) we compute

$$e_\eta = \sum_{\omega \in \text{Irr}(G/N)} e_{\psi \otimes \omega} = \sum_{\omega \in \text{Irr}(G/N)} \sum_{c \in G/H} e_{c \lambda \otimes \omega} = \sum_{c \in G/H} e(\eta_c),$$

where the first and second equality follow from Corollary 2.4 (ii) and Lemma 2.9 (ii), respectively. \square

3. APPLICATIONS TO INTEGRALITY RINGS AND DENOMINATOR IDEALS

A main aim of this section is to prove that $|G'| \in \mathcal{H}(G)$ for every finite group G . This is clear for abelian groups and we will do induction on the group order, where the results established in §2 will play a crucial role.

3.1. Behaviour of the reduced norm under restriction. Let G be a finite group and let K be a field of characteristic 0 that is a splitting field for G and all of its subgroups. Since we want to do induction, we need to know how the reduced norm behaves when we restrict to a subgroup U of G . More precisely, let $H \in M_n(K[G])$ be a matrix with entries in $K[G]$. On the one hand we can compute its reduced norm $\text{nr}_G(H) \in \zeta(K[G])$. On the other hand, we may view H as a $K[G]$ -linear map in the usual way. This map is also $K[U]$ -linear and (after a choice of basis) we obtain a new matrix $H|_U \in M_{n[G:U]}(K[U])$. Its reduced norm $\text{nr}_U(H|_U) \in \zeta(K[U])$ does not depend upon the particular choice of basis, and we want to determine its value in terms of $\text{nr}_G(H)$.

If H is invertible, results in the spirit of Lemma 3.1 below already appear in the literature if K is a number field [CR87, (52.23)] and if $K = \mathbb{Q}_p^c$ [Bre04, §2.2]. Unfortunately, this is not sufficient for our purposes so that we include a proof of Lemma 3.1 for convenience. We mainly follow the argument given in [CR87], but since H might not be invertible, we cannot reduce to the case $n = 1$.

Lemma 3.1. *Let G be a finite group and let U a subgroup of G . Let $H \in M_n(K[G])$ and write $\text{nr}_G(H) = \sum_{\chi \in \text{Irr}(G)} \alpha_\chi e_\chi$ with $\alpha_\chi \in K$. Then we have that*

$$\text{nr}_U(H|_U) = \sum_{\psi \in \text{Irr}(U)} \beta_\psi e_\psi, \text{ where } \beta_\psi = \prod_{\chi \in \text{Irr}(G)} \alpha_\chi^{\langle \text{ind}_U^G \psi, \chi \rangle}.$$

Proof. We consider $K[G]^n$ as a right $K[G]$ -module and let e_1, \dots, e_n be its standard basis. We set $m := [G : U]$ and choose left coset representatives g_1, \dots, g_m of U in G . Then $(e_k g_\ell)_{1 \leq k \leq n, 1 \leq \ell \leq m}$ constitutes a basis of $K[G]^n$ as a right $K[U]$ -module. We define a K -linear map

$$\begin{aligned} \omega_{G,U} : K[G] &\longrightarrow K[U] \\ \sum_{g \in G} x_g g &\mapsto \sum_{g \in U} x_g g \end{aligned}$$

that sends each $g \in G \setminus U$ to zero. This map extends to matrix rings in the obvious way. With respect to the above basis the $K[U]$ -linear map $H|_U$ is given by the block matrix

$$B := (\omega_{G,U}(g_j^{-1}Hg_i))_{1 \leq i,j \leq m}.$$

To see this let $H = (h_{ij})_{1 \leq i,j \leq n}$ so that $He_k g_\ell = \sum_{j=1}^n h_{kj} g_\ell e_j$. We likewise have that

$$Be_k g_\ell = \sum_{i=1}^m \sum_{j=1}^n (e_j g_i) \omega_{G,U}(g_i^{-1} h_{kj} g_\ell).$$

Comparing the coefficients at each e_j we see that it suffices to show that

$$(3.1) \quad xg_\ell = \sum_{i=1}^m g_i \omega_{G,U}(g_i^{-1} xg_\ell)$$

for each $x \in K[G]$. By K -linearity we may assume that $x = g \in G$. Then $\omega_{G,U}(g_i^{-1}gg_\ell)$ vanishes unless $g_i^{-1}gg_\ell \in U$. The latter happens if and only if $gg_\ell \in g_i U$ and therefore for exactly one choice of i . This shows (3.1).

Now let ψ be an irreducible character of U and let π be a representation with character ψ . Then the induced representation is given by (see [CR81, (10.1)], for instance)

$$\text{ind}_U^G(\pi)(x) = (\pi(\omega_{G,U}(g_j^{-1}xg_i)))_{1 \leq i,j \leq m}.$$

In particular, we have $\text{ind}_U^G(\pi)(H) = \pi(B)$ and hence

$$\beta_\psi = \det(\pi(B)) = \det(\text{ind}_U^G(\pi)(H)) = \prod_{\chi \in \text{Irr}(G)} \alpha_\chi^{\langle \text{ind}_U^G \psi, \chi \rangle_G}$$

as desired. \square

3.2. Bounding denominators of reduced norms. We first consider the case of local fields. The crucial result we have to show is the following.

Theorem 3.2. *Let p be prime and let G be a finite group. Let F be a (possibly infinite) extension of \mathbb{Q}_p and $R \subset F$ an integrally closed noetherian ring with field of fractions F . Then for every square matrix H with entries in the R -order $R[G]$ one has*

$$|G'| \text{nr}(H) \in \zeta(R[G]).$$

Proof. We first note that $\text{nr}(H) \in \zeta(F[G])$ so that it suffices to show that $|G'| \text{nr}(H)$ has integral coefficients. Let K be a finite extension of \mathbb{Q}_p that is a splitting field for G and all of its subgroups. By [Eis95, Proposition 13.14] we may and do assume that F contains K .

Choose a normal subgroup N containing the commutator subgroup G' of G . Let η be an irreducible character of N and let χ_1, \dots, χ_s be the irreducible characters of G dividing $\text{ind}_N^G \eta$, which are exactly those whose restriction to N is divisible by η . Hence this is compatible with the notation of §2.2. By Corollary 2.4 (i) we have

$$(3.2) \quad e(\eta) := \sum_{i=1}^s e_{\chi_i} = \sum_{x \in G/G_\eta} e_{x\eta} \in K[N].$$

The ring of integers \mathcal{O}_K in K is contained in R and we clearly have

$$(3.3) \quad |N|e(\eta) \in \mathcal{O}_K[G].$$

Let η' be a second irreducible character of N . We write $\eta' \sim \eta$ if η' is of the form $x\eta$ for some $x \in G$. Then $e(\eta') = e(\eta)$ if and only if $\eta' \sim \eta$, and $e(\eta')e(\eta) = 0$ otherwise. Hence

we can write $1 = \sum_{\eta \in \text{Irr}(N)/\sim} e(\eta)$ and likewise $\text{nr}(H) = \sum_{\eta \in \text{Irr}(N)/\sim} \text{nr}(H)e(\eta)$. We now show that

$$(3.4) \quad |N|\text{nr}(H)e(\eta) \in \zeta(R[G])$$

for each η . Choosing $N = G'$ then proves the theorem.

We do this by induction on $|G|$. If G is abelian, then $\text{nr}(H) \in R[G] = \zeta(R[G])$ and (3.4) follows from (3.3). Now suppose that G is non-abelian. By Corollary 2.4 (iii) we have $e_{\chi_i} = \sum_{x \in G/G_\eta} e_{x\psi_i}$ for all i , where as before $\chi_i = \text{ind}_{G_\eta}^G \psi_i$ for an (irreducible) character ψ_i of G_η . Hence the natural diagonal embeddings $Fe_{\chi_i} \hookrightarrow \bigoplus_{x \in G/G_\eta} Fe_{x\psi_i}$ induce an embedding

$$(3.5) \quad \iota_\eta : \zeta(F[G]e(\eta)) = \bigoplus_{i=1}^s Fe_{\chi_i} \hookrightarrow \bigoplus_{i=1}^s \bigoplus_{x \in G/G_\eta} Fe_{x\psi_i} = \zeta(F[G_\eta]e(\eta)).$$

Since $\chi_i = \text{ind}_{G_\eta}^G \psi_i$ is irreducible, Lemma 3.1 implies that $\iota_\eta(\text{nr}(H)e(\eta)) = \text{nr}(H|_{G_\eta})e(\eta)$. Hence by induction we may assume that $G = G_\eta$ (note that for this step it is not sufficient to consider (3.4) with $N = G'$, as the commutator subgroup of G_η might be strictly smaller than G').

Recall the definition of U_ψ from §2.2 (where $\psi = \psi_1$ by convention). We first assume that U_ψ is a proper subgroup of G . As G/U_ψ is abelian, we may choose a maximal proper subgroup U of G that contains U_ψ . Then $\ell := [G : U]$ is a prime. Let ρ be an irreducible constituent of $\text{res}_U^G \psi$ and define ρ_i as in (2.6). Then there is a positive integer f such that $\text{res}_U^G \psi_i = f \sum_{x \in G/G_\rho} {}^x \rho_i$ by (2.5). By Lemma 2.6 we have that $[G_\rho : U] = f^2$ is a square; but $[G_\rho : U]$ divides $[G : U] = \ell$ so that we must have $G_\rho = U$ and $f = 1$. In complete analogy to the definition of ι_η , but using Corollary 2.7, we obtain an embedding

$$(3.6) \quad \iota_\rho : \zeta(F[G]e(\eta)) = \bigoplus_{i=1}^s Fe_{\psi_i} \hookrightarrow \bigoplus_{i=1}^s \bigoplus_{x \in G/G_\rho} Fe_{x\rho_i} = \zeta(F[G_\rho]e(\eta)).$$

We have $\psi_i = \text{ind}_{G_\rho}^G \rho_i$ for all i by Frobenius reciprocity so that again $\iota_\rho(\text{nr}(H)e(\eta)) = \text{nr}(H|_{G_\rho})e(\eta)$ by Lemma 3.1. Hence (3.4) holds by induction.

We are left with the case $U_\psi = G = G_\eta$. By Lemma 2.5 the twists of ψ by irreducible characters $\omega \in \text{Irr}(G/N)$ are pairwise distinct. Since the set of these twists coincides with the set $\{\psi_1, \dots, \psi_s\}$ by Proposition 2.3 (i) and (ii), we have $s = [G : N]$ and thus $m = 1$ by part (iii) of the same result. Here we recall that $m = m_i$ denotes the multiplicity of η in $\text{res}_N^G \psi_i$. By [Hup67, Kapitel V, Satz 12.5] we can choose an integral representation

$$\pi : G \rightarrow \text{GL}_{\psi(1)}(\mathcal{O}_K)$$

with character ψ . Let $\omega \in \text{Irr}(G/N)$. Then likewise $\pi \otimes \omega$ is an integral representation with character $\psi \otimes \omega$. Let us denote the image of $g \in G$ in G/N by \bar{g} . Note that we have an isomorphism $F[G/N] \simeq \bigoplus_{\omega \in \text{Irr}(G/N)} F$. Hence the representations $\pi \otimes \omega$ induce an isomorphism of rings

$$(3.7) \quad \iota_\pi : F[G]e(\eta) \simeq \bigoplus_{\omega \in \text{Irr}(G/N)} M_{\psi(1)}(F) \simeq M_{\psi(1)}(F[G/N])$$

which sends $ge(\eta)$ to $\bar{g}\pi(g)$ for all $g \in G$. This extends to matrix rings in the obvious way and for each $n \in \mathbb{N}$ we obtain a commutative diagram

$$\begin{array}{ccc} M_n(F[G]e(\eta)) & \xrightarrow[\simeq]{\iota_\pi} & M_{n\psi(1)}(F[G/N]) \\ \downarrow \text{nr} & & \downarrow \text{det} \\ \zeta(F[G]e(\eta)) & \xrightarrow[\simeq]{\iota_\pi} & F[G/N]. \end{array}$$

Now if $H \in M_n(R[G])$ has integral coefficients, then the image of $He(\eta)$ under ι_π is contained in $M_{n\psi(1)}(R[G/N])$, since π was chosen to be integral. Hence we obtain

$$(3.8) \quad \iota_\pi(\text{nr}(He(\eta))) = \det(\iota_\pi(He(\eta))) \in R[G/N].$$

To finish the proof of (3.4) it now suffices to show that $\iota_\pi^{-1}(\bar{g}) \in |N|^{-1}R[G]$ for all $g \in G$. We compute

$$\begin{aligned} \iota_\pi^{-1}(\bar{g}) &= \sum_{\omega \in \text{Irr}(G/N)} \omega(\bar{g}) e_{\psi \otimes \omega} \\ &= \sum_{\omega \in \text{Irr}(G/N)} \omega(\bar{g}) \frac{\psi(1)}{|G|} \sum_{h \in G} (\psi \otimes \omega)(h^{-1}) h \\ &= \frac{\psi(1)}{|N|} \sum_{h \in G: \bar{h} = \bar{g}} \psi(h^{-1}) h, \end{aligned}$$

where the last equality follows from the well-known orthogonality relation

$$\frac{1}{[G : N]} \sum_{\omega \in \text{Irr}(G/N)} \omega(\bar{g}) \omega(\bar{h}^{-1}) = \delta_{\bar{g}, \bar{h}}.$$

This finishes the proof of the theorem. \square

Corollary 3.3. *Let G be a finite group with commutator subgroup G' . Then the following holds.*

- (i) *For each prime p we have an inclusion $|G'|\mathcal{I}_p(G) \subseteq \zeta(\mathbb{Z}_p[G])$.*
- (ii) *We have an inclusion $|G'|\mathcal{I}(G) \subseteq \zeta(\mathbb{Z}[G])$.*

Proof. (i) follows from Theorem 3.2 with $F = \mathbb{Q}_p$. (i) for all primes p implies (ii). \square

Corollary 3.4. *Let G be a finite group with commutator subgroup G' . Then the following holds.*

- (i) *For each prime p we have $|G'| \in \mathcal{H}_p(G)$.*
- (ii) *We have that $|G'| \in \mathcal{H}(G)$.*

Proof. (ii) is again implied by (i) for all p so that it suffices to prove (i). So let p be a prime and let $H \in M_n(\mathbb{Z}_p[G])$ be a matrix with reduced characteristic polynomial $f_H(X)$. Then the generalised adjoint H^* a priori has entries in $\mathbb{Q}_p[G]$ and we have to show that $|G'|H^* \in M_n(\mathbb{Z}_p[G])$. As before we let K be a finite extension of \mathbb{Q}_p such that K is a splitting field for G and all of its subgroups. The polynomial ring $\mathcal{O}_K[X]$ is noetherian and integrally closed. So we may apply Theorem 3.2 with $R = \mathcal{O}_K[X]$ and the matrix $\mathbf{1}_{n \times n} \otimes X - H \otimes 1 \in M_n(\mathcal{O}_K[X][G])$. Taking (1.1) into account we see that $|G'|f_H(X) \in \zeta(\mathcal{O}_K[X][G])$. Indeed by (3.4) we know that for each irreducible character η of G' we have that

$$|G'|f_H(X)e(\eta) \in \zeta(\mathcal{O}_K[X][G]),$$

where $e(\eta)$ was defined in (3.2). The characters χ_1, \dots, χ_s that occur in this definition are all of the same degree d . In particular, if we write $f_H(X)e(\eta) = \sum_{j=0}^{dn} \alpha_j X^j$ with $\alpha_j \in \zeta(K[G]e(\eta))$, then $|G'|\alpha_j$ has integral coefficients for all j , whence

$$|G'|H^*e(\eta) = |G'|(-1)^{dn+1} \sum_{j=1}^{dn} \alpha_j e(\eta) H^{j-1} \in M_n(\mathcal{O}_K[G]).$$

As this holds for all η , we get

$$|G'|H^* \in M_n(\mathbb{Q}_p[G]) \cap M_n(\mathcal{O}_K[G]) = M_n(\mathbb{Z}_p[G])$$

as desired. \square

Proof of Theorem 1. This is now immediate from Corollary 2.4 and Example 1.7. \square

Remark 3.5. An inspection of the proof of Theorem 3.2 shows that for each normal subgroup N of G containing G' and each character $\eta \in \text{Irr}(N)$ one has

$$\frac{|N|}{\eta(1)} e(\eta) \in \mathcal{H}(R[G]).$$

3.3. More on integrality rings. The aim of this subsection is to show that the integrality ring $\mathcal{I}(G)$ is strictly larger than $\zeta(\mathbb{Z}[G])$ for every non-abelian finite group G . This is not needed in later sections and may be skipped by the reader who is only interested in our applications to number theory. We first state the main result of this subsection.

Theorem 3.6. *Let G be a finite group. Then $\mathcal{I}(G) = \zeta(\mathbb{Z}[G])$ if and only if G is abelian.*

If G is abelian, then clearly $\mathcal{I}(G) = \zeta(\mathbb{Z}[G]) = \mathbb{Z}[G]$. So the main task of this subsection is to show that the integrality ring is strictly larger than $\zeta(\mathbb{Z}[G])$ whenever G is non-abelian. This means that we aim to construct elements $x \in \mathbb{Z}[G]$ such that their reduced norm $\text{nr}(x)$ is not integral.

For a finite group G and a positive integer d we set

$$E_d = E_{G,d} := \sum_{\chi(1)=d} e_\chi,$$

where the sum runs over all irreducible characters of G of degree d . In particular, one has $E_1 = |G'|^{-1} \text{Tr}_{G'}$. We will need the following interesting consequence of Corollary 2.4.

Lemma 3.7. *Let G be a finite group. Then for each $d \in \mathbb{N}$ one has*

$$|G'|E_d \in \mathbb{Z}[G].$$

Proof. Let K be a finite extension of \mathbb{Q} that is a splitting field for G and all of its subgroups. Let X_d be the set of all irreducible characters η of G' such that there is a $\chi \in \text{Irr}(G)$ of degree d with $\langle \text{res}_{G'}^G \chi, \eta \rangle > 0$. Let $\eta \in X_d$. As before we set $e(\eta) := \sum_{i=1}^s e_{\chi_i}$, where the sum runs over all irreducible characters of G whose restriction to G' is divisible by η . The characters χ_1, \dots, χ_s then all have degree d and Corollary 2.4 (i) implies that $|G'|e(\eta) \in \mathcal{O}_K[G']$, as was essentially already observed in (3.3). As in the proof of Theorem 3.2 we write $\eta \sim \eta'$ if and only if $\eta' = {}^x \eta$ for some $x \in G$ if and only if $e(\eta) = e(\eta')$. Then we have $E_d = \sum_{\eta \in X_d / \sim} e(\eta)$ and thus $|G'|E_d \in \mathcal{O}_K[G']$. Since $\sigma(E_d) = E_d$ for each Galois automorphism $\sigma \in G_{\mathbb{Q}}$, we see that $|G'|E_d \in \mathbb{Q}[G] \cap \mathcal{O}_K[G'] = \mathbb{Z}[G']$ as desired. \square

Corollary 3.8. *Let G be a finite group. Then the following holds.*

- (i) *If $\mathcal{I}(G) = \zeta(\mathbb{Z}[G])$, then the order of the commutator subgroup is squarefree.*

- (ii) If p is a prime such that $\mathcal{I}_p(G) = \zeta(\mathbb{Z}_p[G])$, then $|G'|$ is not divisible by p^2 .
- (iii) If $\mathcal{I}(G) = \zeta(\mathbb{Z}[G])$, then G is solvable.

Proof. Assume that p is a prime such that p^2 divides $|G'|$. Then $p^{-1}|G'|$ is an integer and

$$\begin{aligned} \text{nr}(p^{-1}|G'|) &= \sum_{\chi \in \text{Irr}(G)} (p^{-1}|G'|)^{\chi(1)} e_\chi \\ &= p^{-1}|G'|E_1 + \sum_{d \geq 2} (p^{-1}|G'|)^d E_d \\ &= p^{-1}\text{Tr}_{G'} + \sum_{d \geq 2} (p^{-1}|G'|)^{d-2} \frac{|G'|}{p^2} |G'| E_d. \end{aligned}$$

By Lemma 3.7 the sum on the right-hand side is integral. Since the first term has denominator p , we see that $\text{nr}(p^{-1}|G'|) \in \mathcal{I}(G) \subseteq \mathcal{I}_p(G)$ is not integral. This shows (i) and (ii). If $\mathcal{I}(G) = \zeta(\mathbb{Z}[G])$, then the cardinality of G' is squarefree by (i) and thus all Sylow subgroups of G' are cyclic. Then G' is solvable by Hölder's Theorem [Hup67, Kapitel IV, Satz 2.9]. Hence also G is solvable. \square

We need a few more preliminary results. In particular, the induction argument will lead to two types of rather special groups that we have to treat separately.

Lemma 3.9. *Let p be a prime and let G be a finite p -group. Suppose that G' is cyclic of order p and contained in the centre $Z(G)$ of G . Moreover, assume that every non-linear irreducible character of G is faithful. Then for every $x \in G \setminus Z(G)$ one has $\text{nr}(x) \notin \mathbb{Z}_p[G]$.*

Proof. Let χ be an irreducible (\mathbb{C}_p -valued) character of G of degree $d > 1$. Choose a representation $\pi_\chi : G \rightarrow \text{GL}_d(\mathbb{C}_p)$ with character χ . Since G' is central in G , also $\pi_\chi(g)$ is central for every $g \in G'$ and thus of the form $\lambda(g)\mathbf{1}_{d \times d}$, where $\lambda : G' \rightarrow \mathbb{C}_p^\times$ is a linear character of G' . In other words, we have $\text{res}_{G'}^G \chi = \chi(1)\lambda$. Since χ is faithful, the character λ cannot be trivial.

Now let $x \in G \setminus Z(G)$. As x is not central, we may choose $y \in G$ such that the commutator $[x, y] := xyx^{-1}y^{-1}$ is non-trivial. Since χ is a class function and $[x, y] \in G'$, we see that

$$\chi(x^{-1}) = \chi(yx^{-1}y^{-1}) = \chi(x^{-1}[x, y]) = \chi(x^{-1})\lambda([x, y]).$$

By our choice of x and y we have $\lambda([x, y]) \neq 1$ so that $\chi(x^{-1})$ vanishes. In particular, the coefficient of the associated idempotent e_χ at x vanishes. Hence the coefficient of $\text{nr}(x)$ at x is given by

$$\sum_{\substack{\chi \in \text{Irr}(G) \\ \chi(1)=1}} \chi(x) \frac{1}{|G|} \chi(x^{-1}) = \frac{1}{|G|} = \frac{1}{p} \notin \mathbb{Z}_p$$

and thus $\text{nr}(x) \notin \mathbb{Z}_p[G]$ as desired. \square

We recall the definition and some basic facts about Frobenius groups.

Definition 3.10. A *Frobenius group* is a finite group G with a proper nontrivial subgroup H such that $H \cap gHg^{-1} = \{1\}$ for all $g \in G \setminus H$, in which case H is called a *Frobenius complement*.

Theorem 3.11. *Let G be a Frobenius group with Frobenius complement H . Then the following holds.*

- (i) *There is a unique normal subgroup N in G , known as the Frobenius kernel, such that G is a semi-direct product $N \rtimes H$.*

- (ii) If χ is an irreducible character of G such that $N \not\subseteq \ker(\chi)$ then $\chi = \text{ind}_N^G \psi$ for some $1 \neq \psi \in \text{Irr}(N)$.

Proof. Part (i) is due to Frobenius, see [Hup67, Kapitel V, Satz 7.6]. For (ii) see [CR81, Proposition 14.4]. \square

Lemma 3.12. *Let $G \simeq G' \rtimes A$ be a Frobenius group with Frobenius kernel G' and (necessarily abelian) Frobenius complement A . Then for every $x \in G \setminus G'$ one has that $\text{nr}(x) \notin \mathbb{Z}[G]$.*

Proof. Let $\chi \in \text{Irr}(G)$ be non-linear. Then G' is not contained in the kernel of χ so that χ is induced from a non-trivial irreducible character of G' by Theorem 3.11 (ii). Since G' is normal, we have $\chi(x^{-1}) = 0$ for every $x \in G \setminus G'$. It follows as in the proof of Lemma 3.9 that the coefficient of $\text{nr}(x)$ at x is given by $|G'|^{-1}$. Hence $\text{nr}(x)$ is not integral. \square

Example 3.13. Let $G = S_3$ be the symmetric group on 3 letters. Then G is a Frobenius group with Frobenius kernel $G' = A_3$, the alternating group on 3 letters, and the proof of Lemma 3.12 shows that the coefficient of $\text{nr}(\tau)$ at τ equals $1/3$ for every transposition τ .

Example 3.14. More generally, let $q \neq 2$ be a prime power. Then the group of affine transformations $\text{Aff}(q) = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$ on the finite field \mathbb{F}_q is a Frobenius group with Frobenius kernel $\text{Aff}(q)' = \mathbb{F}_q$ and Frobenius complement \mathbb{F}_q^\times . Then the coefficient of $\text{nr}(x)$ at x equals $1/q$ for every $x \in \text{Aff}(q) \setminus \text{Aff}(q)'$. Note that $\text{Aff}(3) \simeq S_3$ and $\text{Aff}(4) \simeq A_4$, the alternating group on 4 letters.

Lemma 3.15. *Let G be a finite, non-abelian, solvable group. Let p be a prime and assume that G' is the unique minimal normal subgroup of G and is cyclic of order p . Then either (i) G is a p -group with $G' \subseteq Z(G)$ or (ii) G is a Frobenius group with Frobenius kernel G' (and abelian Frobenius complement A).*

Proof. Let us first assume that G is nilpotent. Then G decomposes into a direct product $G = P \times H$, where P is a p -group and the cardinality of H is coprime to p . Since G' is contained in every normal subgroup of G and is cyclic of order p , we must have $H = 1$. So G is a p -group. The centre of a p -group is a non-trivial normal subgroup so that it contains the unique minimal normal subgroup G' . Hence we are in case (i).

Recall that the Frattini subgroup $\Phi(G)$ is the intersection of all maximal subgroups of G . If G is not nilpotent, then the commutator subgroup G' is not contained in $\Phi(G)$ by a theorem of Wielandt [Hup67, Kapitel III, Satz 3.11]. Hence $\Phi(G)$ is trivial and there is a maximal subgroup A that does not contain G' . Then $G = AG'$ by maximality of A and $A \cap G' = 1$, as $G' \subseteq A$ otherwise. Hence G is a semi-direct product $G' \rtimes A$ so that $A \simeq G/G'$ is abelian. It remains to show that for every $g \in G \setminus A$ we have $A \cap g^{-1}Ag = 1$. Suppose to the contrary that there is an $a \in A$ such that $g^{-1}ag \in A$. Then $a^{-1}g^{-1}ag \in G' \cap A = 1$ so that a commutes with g . Since A is a maximal subgroup of G and is abelian, the element a is indeed central. In particular, it generates a normal subgroup of G and thus contains G' . This contradicts the fact that $G' \cap A = 1$. So if G is not nilpotent, we are in case (ii). \square

Proof of Theorem 3.6. We have to show that the integrality ring $\mathcal{I}(G)$ is strictly larger than the centre $\zeta(\mathbb{Z}[G])$. By Corollary 3.8 we may and do assume that G is solvable and that $|G'|$ is squarefree. We will now show by induction that there is an $x \in \mathbb{Z}[G]$ such that $\text{nr}(x) = \text{nr}_G(x) \notin \mathbb{Z}[G]$. Suppose first that G contains a proper normal subgroup N such that G/N is non-abelian. Then we may assume by induction that there is an $\bar{x} \in \mathbb{Z}[G/N]$

such that $\text{nr}_{G/N}(\bar{x}) \notin \mathbb{Z}[G/N]$. Let $x \in \mathbb{Z}[G]$ be any lift of \bar{x} . Then $\text{nr}_G(x)$ cannot be integral as it maps to $\text{nr}_{G/N}(\bar{x})$ under the canonical projection map $\mathbb{Q}[G] \rightarrow \mathbb{Q}[G/N]$. If no such N exists, the commutator subgroup is contained in every non-trivial normal subgroup, i.e. it is the unique minimal normal subgroup of G . By [Hup67, Kapitel I, Satz 9.13] the minimal normal subgroups of a solvable group are p -groups. Since $|G'|$ is squarefree, it must be cyclic of order p . Moreover, the non-linear irreducible characters of G are faithful, as their kernel is a normal subgroup that does not contain G' . So it remains to treat the two cases given by Lemma 3.15. These are covered by Lemmas 3.9 and 3.12, respectively. \square

Let us briefly discuss the case of p -adic groups rings. If p does not divide the cardinality of the commutator subgroup, then $\mathcal{I}_p(G) = \zeta(\mathbb{Z}_p[G])$ by Proposition 1.6. The converse implication seems to be likely, but the above argument fails in several respects. For instance, we can only conclude that $|G'|$ is not divisible by p^2 , but this does not imply that G' is solvable. Moreover, if N is a normal subgroup of G such that G/N is non-abelian, the commutator subgroup of the quotient may be of cardinality coprime to p even if this does not hold for G' . What we can offer, is the following variant of Theorem 3.6.

Proposition 3.16. *Let G be a finite nilpotent group and let p be a prime. Then $\mathcal{I}_p(G) = \zeta(\mathbb{Z}_p[G])$ if and only if $p \nmid |G'|$.*

Proof. Let G be a finite nilpotent group such that $|G'|$ is divisible by the prime p . It suffices to construct an element $x \in \mathbb{Z}_p[G]$ such that $\text{nr}_G(x) \notin \mathbb{Z}_p[G]$. The group G decomposes into a direct product $G \simeq P \times H$, where H is of cardinality prime to p and P is a finite non-abelian p -group. If there is $\bar{x} \in \mathbb{Z}_p[P]$ such that $\text{nr}_P(\bar{x}) \notin \mathbb{Z}_p[P]$, then any lift $x \in \mathbb{Z}_p[G]$ of \bar{x} under the projection $\mathbb{Z}_p[G] \rightarrow \mathbb{Z}_p[P]$ will do. So we may assume that G is a p -group. By a similar argument we may assume that there is no proper normal subgroup N of G such that G/N is non-abelian. Hence G' is contained in every non-trivial normal subgroup of G and every non-linear irreducible character of G must be faithful. By Corollary 3.8 we may assume that p^2 does not divide $|G'|$. Thus G' is a cyclic group of order p . Now Lemma 3.9 implies the result. \square

We are not aware of any example of a finite group G , where the conclusion of Proposition 3.16 fails. For instance, we have seen in Example 1.2 that for the special linear group $\text{SL}_2(\mathbb{F}_3)$ the commutator subgroup is of order 8 and $\mathcal{I}_2(\text{SL}_2(\mathbb{F}_3))$ is indeed strictly larger than $\zeta(\mathbb{Z}_2[\text{SL}_2(\mathbb{F}_3)])$. We give two further examples.

Example 3.17. Let $G = G' \rtimes A$ be a Frobenius group with Frobenius kernel G' such as the group of affine transformations considered in Example 3.14. Then the proof of Lemma 3.12 shows that $\text{nr}(x)$ has a denominator $|G'|$ at x for every $x \in G \setminus G'$. In particular, $\mathcal{I}_p(G)$ is strictly larger than $\zeta(\mathbb{Z}_p[G])$ whenever $|G'|$ is divisible by p .

Example 3.18. Let $G = M$ be the Monster group, which is the largest sporadic simple group. We show that $\mathcal{I}_p(M)$ is strictly larger than $\zeta(\mathbb{Z}_p[M])$ whenever p divides

$$|M'| = |M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

By Corollary 3.8 (ii) we only have to consider primes p in the set P of those primes that divide $|M|$ exactly once. For $n \in \mathbb{Z}$ the coefficient of the reduced norm $\text{nr}_M(n)$ at 1 is given by $|M|^{-1}A(n)$ where

$$A(n) := \sum_{\chi \in \text{Irr}(M)} n^{\chi(1)} \chi(1)^2.$$

The degrees of the characters are too large to compute this expression easily. However, it suffices to show that for each $p \in P$ there is an n such that $A(n) \bmod p \neq 0$. Choosing $n = -1$ we computed the following values for $A(-1) \bmod p$ using Python:

| | | | | | | | | | |
|-----------------|----|----|----|----|----|----|----|----|----|
| p | 17 | 19 | 23 | 29 | 31 | 41 | 47 | 59 | 71 |
| $A(-1) \bmod p$ | 1 | 1 | 9 | 15 | 10 | 5 | 17 | 31 | 51 |

In particular, we see that $A(-1)$ is not divisible by any $p \in P$ as desired.

4. CONJECTURES AND RESULTS ON INTEGRALITY AND ANNIHILATION

4.1. Equivariant Artin L -functions. Let L/K be a finite Galois extension of number fields with Galois group G . For each place v of K we fix a place w of L above v and write G_w and I_w for the decomposition group and the inertia subgroup of G at w , respectively. When w is a finite place, we choose a lift $\sigma_w \in G_w$ of the Frobenius automorphism at w . We write \mathfrak{P}_w for the associated prime ideal in L and $L(w)$ for the residue field at w . For a finite place v of K we denote the cardinality of its residue field $K(v)$ by Nv .

Let S be a finite set of places of K containing the set S_∞ of all archimedean places of K . For $\chi \in \text{Irr}_{\mathbb{C}}(G)$ let V_χ be a $\mathbb{C}[G]$ -module with character χ . The S -truncated Artin L -function $L_S(s, \chi)$ is defined as the meromorphic extension to the whole complex plane of the holomorphic function given by the Euler product

$$L_S(s, \chi) = \prod_{v \notin S} \det(1 - (Nv)^{-s} \sigma_w | V_\chi^{I_w})^{-1}, \quad \text{Re}(s) > 1.$$

Recall that there is a canonical isomorphism $\zeta(\mathbb{C}[G]) \simeq \prod_{\chi \in \text{Irr}_{\mathbb{C}}(G)} \mathbb{C}$. The equivariant S -truncated Artin L -function is defined to be the meromorphic $\zeta(\mathbb{C}[G])$ -valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \text{Irr}_{\mathbb{C}}(G)}.$$

Now suppose that T is a second finite set of places of K such that $S \cap T = \emptyset$. Then we define

$$(4.1) \quad \delta_T(s, \chi) := \prod_{v \in T} \det(1 - (Nv)^{1-s} \sigma_w^{-1} | V_\chi^{I_w}) \quad \text{and} \quad \delta_T(s) := (\delta_T(s, \chi))_{\chi \in \text{Irr}_{\mathbb{C}}(G)}.$$

Let $x \mapsto x^\#$ denote the anti-involution on $\mathbb{C}[G]$ induced by $g \mapsto g^{-1}$ for $g \in G$. The (S, T) -modified G -equivariant Artin L -function is defined to be

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\#.$$

Note that $L_S(s)^\# = (L_S(s, \tilde{\chi}))_{\chi \in \text{Irr}_{\mathbb{C}}(G)}$ where $\tilde{\chi}$ denotes the character contragredient to χ . Evaluating $\Theta_{S,T}(s)$ at integers $r \leq 0$ gives (S, T) -modified Stickelberger elements

$$\theta_S^T(r) := \Theta_{S,T}(r) \in \zeta(\mathbb{Q}[G]).$$

Note that a priori we only have $\theta_S^T(r) \in \zeta(\mathbb{C}[G])$, but by a result of Siegel [Sie70] we know that $\theta_S^T(r)$ in fact belongs to $\zeta(\mathbb{Q}[G])$. If T is empty, we abbreviate $\theta_S^T(r)$ to $\theta_S(r)$. If the extension L/K is not clear from context, we will also write $\theta_S^T(L/K, r)$, $L_S(L/K, s)$, $\delta_T(L/K, s)$ etc.

4.2. The integrality conjecture. If S is any finite set of places of K , we write $S(L)$ for the set of places of L lying above those in S . Assume that S contains S_∞ and let T be a second finite set of places of K such that T and S are disjoint. We denote the ring of $S(L)$ -integers in L by $\mathcal{O}_{L,S}$ and let $\mathcal{O}_{L,S,T}^\times$ be the subgroup of $\mathcal{O}_{L,S}^\times$ comprising those $S(L)$ -units which are congruent to 1 modulo each place in $T(L)$.

We will usually impose the following hypotheses on the sets S and T . Note that condition (iii) in particular implies that T is non-empty.

Hypothesis. Let S and T be finite sets of places of K . We say that $\text{Hyp}(S, T)$ is satisfied if (i) $S_{\text{ram}} \cup S_{\infty} \subseteq S$, (ii) $S \cap T = \emptyset$, and (iii) $\mathcal{O}_{L,S,T}^{\times}$ is torsionfree.

If G is abelian and $r \leq 0$ is an integer, then independent work of Pi. Cassou-Noguès [CN79] and of Deligne and Ribet [DR80] (see also Barsky [Bar78]) shows that

$$\theta_S^T(r) \in \mathbb{Z}[G]$$

whenever $\text{Hyp}(S, T)$ is satisfied. For general G this is no longer true, but we have the following conjecture due to the second named author. It originates from the more general [Nic11b, Conjecture 2.1] (if $r = 0$) and [Nic11a, Conjecture 2.11] (if $r < 0$). See also [Nic16, §3].

Conjecture 4.1 (Integrality Conjecture). Let L/K be a Galois extension of number fields with Galois group G and let $r \leq 0$ be an integer. Then for every pair S, T of finite sets of places of K satisfying $\text{Hyp}(S, T)$ we have $\theta_S^T(r) \in \mathcal{I}(G)$.

Remark 4.2. Let d_G be the lowest common multiple of $|G'|$ and the cardinalities of the conjugacy classes of G . If Conjecture 4.1 holds, then Corollary 3.3 implies that the denominators of $\theta_S^T(r)$ are bounded by $|G'|$, that is $|G'| \theta_S^T(r) \in \zeta(\mathbb{Z}[G])$. A variant of this conjecture in the case $r = 0$ has been formulated by Dejou and Roblot [DR14], which suggests that one has to replace $|G'|$ by the weaker bound d_G . This was motivated by the following example which is taken from [Dej11, p. 83]. Let L be the splitting field of the polynomial $\mathbf{x}^8 - 2\mathbf{x}^7 + \mathbf{x}^6 + \mathbf{x}^5 - \mathbf{x}^4 + 2\mathbf{x}^3 + 4\mathbf{x}^2 - 16\mathbf{x} + 16$. Then L/\mathbb{Q} is a Galois extension with Galois group $G \simeq \text{SL}_2(\mathbb{F}_3)$ so that $|G'| = 8$ and $d_G = 24$. The only roots of unity in L are ± 1 and the only ramified prime is 853. Using the notation of Example 1.2 one has

$$\theta_{S_{\infty} \cup S_{\text{ram}}}(0) = \frac{1}{3}(C_1 - C_2)(-42C_1 + 2C_3 - 22C_5 - 21C_6).$$

Because of the factor $1/3$ it seems that one has to multiply by d_G rather than $|G'|$. (Note that there is no set T in this example, but the number of roots of unity in L is 2 so that there is a totally decomposed prime $p \neq 2$ such that $\delta_{\{p\}}(0) \in \zeta(\mathbb{Z}_3[G])^{\times}$ by [Nic11b, Lemma 2.2]; hence the 3-adic denominators do not change if we multiply the equation by $\delta_{\{p\}}(0)$.) However, using equalities of the form $C_3 = C_2C_5$ this can be rewritten as

$$\theta_{S_{\infty} \cup S_{\text{ram}}}(0) = -14C_1 + 14C_2 + 8C_3 + 7C_4 - 8C_5 - 7C_6$$

which is indeed integral.

4.3. Brumer's conjecture and generalisations. Let L/K be a finite Galois extension of number fields with Galois group G . Assume that S contains S_{∞} and let T be a second finite set of places of K such that T and S are disjoint. Write $S_f := S \setminus S_{\infty}$ for the subset of S comprising all finite places in S . We let cl_L^T be the ray class group of L associated to the modulus $\mathfrak{M}_L^T := \prod_{w \in T(L)} \mathfrak{P}_w$ and write $\text{cl}_{L,S}^T$ for the cokernel of the natural map $\mathbb{Z}[S_f(L)] \rightarrow \text{cl}_L^T$ which sends each place $w \in S_f(L)$ to the class of the associated prime ideal \mathfrak{P}_w . If T is empty we abbreviate $\text{cl}_{L,S}^T$ and cl_L^T to $\text{cl}_{L,S}$ and cl_L , respectively. All these modules are equipped with a natural G -action and we have the following exact sequence of finitely generated $\mathbb{Z}[G]$ -modules

$$(4.2) \quad 0 \longrightarrow \mathcal{O}_{L,S,T}^{\times} \longrightarrow \mathcal{O}_{L,S}^{\times} \longrightarrow \prod_{w \in T(L)} L(w)^{\times} \xrightarrow{\nu} \text{cl}_{L,S}^T \longrightarrow \text{cl}_{L,S} \longrightarrow 0,$$

where the map ν lifts an element $\bar{x} \in \prod_{w \in T(L)} L(w)^\times \simeq (\mathcal{O}_L/\mathfrak{M}_L^T)^\times$ to $x \in \mathcal{O}_L$ and sends it to the ideal class $[(x)] \in \text{cl}_{L,S}^T$ of the principal ideal (x) .

If G is abelian, then Brumer's conjecture simply asserts that $\theta_S^T(0)$ annihilates the class group cl_L . However, we will mainly deal with the following strengthening that was stated by Tate [Tat84] and is known as the Brumer–Stark conjecture.

Conjecture 4.3. Let L/K be an abelian extension of number fields with Galois group G . Then for every pair S, T of finite sets of places of K satisfying $\text{Hyp}(S, T)$ we have $\theta_S^T(0) \in \text{Ann}_{\mathbb{Z}[G]}(\text{cl}_L^T)$.

By groundbreaking work of Dasgupta and Kakde [DK23, Corollary 3.8] Conjecture 4.3 holds away from its 2-primary part.

Theorem 4.4 (Dasgupta–Kakde). *Let L/K be an abelian extension of number fields with Galois group G . Then for every pair S, T of finite sets of places of K satisfying $\text{Hyp}(S, T)$ we have $\theta_S^T(0) \in \text{Ann}_{\mathbb{Z}[1/2][G]}(\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \text{cl}_L^T)$.*

Non-abelian generalisations of Brumer's conjecture have been formulated by the second named author [Nic11b], by Dejou and Roblot [DR14] and in even greater generality by Burns [Bur11]. We consider the following generalisation of Conjecture 4.3.

Conjecture 4.5. Let L/K be a Galois extension of number fields with Galois group G . Then for every pair S, T of finite sets of places of K satisfying $\text{Hyp}(S, T)$ we have

$$\mathcal{H}(G) \cdot \theta_S^T(0) \subseteq \text{Ann}_{\zeta(\mathbb{Z}[G])}(\text{cl}_L^T).$$

Note that the Integrality Conjecture implies that the set on the left is indeed contained in the centre of $\mathbb{Z}[G]$. By Corollary 3.3 we thus expect that in particular $|G'| \theta_S^T(0)$ has integral coefficients and annihilates the ray class group cl_L^T .

4.4. Derived categories and Galois cohomology. Let Λ be a noetherian ring and $\text{PMod}(\Lambda)$ be the category of all finitely generated projective Λ -modules. We write $\mathcal{D}(\Lambda)$ for the derived category of Λ -modules and $\mathcal{C}^b(\text{PMod}(\Lambda))$ for the category of bounded complexes of finitely generated projective Λ -modules. Recall that a complex of Λ -modules is called perfect if it is isomorphic in $\mathcal{D}(\Lambda)$ to an element of $\mathcal{C}^b(\text{PMod}(\Lambda))$. We denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$ comprising perfect complexes by $\mathcal{D}^{\text{perf}}(\Lambda)$.

If M is a Λ -module and n is an integer, we write $M[n]$ for the complex

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow 0 \rightarrow \cdots,$$

where M is placed in degree $-n$.

Let L/K be a Galois extension of number fields with Galois group G . For a finite set S of places of K containing S_∞ we let $G_{L,S}$ be the Galois group over L of the maximal extension of L that is unramified outside $S(L)$. For any topological $G_{L,S}$ -module M we write $R\Gamma(\mathcal{O}_{L,S}, M)$ for the complex of continuous cochains of $G_{L,S}$ with coefficients in M . If F is a field and M is a topological G_F -module, we likewise define $R\Gamma(F, M)$ to be the complex of continuous cochains of G_F with coefficients in M .

If F is a global or a local field of characteristic zero, and M is a discrete or a compact G_F -module, then for $n \in \mathbb{Z}$ we denote the n -th Tate twist of M by $M(n)$. Now let p be a prime and suppose that S also contains all p -adic places of K . Then we will consider the complexes $R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(n))$ in $\mathcal{D}(\mathbb{Z}_p[G])$. Note that for an integer i the cohomology group in degree i of $R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(n))$ naturally identifies with $H_{\text{ét}}^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(n))$, the i -th étale cohomology group of the affine scheme $\text{Spec}(\mathcal{O}_{L,S})$ with coefficients in the étale p -adic sheaf $\mathbb{Z}_p(n)$.

4.5. The Coates–Sinnott conjecture and generalisations. For an integer $n \geq 0$ we let $K_n(\mathcal{O}_{L,S})$ denote the Quillen K -groups of $\mathcal{O}_{L,S}$. Coates and Sinnott [CS74] formulated the following analogue of Brumer’s conjecture for higher K -groups.

Conjecture 4.6 (Coates–Sinnott). Let L/K be a finite abelian extension of number fields with Galois group G . Let r be a negative integer. Then for every pair S, T of finite sets of places of K satisfying $\text{Hyp}(S, T)$ we have $\theta_S^T(r) \in \text{Ann}_{\mathbb{Z}[G]}(K_{-2r}(\mathcal{O}_{L,S}))$.

Building upon work of Dasgupta and Kakde [DK23] it has recently been shown by Johnston and the second named author that the Coates–Sinnott conjecture holds away from its 2-primary part.

Theorem 4.7 ([JN21, Corollary 12.6]). *Let L/K be a finite abelian extension of number fields with Galois group G . Let r be a negative integer. Then for every pair S, T of finite sets of places of K satisfying $\text{Hyp}(S, T)$ we have $\theta_S^T(r) \in \text{Ann}_{\mathbb{Z}[\frac{1}{2}][G]}(\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} K_{-2r}(\mathcal{O}_{L,S}))$.*

The following is a non-abelian generalisation of Conjecture 4.6 and is a consequence of [Nic11a, Conjecture 2.11] away from its 2-primary part.

Conjecture 4.8. Let L/K be a finite Galois extension of number fields with Galois group G . Let r be a negative integer. Then for every pair S, T of finite sets of places of K satisfying $\text{Hyp}(S, T)$ we have

$$\mathcal{H}(G) \cdot \theta_S^T(r) \subseteq \text{Ann}_{\zeta(\mathbb{Z}[G])}(K_{-2r}(\mathcal{O}_{L,S})).$$

Let p be an odd prime and suppose in addition that S contains the set S_p of all p -adic places of K . For any negative integer r and $i = 0, 1$ Soulé [Sou79] has constructed canonical G -equivariant p -adic Chern class maps

$$(4.3) \quad \mathbb{Z}_p \otimes_{\mathbb{Z}} K_{i-2r}(\mathcal{O}_{L,S}) \longrightarrow H_{\text{ét}}^{2-i}(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)).$$

Soulé proved surjectivity and by the norm residue isomorphism theorem [Wei09] (formerly known as the Quillen–Lichtenbaum Conjecture) these maps are actually isomorphisms. Hence the p -parts of Conjectures 4.6 and 4.8 can be formulated in terms of étale cohomology. As in the case of Brumer’s conjecture we will consider a strengthening as follows. For each $n \in \mathbb{Z}$ define a complex

$$R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(n)) := \text{cone}(R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(n)) \rightarrow \bigoplus_{w \in T(K)} R\Gamma(L(w), \mathbb{Z}_p(n)))[-1]$$

in $\mathcal{D}(\mathbb{Z}_p[G])$. Note that the associated long exact sequence in cohomology gives rise to the p -completion of sequence (4.2) if $n = 1$. We denote the cohomology group of this complex in degree i by $H_T^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(n))$. Let $n > 1$. Then these groups vanish unless $i = 1$ or $i = 2$ by [Nic13, Lemma 5.2]; moreover, the second cohomology group $H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(n))$ is finite.

The following result is the (p -adic) analogue of Theorem 4.4 and has essentially been shown by Johnston and the second named author in the course of the proof of Theorem 4.7.

Theorem 4.9. *Let L/K be a finite abelian extension of number fields with Galois group G . Let r be a negative integer and let p be an odd prime. Let S and T be two finite non-empty sets of places of K such that S contains $S_{\infty} \cup S_{\text{ram}} \cup S_p$ and $S \cap T = \emptyset$. Then we have $\theta_S^T(r) \in \text{Ann}_{\mathbb{Z}_p[G]}(H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)))$.*

Proof. We actually show the stronger statement $\theta_S^T(r) \in \text{Fitt}_{\mathbb{Z}_p[G]}(H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)))$, where $\text{Fitt}_R(M)$ denotes the initial Fitting ideal of a finitely presented R -module M . By [Nic13, Lemma 5.8] we may assume that L/K is a CM extension, in which case the result follows from [JN21, Theorem 12.5]. \square

The following conjecture generalises Theorem 4.9 to the non-abelian case and is the (p -adic) analogue of Conjecture 4.3. It is apparently implied by [Nic13, Conjecture 5.1].

Conjecture 4.10. Let L/K be a finite Galois extension of number fields with Galois group G . Let r be a negative integer and let p be an odd prime. Let S and T be two finite non-empty sets of places of K such that S contains $S_\infty \cup S_{\text{ram}} \cup S_p$ and $S \cap T = \emptyset$. Then we have

$$\mathcal{H}_p(G) \cdot \theta_S^T(r) \subseteq \text{Ann}_{\zeta(\mathbb{Z}_p[G])}(H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))).$$

4.6. Results. In this subsection we prove the annihilation results from the introduction. More generally, we show the following.

Theorem 4.11. *Let L/K be a Galois extension of number fields such that $G = \text{Gal}(L/K)$ is nilpotent. Let S, T be a pair of finite sets of places of K such that $\text{Hyp}(S, T)$ is satisfied. Let $r \leq 0$ be an integer. Then we have*

$$|G'| \theta_S^T(r) \in \zeta(\mathbb{Z}[G])$$

and moreover the following holds.

- (i) If $r = 0$, then $|G'| \theta_S^T(0)$ annihilates $\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \text{cl}_L^T$.
- (ii) If $r < 0$, then $|G'| \theta_S^T(r)$ annihilates $\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} K_{-2r}(\mathcal{O}_{L,S})$.
- (iii) Let p be an odd prime and assume in addition that $S_p \subseteq S$. If $r < 0$, then $|G'| \theta_S^T(r)$ annihilates $H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$.

Proof. Let N be a normal subgroup of G containing the commutator subgroup G' of G . Let χ be an irreducible complex-valued character of G and let η be an irreducible constituent of $\text{res}_N^G \chi$ so that $\text{res}_N^G \chi = m \sum_{x \in G/G_\eta} x \eta$ for some positive integer m . Let $\mathbb{Q}(\text{res}_N^G \chi) := \mathbb{Q}(\chi(n) \mid n \in N)$ which is a finite abelian extension of \mathbb{Q} . We denote its Galois group by $\mathcal{A}_{\chi, N}$. Recall from (3.2) that we have defined an idempotent

$$e(\eta) = \sum_{x \in G/G_\eta} e_{x\eta} = \frac{\eta(1)}{m|N|} \sum_{g \in N} \chi(g^{-1})g \in \mathbb{Q}(\text{res}_N^G \chi)[N]$$

which is actually central in $\mathbb{Q}(\text{res}_N^G \chi)[G]$. Now let χ' be a second irreducible character of G . We say that χ' is equivalent to χ if there is a $\sigma \in \mathcal{A}_{\chi, N}$ such that $\text{res}_N^G \chi' = \sigma(\text{res}_N^G \chi)$. Then $\mathcal{A}_{\chi, N} = \mathcal{A}_{\chi', N}$ and $G_\eta = G_{\eta'}$ for every irreducible constituent η' of $\text{res}_N^G \chi'$. We set

$$\epsilon_\chi := \sum_{\chi'} e_{\chi'} = \sum_{\sigma \in \mathcal{A}_{\chi, N}} \sigma(e(\eta)) \in \mathbb{Q}[N]$$

where the first sum runs over all χ' which are equivalent to χ , and the equality follows from Corollary 2.4 (i).

We now show by induction on the group order that $|N| \theta_S^T(r) \epsilon_\chi$ is contained in $\zeta(\mathbb{Z}[G])$ and annihilates the Galois modules occurring in claims (i), (ii) and (iii). Taking $N = G'$ then finishes the proof. If G is abelian, then $|N| \epsilon_\chi \in \mathbb{Z}[G]$ and the claims follow from Theorems 4.4, 4.7 and 4.9, respectively.

Now let G be arbitrary. We first follow the strategy of the proof of Theorem 3.2. Let F be a splitting field for G and all of its subgroups. Recall the embedding ι_η from (3.5). For

each $\sigma \in \mathcal{A}_{\chi, N}$ we likewise have an embedding $\iota_\eta^\sigma : \zeta(F[G]\sigma(e(\eta))) \hookrightarrow \zeta(F[G_\eta]\sigma(e(\eta)))$. Their product then induces an embedding

$$\iota_\chi : \zeta(F[G]\epsilon_\chi) \hookrightarrow \zeta(F[G_\eta]\epsilon_\chi)$$

which is actually the canonical inclusion: The coefficient of an element on the left hand side at $g \in G$ equals 0 whenever $g \notin G_\eta$. Let ψ' be an irreducible character of G_η such that $e_{\psi'}$ divides ϵ_χ . Then Corollary 2.4 implies that $\chi' := \text{ind}_{G_\eta}^G \psi'$ is an irreducible character of G which is equivalent to χ (recall that in the notation of the Corollary we have that $\chi = \text{ind}_{G_\eta}^G \psi = \text{ind}_{G_\eta}^G x \psi$ for each $x \in G$). We set $K_\eta := L^{G_\eta}$ and let S_η and T_η be the sets of places of K_η lying above those in S and T , respectively. Since $L_S(r, \check{\chi}') = L_{S_\eta}(r, \check{\psi}')$ and likewise $\delta_T(r, \chi') = \delta_{T_\eta}(r, \psi')$, we have

$$\iota_\chi(\theta_S^T(L/K, r)\epsilon_\chi) = \theta_{S_\eta}^{T_\eta}(L/K_\eta, r)\epsilon_\chi.$$

As we may consider each G -module as G_η -module by restriction, we may and do assume that $G = G_\eta$. Hence $\chi = \psi$ is an irreducible character of $G = G_\eta$ and we consider the subgroup U_ψ , which has been defined before Lemma 2.5. We may generalize the second step of the proof of Theorem 3.2 just as above and assume that $G = U_\psi$.

If $\chi = \psi$ is linear, then so is $\eta = \text{res}_N^G \psi$. We consider the factor groups $\overline{G} := G/G'$ and $\overline{N} := N/G'$. Then $\chi = \text{infl}_{\overline{G}}^G \bar{\chi}$ and $\eta = \text{infl}_{\overline{N}}^N \bar{\eta}$ for (linear) characters $\bar{\chi}$ and $\bar{\eta}$ of \overline{G} and \overline{N} , respectively. The natural projection $\mathbb{Q}[G] \rightarrow \mathbb{Q}[\overline{G}]$ is split by the map $\phi : 1 \mapsto |G'|^{-1} \text{Tr}_{G'}$. In particular, the image of $\mathbb{Z}[\overline{G}]$ under ϕ is contained in $|G'|^{-1} \mathbb{Z}[G]$. Since \overline{G} is abelian, we know that $|\overline{N}| \theta_S^T(L^{G'}/K, r) \epsilon_{\bar{\chi}} \in \mathbb{Z}[\overline{G}]$, where $L^{G'}$ is the maximal abelian subextension of K in L . Thus we obtain that

$$|N| \theta_S^T(L/K, r) \epsilon_\chi = |G'| \phi(|\overline{N}| \theta_S^T(L^{G'}/K, r) \epsilon_{\bar{\chi}}) \in \mathbb{Z}[G].$$

Multiplication by $|N| \theta_S^T(L/K, 0) \epsilon_\chi = |\overline{N}| \theta_S^T(L/K, 0) \epsilon_{\bar{\chi}} \text{Tr}_{G'}$ on cl_L^T factors as

$$\text{cl}_L^T \xrightarrow{\text{Tr}_{G'}} \text{cl}_{L^{G'}}^T \xrightarrow{|\overline{N}| \theta_S^T(L^{G'}/K, 0) \epsilon_{\bar{\chi}}} \text{cl}_{L^{G'}}^T \longrightarrow \text{cl}_L^T,$$

where the first and last map are induced by the usual trace and inclusion maps on ideals. Since the middle arrow annihilates $\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \text{cl}_{L^{G'}}^T$ by Theorem 4.4, we are done in case (i). We now consider case (iii) and for this purpose we temporarily assume that S contains S_p . By [Nic13, (22)] there is a natural isomorphism of $\mathbb{Z}_p[\overline{G}]$ -modules

$$H_T^2(\mathcal{O}_{L, S}, \mathbb{Z}_p(1-r))_{G'} \simeq H_T^2(\mathcal{O}_{L^{G'}, S}, \mathbb{Z}_p(1-r)).$$

Since multiplication by $\text{Tr}_{G'}$ factors through the quotient $H_T^2(\mathcal{O}_{L, S}, \mathbb{Z}_p(1-r))_{G'}$ and $|\overline{N}| \theta_S^T(L^{G'}/K, r) \epsilon_{\bar{\chi}}$ annihilates $H_T^2(\mathcal{O}_{L^{G'}, S}, \mathbb{Z}_p(1-r))$ by Theorem 4.9, this shows the claim in case (iii). By (4.3) and the fact that $(1 - (Nv)^r \sigma_w) \in \mathbb{Z}_p[G]^\times$ for $v \in S_p$ and $r < 0$ (see the proof of [GP15, Lemma 6.13], for instance), this also implies (ii).

We are left with the case where $\chi = \psi$ is non-linear. By [CR81, Theorem 11.3] there is a necessarily proper subgroup \tilde{H} of G and a linear character $\tilde{\lambda}$ of \tilde{H} such that $\psi = \text{ind}_{\tilde{H}}^G \tilde{\lambda}$. By [Rob96, 5.2.4] there is a proper *normal* subgroup H containing \tilde{H} . We set $\lambda := \text{ind}_{\tilde{H}}^H \tilde{\lambda}$ so that $\psi = \text{ind}_H^G \lambda$ and we are in the situation of §2.3.

Since $G = G_\eta$ we have $e(\eta) = e_\eta$ which is a primitive central idempotent in $F[N]$ and which is still central in $F[G]$. As $G = U_\psi$ we have a decomposition into primitive central idempotents $e_\eta = \sum_{\omega \in \text{Irr}(G/N)} e_{\psi \otimes \omega}$ in $F[G]$ by Proposition 2.3 and Corollary 2.4 (ii). Moreover, the idempotent e_η is indeed in $F[H]$, where we have a decomposition into

primitive central idempotents $e_\eta = \sum_{c \in G/H} \sum_{\omega \in \text{Irr}(G/N)} e_{c\lambda \otimes \omega}$ by Lemma 2.10 (ii). We obtain the diagonal embedding

$$\iota_\eta^H : \zeta(F[G]e_\eta) = \bigoplus_{\omega \in \text{Irr}(G/N)} F e_{\psi \otimes \omega} \hookrightarrow \bigoplus_{\omega \in \text{Irr}(G/N)} \bigoplus_{c \in G/H} F e_{c\lambda \otimes \omega} = \zeta(F[H]e_\eta).$$

As above we have a similar embedding for $\sigma(e_\eta)$ for each $\sigma \in \mathcal{A}_{\chi,N}$ so that the product over all σ induces an embedding

$$\iota_\chi^H : \zeta(F[G]\epsilon_\chi) \hookrightarrow \zeta(F[H]\epsilon_\chi).$$

By Lemma 2.9 (i) we have that $\text{ind}_H^G({}^c\lambda \otimes \omega) = \text{ind}_H^G(\lambda \otimes \omega) = \psi \otimes \omega$ for each $c \in G/H$, $\omega \in \text{Irr}(G/N)$. Since the L -functions behave well under induction of characters, we see that

$$\iota_\chi^H(\theta_S^T(L/K, r)\epsilon_\chi) = \theta_{S(L^H)}^{T(L^H)}(L/L^H, r)\epsilon_\chi.$$

Note that the inclusion ι_χ^H actually leaves any element z in $\zeta(F[G]\epsilon_\chi)$ unchanged; the coefficients of z at any $g \in G \setminus H$ must vanish. We may therefore view our G -modules as H -modules by restriction and replace the triple (G, N, χ) by the triples $(H, N \cap H, {}^c\lambda)$, $c \in G/H$. Since H is a proper subgroup of G and ϵ_χ is a sum of analogously defined idempotents $\epsilon_{c\lambda}$, where c runs over a possibly proper subset of G/H , this completes the proof. \square

Recall that E_d was defined as the sum over all primitive central idempotents e_χ associated to irreducible characters of G of degree d . Then E_d may be written as a sum of idempotents of the form ϵ_χ and the proof of Theorem 4.11 indeed shows the following.

Corollary 4.12. *Let L/K be a Galois extension of number fields such that $G = \text{Gal}(L/K)$ is nilpotent. Let S, T be a pair of finite sets of places of K such that $\text{Hyp}(S, T)$ is satisfied. Let $r \leq 0$ and $d > 0$ be integers. Then we have*

$$|G'| \theta_S^T(r) E_d \in \zeta(\mathbb{Z}[G])$$

and moreover the following holds.

- (i) If $r = 0$, then $|G'| \theta_S^T(0) E_d$ annihilates $\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \text{cl}_L^T$.
- (ii) If $r < 0$, then $|G'| \theta_S^T(r) E_d$ annihilates $\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} K_{-2r}(\mathcal{O}_{L,S})$.
- (iii) Let p be an odd prime and assume in addition that $S_p \subseteq S$. If $r < 0$, then $|G'| \theta_S^T(r) E_d$ annihilates $H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$.

Remark 4.13. Let N be a normal subgroup of G containing G' . Similar to Remark 3.5, one can show that indeed

$$\left(\frac{|N|}{\eta(1)} \epsilon_\chi \right) \theta_S^T(r) \in \zeta(\mathbb{Z}[G])$$

and annihilates the corresponding Galois module. This follows along the lines of the proof of Theorem 4.11. One only has to observe in the last step of the proof that $|N|/\eta(1) = |N \cap H|/\eta_c(1)$ for all $c \in G/H$, as $\eta(1) = [G : H]\eta_c(1)$ and $H/(H \cap N) \simeq G/N$ by Lemma 2.9.

5. THE RELATION TO THE EQUIVARIANT TAMAGAWA NUMBER CONJECTURE

5.1. Algebraic K -theory. We briefly recall some basic notions from algebraic K -theory. For more details we refer the reader to [CR87] and [Swa68].

Let R be any ring. Recall that $\text{PMod}(R)$ denotes the category of finitely generated projective left R -modules. We write $K_0(R)$ for the Grothendieck group of $\text{PMod}(R)$ and $K_1(R)$ for the Whitehead group (see [CR87, §38 and §40]).

Let \mathcal{O} be a noetherian integral domain of characteristic 0 with field of fractions F . Let A be a finite-dimensional semisimple F -algebra and let Λ be an \mathcal{O} -order in A . For any field extension E of F we set $A_E := E \otimes_F A$. Let $K_0(\Lambda, E)$ denote the relative algebraic K -group associated to the ring homomorphism $\Lambda \hookrightarrow A_E$. We recall that $K_0(\Lambda, E)$ is an abelian group with generators $[X, g, Y]$ where X and Y are finitely generated projective Λ -modules and $g : E \otimes_R X \rightarrow E \otimes_R Y$ is an isomorphism of A_E -modules; for a full description in terms of generators and relations, we refer the reader to [Swa68, p. 215]. Furthermore, there is a long exact sequence of relative K -theory (see [Swa68, Chapter 15])

$$(5.1) \quad K_1(\Lambda) \longrightarrow K_1(A_E) \xrightarrow{\partial_{\Lambda, E}} K_0(\Lambda, E) \longrightarrow K_0(\Lambda) \longrightarrow K_0(A_E).$$

Let p be a prime and let G be a finite group. Let F be a finite extension of \mathbb{Q}_p with ring of integers $\mathcal{O} = \mathcal{O}_F$ and let $e \in \mathcal{O}[G]$ be a central idempotent. It follows from a theorem of Swan [CR81, Theorem 32.1] that the map $K_0(e\mathcal{O}[G]) \rightarrow K_0(eF[G])$ is injective. Hence from (5.1) we obtain a right-exact sequence

$$(5.2) \quad K_1(e\mathcal{O}[G]) \rightarrow K_1(eF[G]) \rightarrow K_0(e\mathcal{O}[G], F) \rightarrow 0.$$

Moreover, by [CR87, Theorems 45.3 and 40.31] the reduced norm induces an isomorphism $K_1(eF[G]) \simeq \zeta(eF[G])^\times$ and $\text{nr}(K_1(e\mathcal{O}[G])) = \text{nr}(e\mathcal{O}[G]^\times)$. So from (5.2) we obtain an isomorphism

$$(5.3) \quad K_0(e\mathcal{O}[G], F) \simeq \zeta(eF[G])^\times / \text{nr}(e\mathcal{O}[G]^\times).$$

Choose a maximal \mathcal{O} -order $\mathcal{M}(G)$ in $F[G]$ containing $\mathcal{O}[G]$. Then a similar reasoning using [CR87, Proposition 45.8] yields an isomorphism

$$(5.4) \quad K_0(e\mathcal{M}(G), F) \simeq \zeta(eF[G])^\times / \zeta(e\mathcal{M}(G))^\times.$$

Now let E be a finite Galois extension of F . We let $\sigma \in \text{Gal}(E/F)$ act on $x = \sum_{g \in G} x_g g \in E[G]$ by $\sigma(x) = \sum_{g \in G} \sigma(x_g)g$.

Lemma 5.1. *Let p be a prime and let G be a finite p -group. Let F be a finite extension of \mathbb{Q}_p and let E be a finite unramified extension of F such that p and $[E : F]$ are coprime. Then the norm map*

$$N_{E/F} : \mathcal{O}_E[G]^\times \rightarrow \mathcal{O}_F[G]^\times, \quad x \mapsto \prod_{\sigma \in \text{Gal}(E/F)} \sigma(x)$$

is surjective.

Proof. We denote the radical of a ring Λ by $\text{rad}(\Lambda)$. Let \mathbb{E} and \mathbb{F} be the residue fields of E and F , respectively. By [CR81, Corollary 5.25] the radical of $\mathcal{O}_F[G]$ is generated by a uniformizer of \mathcal{O}_F and the augmentation ideal, which is the kernel of the natural \mathcal{O}_F -linear map $\mathcal{O}_F[G] \rightarrow \mathcal{O}_F$ that sends each $g \in G$ to 1. In particular, we have an isomorphism $\mathcal{O}_F[G]/\text{rad}(\mathcal{O}_F[G]) \simeq \mathbb{F}$. Similar observations hold with F replaced by E .

Since an element is invertible if and only if it is invertible modulo the radical, we obtain a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & 1 + \text{rad}(\mathcal{O}_E[G]) & \longrightarrow & \mathcal{O}_E[G]^\times & \longrightarrow & \mathbb{E}^\times \longrightarrow 1 \\ & & \downarrow N_{E/K} & & \downarrow N_{E/K} & & \downarrow N_{E/K} \\ 1 & \longrightarrow & 1 + \text{rad}(\mathcal{O}_F[G]) & \longrightarrow & \mathcal{O}_F[G]^\times & \longrightarrow & \mathbb{F}^\times \longrightarrow 1. \end{array}$$

As E/F is unramified, the rightmost vertical map is surjective. Since $N_{E/K}(1+x) = (1+x)^{[E:F]}$ for $x \in \text{rad}(\mathcal{O}_F[G])$ and $1 + \text{rad}(\mathcal{O}_F[G])$ is a pro- p -group (see [CR87, Example 45.29]), the leftmost vertical map is also surjective. The result follows. \square

Corollary 5.2. *Let p be a prime and let G be a finite abelian p -group. Let F be a finite extension of \mathbb{Q}_p and let D be a skewfield with centre F such that $[D : F]$ is finite and coprime to p . Let \mathcal{O}_D be the unique maximal \mathcal{O}_F -order in D . Then we have an equality*

$$\text{nr}(K_1(\mathcal{O}_D[G])) = \mathcal{O}_F[G]^\times.$$

Proof. Let s be the Schur index of D so that $s^2 = [D : F]$. Then s is prime to p by assumption. By [Rei03, §14] there is an unramified extension E of F of degree s , which is contained in D . Then E is a maximal subfield of D and splits D . Moreover, the ring of integers \mathcal{O}_E is contained in \mathcal{O}_D and the matrix representation that occurs in the proof of [Rei03, Theorem 14.6] (see [Rei03, (14.7)]) shows that $\text{nr}(x) = N_{E/F}(x)$ for $x \in E$. More precisely, the matrix corresponding to x is diagonal with the Galois conjugates of x as entries. We have isomorphisms of rings

$$(5.5) \quad E \otimes_F D[G] \simeq (E \otimes_F D)[G] \simeq M_s(E)[G] \simeq M_s(E[G]),$$

where the second isomorphism is induced by just that matrix representation. Let $\sigma \in \text{Gal}(E/F)$ be the unique lift of the Frobenius automorphism. Choose $1 \leq r \leq s$ such that r/s is the Hasse invariant of D . Then r is prime to s and $\theta := \sigma^r$ is also a generator of $\text{Gal}(E/F)$. Then θ acts upon $a \otimes x$, $a \in E$, $x \in D[G]$ by $\theta(a \otimes x) = \theta(a) \otimes x$. If $A = (\alpha_{i,j})_{1 \leq i,j \leq s}$ is a matrix in $M_s(E[G])$, then we let θ act on A by $\theta(A) = (\beta_{i,j})_{i,j}$ with $\beta_{i,j} = \theta(\alpha_{i-1,j-1})$, where the indices are understood to be considered modulo s . By inspection of [Rei03, (14.7)] the composite map (5.5) is then $\text{Gal}(E/F)$ -equivariant. Let $x \in \mathcal{O}_D[G]^\times$. Then the image A_x of $1 \otimes x$ under (5.5) is invariant under the action of $\text{Gal}(E/F)$ and a matrix with entries in $\mathcal{O}_E[G]$. Hence

$$\text{nr}(x) = \det(A_x) \in (\mathcal{O}_E[G]^\times)^{\text{Gal}(E/F)} = \mathcal{O}_F[G]^\times$$

so that $\text{nr}(K_1(\mathcal{O}_D[G])) = \text{nr}(\mathcal{O}_D[G]^\times)$ is contained in $\mathcal{O}_F[G]^\times$. For $x \in \mathcal{O}_E[G]$ its image A_x is a diagonal matrix whose entries are again the Galois conjugates of x . Hence $\text{nr}(x) = N_{E/K}(x)$ and the result follows from Lemma 5.1. \square

5.2. The relation to the equivariant Tamagawa number conjecture. Let L/K be a Galois extension of number fields with Galois group $G = \text{Gal}(L/K)$ and let r be an integer. The equivariant Tamagawa number conjecture (ETNC for short) for the pair $(h^0(\text{Spec}(L))(r), \mathbb{Z}[G])$ has been formulated by Burns and Flach [BF01] and asserts that a certain canonical element $T\Omega(L/K, \mathbb{Z}[G], r)$ in the relative algebraic K -group $K_0(\mathbb{Z}[G], \mathbb{R})$ vanishes. We refrain from giving a more precise statement, as we will only exploit known cases and functorial properties of the conjecture.

We henceforth assume that L/K is a CM extension. So L is a totally complex extension of the totally real field K such that complex conjugation induces a unique automorphism τ of L which is central in G . Note that this is the essential case for the integrality and annihilation conjectures considered in this article (see [Nic16, §4], for instance).

We define a central idempotent $e_r := \frac{1-(-1)^r\tau}{2}$ in $\mathbb{Z}[\frac{1}{2}][G]$, where $\tau \in G$ denotes complex conjugation. The ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathbb{Z}[\frac{1}{2}][G])$ then likewise asserts that a canonical element $T\Omega(L/K, r)$ in $K_0(e_r\mathbb{Z}[\frac{1}{2}][G], \mathbb{R})$ vanishes. (It would be more accurate to denote $T\Omega(L/K, r)$ by $T\Omega(L/K, e_r\mathbb{Z}[\frac{1}{2}][G], r)$, but in order to avoid too heavy notation we will drop the reference to the order $e_r\mathbb{Z}[\frac{1}{2}][G]$.) This corresponds to the plus or minus part of the ETNC (away from 2) if r is odd or even, respectively.

Now assume that $r \leq 0$. Then a result of Siegel [Sie70] implies that $T\Omega(L/K, r)$ actually belongs to the subgroup

$$(5.6) \quad K_0(e_r\mathbb{Z}[\tfrac{1}{2}][G], \mathbb{Q}) \simeq \bigoplus_{p \text{ odd}} K_0(e_r\mathbb{Z}_p[G], \mathbb{Q}_p)$$

and we say that the p -part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathbb{Z}[\tfrac{1}{2}][G])$ holds if its image $T\Omega_p(L/K, r)$ in $K_0(e_r\mathbb{Z}_p[G], \mathbb{Q}_p)$ vanishes. More generally, if \mathcal{M} is an order containing $e_r\mathbb{Z}[\tfrac{1}{2}][G]$, we say that the p -part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), \mathcal{M})$ holds if $T\Omega_p(L/K, r)$ is contained in the kernel of the natural map $K_0(e_r\mathbb{Z}_p[G], \mathbb{Q}_p) \rightarrow K_0(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{M}, \mathbb{Q}_p)$.

5.3. Functorialities. By [BF01, Theorem 4.1] the ETNC behaves well under base change. In particular, the following holds in our situation.

Let H be a subgroup of G which contains τ . Then there is a natural restriction map

$$\text{res}_H^G : K_0(e_r\mathbb{Z}[\tfrac{1}{2}][G], \mathbb{R}) \rightarrow K_0(e_r\mathbb{Z}[\tfrac{1}{2}][H], \mathbb{R}).$$

such that for every $r \in \mathbb{Z}$ one has

$$\text{res}_H^G(T\Omega(L/K, r)) = T\Omega(L/L^H, r).$$

Likewise, if N is a normal subgroup of G which does not contain τ , then there is a natural quotient map

$$\text{quot}_{G/N}^G : K_0(e_r\mathbb{Z}[\tfrac{1}{2}][G], \mathbb{R}) \rightarrow K_0(e_r\mathbb{Z}[\tfrac{1}{2}][G/N], \mathbb{R})$$

such that

$$\text{quot}_{G/N}^G(T\Omega(L/K, r)) = T\Omega(L^N/K, r).$$

If $r \leq 0$ and p is an odd prime, then similar observations hold with $T\Omega(L/K, r)$ replaced by $T\Omega_p(L/K, r)$.

5.4. Results on the ETNC. The following result and variants thereof will be important for our purposes. Note that parts (i) and (ii) are essentially known, but part (iii) is new and generalizes [Nic16, Theorem 1.2].

Theorem 5.3. *Let L/K be a Galois CM extension of number fields with Galois group G . Let p be an odd prime and let $r \leq 0$ be an integer. Let S and T be two finite sets of places of K such that $\text{Hyp}(S, T)$ is satisfied. Then the following holds.*

- (i) *If the p -part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathbb{Z}[\tfrac{1}{2}][G])$ holds, then $\theta_S^T(L/K, r) \in \mathcal{I}_p(G)$. In particular, we then have that $|G'|\theta_S^T(L/K, r) \in \zeta(\mathbb{Z}_p[G])$.*
- (ii) *Let $\mathcal{M}(G)$ be a maximal order containing $\mathbb{Z}[\tfrac{1}{2}][G]$. Then the p -part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathcal{M}(G))$ holds. In particular, we have $\theta_S^T(L/K, r) \in \zeta(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{M}(G))$ and hence $|G|\theta_S^T(L/K, r) \in \zeta(\mathbb{Z}_p[G])$.*
- (iii) *Assume that G decomposes into a direct product $G = N \times \Delta$ of a finite group N and a finite abelian p -group Δ . Let $\mathcal{M}(N)$ be a maximal order containing $\mathbb{Z}[\tfrac{1}{2}][N]$. Then the p -part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathcal{M}(N)[\Delta])$ holds. In particular, we have $\theta_S^T(L/K, r) \in \zeta(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{M}(N))[\Delta]$ and hence $|N|\theta_S^T(L/K, r) \in \zeta(\mathbb{Z}_p[G])$.*

Proof. If the p -power roots of unity $\mu_L(p)$ in L form a cohomologically trivial G -module, then the first claim in (i) is due to the second named author [Nic11b, Theorem 5.1]; this includes the cases $\mu_L(p) = 1$ and $p \nmid |G|$. The general case follows from more recent work of Burns [Bur20, Proof of Corollary 3.11 (iii)]. The final claim is a consequence of Theorem 1.

For (ii) we first observe that it suffices to prove the p -part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathcal{M}(G))$. This follows from (the proofs of) [Nic11b, Theorem 4.1] if $r = 0$ and [Nic11a, Theorem 5.7] if $r < 0$. The relevant special cases of the ETNC indeed hold as follows from work of the second named author [Nic24, Theorem 1] if $r = 0$ and of Burns [Bur15, Corollary 2.10] if $r < 0$ (see also [JN21, Theorem 12.3 (i)] for details).

For part (iii) it again suffices to show the claim on the ETNC. The final assertion can then be deduced as in (i) and (ii). If G is abelian, the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathbb{Z}[\frac{1}{2}][G])$ holds unconditionally by recent work of Bullach, Burns, Daoud and Seo [BBDS21] if $r = 0$ and of Johnston and the second named author [JN21, Theorem 1.3] if $r < 0$. From this we now deduce the p -part of the ETNC over the order $e_r\mathcal{M}(N)[\Delta]$ by using Brauer induction as follows. Since Δ is a p -group for an odd prime p , we have $\tau \in N$. We then have a decomposition

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} e_r\mathcal{M}(N) \simeq \prod_{\chi} \mathcal{M}_p(\chi)$$

where the product runs over all irreducible odd (resp. even) characters of N up to Galois action if r is even (resp. odd). Each $\mathcal{M}_p(\chi)$ is a maximal \mathbb{Z}_p -order with centre $\mathbb{Z}_p(\chi)$, the ring of integers in $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(n) \mid n \in N)$, in a central simple $\mathbb{Q}_p(\chi)$ -algebra. The latter is a matrix ring over a skewfield whose Schur index is divisible by $(p-1)$ by [Yam71, Theorem 1] and thus in particular is prime to p . By Corollary 5.2 the above decomposition and the reduced norm induce isomorphisms

$$K_0(\mathbb{Z}_p \otimes_{\mathbb{Z}} e_r\mathcal{M}(N)[\Delta], \mathbb{Q}_p) \simeq \prod_{\chi} K_0(\mathcal{M}_p(\chi)[\Delta], \mathbb{Q}_p) \simeq \prod_{\chi} \mathbb{Q}_p(\chi)[\Delta]^{\times} / \mathbb{Z}_p(\chi)[\Delta]^{\times}.$$

Now let U be a subgroup of N containing τ . Then there is a natural restriction map

$$K_0(\mathbb{Z}_p \otimes_{\mathbb{Z}} e_r\mathcal{M}(N)[\Delta], \mathbb{Q}_p) \rightarrow K_0(\mathbb{Z}_p \otimes_{\mathbb{Z}} e_r\mathcal{M}(U)[\Delta], \mathbb{Q}_p)$$

that maps a tuple $(f_{\chi})_{\chi}$ to the tuple $(g_{\psi})_{\psi} \in \prod_{\psi} \mathbb{Q}_p(\psi)[\Delta]^{\times} / \mathbb{Z}_p(\psi)[\Delta]^{\times}$, where the product is over all irreducible characters ψ of U of the same parity as χ modulo Galois action, and

$$g_{\psi} = \prod_{\chi} f_{\chi}^{\langle \chi, \text{ind}_{\tilde{U}}^N \psi \rangle_N}$$

by a (variant of) Lemma 3.1. Let us fix one of the characters χ . By Brauer's induction theorem [CR81, Theorem 15.9] there are integers z_i and linear characters λ_i of subgroups U_i of N such that

$$(5.7) \quad \chi = \sum_i z_i \text{ind}_{U_i}^N \lambda_i.$$

We may assume that $\tau \in U_i$ for all i . To see this assume that $\tau \notin U_i$ and set $\tilde{U}_i := U_i \times \langle \tau \rangle$. We write $\text{ind}_{U_i}^N \lambda_i = \text{ind}_{\tilde{U}_i}^N (\text{ind}_{U_i}^{\tilde{U}_i} \lambda_i)$ and observe that $\text{ind}_{\tilde{U}_i}^N \lambda_i$ is the sum of two linear characters. Since the parity of a character is preserved under induction, we may assume in addition that all λ_i are of the same parity as χ . We compute

$$f_{\chi} = \prod_{\chi'} f_{\chi'}^{\langle \chi', \chi \rangle_N} = \prod_i \prod_{\chi'} f_{\chi'}^{z_i \langle \chi', \text{ind}_{U_i}^N \lambda_i \rangle_N} = \prod_i g_{\lambda_i}^{z_i}.$$

If all g_{λ_i} are in $\mathbb{Z}_p(\lambda_i)[\Delta]^{\times}$, then f_{χ} must belong to $\mathbb{Z}_p(\chi)[\Delta]^{\times}$. Since the relevant special case of the ETNC holds for abelian CM extensions, the functorial behaviour of the ETNC, as discussed in §5.3, implies the result. \square

This theorem turns out to be sufficient in many, but not all cases of interest to us. In order to handle the general case, we need a further variant in the case where G decomposes into a semi-direct product $G = N \rtimes \Delta$, where Δ is a cyclic p -group. We start with some technical results which are inspired by work of Ritter and Weiss [RW04].

Lemma 5.4. *Let G be a finite group. Let N be a normal subgroup of G such that $\Delta := G/N$ is a cyclic group. Choose a generator $\bar{\delta}$ of Δ . Let F be a field of characteristic 0 such that F is a splitting field for G and all of its subgroups. Let V_χ be an irreducible $F[G]$ -module with character χ and let η be an irreducible constituent of $\text{res}_N^G \chi$. Then the following holds.*

- (i) *The idempotent $e(\eta) := \sum_{x \in G/G_\eta} e_{x\eta} \in F[N]$ is central in $F[G]$ and there exists a unique $\delta_\chi \in \zeta(F[G]e(\eta))$ with all of the following properties.*
 - (a) *We have $\delta_\chi = \delta^{w_\chi} \cdot c = c \cdot \delta^{w_\chi}$, where $\delta \in G$ maps to $\bar{\delta}$ under the canonical map $G \rightarrow G/N = \Delta$, $c \in (F[N]e(\eta))^\times$ and $w_\chi = [G : G_\eta]$.*
 - (b) *The action of δ_χ on V_χ is trivial.*
 - (c) *One has $\zeta(F[G]e(\eta)) = F[\Delta_\chi]$, where Δ_χ is a cyclic group of order $[G_\eta : N]$, generated by δ_χ .*
- (ii) *Let χ' be a second irreducible character of G and η' be an irreducible constituent of $\text{res}_N^G \chi'$. Then $e(\eta) = e(\eta')$ if and only if $\chi' = \chi \otimes \rho$ for a linear character ρ of G such that $N \subset \ker(\rho)$. If this is the case, then $\delta_{\chi \otimes \rho} = \delta_\chi \cdot \rho(\delta)^{-w_\chi}$.*
- (iii) *If F is a finite extension of either \mathbb{Q} or \mathbb{Q}_p for some prime p and H is a matrix with entries in $\mathcal{O}_F[G]e(\eta)$, then $\text{nr}(H) \in \mathcal{O}_F[\Delta_\chi]$.*

Proof. It is clear that $e(\eta)$ is central in G (see also Corollary 2.4 (i)). The existence of δ_χ with properties (a) and (b) can be shown by the argument of Ritter and Weiss [RW04, Proposition 5 (2)]. Note that this actually shows that (a) and (b) determine δ_χ uniquely. We warn the reader that our e_η and $e(\eta)$ are denoted by $e(\eta)$ and e_χ in [RW04], respectively. The main point is that in the notation of §2 we have $m = 1$ and $U_\psi = G_\eta$ by [Hup67, Satz 17.12] because Δ is a cyclic group. For (c) one first has to check that $\delta_\chi^{[G_\eta : N]} = e(\eta)$ once (a) and (b) are established. Here the argument slightly differs from that given in [RW04] and runs as follows.

Since c and δ^{w_χ} commute and $w_\chi \cdot [G_\eta : N] = [G : N] = |\Delta|$, we certainly have that $\delta_\chi^{[G_\eta : N]} = n \cdot c^{[G_\eta : N]}$ for some $n \in N$. In particular, $\delta_\chi^{[G_\eta : N]}$ is in $(F[N]e(\eta))^\times$ and acts trivially upon V_χ . However, the module $F[N]e(\eta)$ is a submodule of some multiple of $\text{res}_N^G V_\chi$ so that the only such element is $e(\eta)$.

The index $[G_\eta : N]$ is actually the minimal positive integer z such that $\delta_\chi^z = e(\eta)$ as the latter equality in particular implies that $\delta^{w_\chi z} \in N$. Hence δ_χ indeed generates a cyclic group of order $[G_\eta : N]$. Now the inclusion $F[\Delta_\chi] \subseteq \zeta(F[G]e(\eta))$ must be an equality as we already know that the right hand side is an F -vector space of dimension $[G_\eta : N]$ by Proposition 2.3 (iii). This finishes the proof of (i).

Part (ii) can be shown along the lines of [RW04, Corollary on p. 556 and Proposition 5 (3)].

To show the final claim, we give a second argument which reproves a large part of (c) and clarifies the relation to our results of §2. We have seen in the proof of Theorem 3.2 that the embedding (3.5) allows us to assume $G = G_\eta$ by Lemma 3.1. Since $G_\eta = U_\psi$, we then have developed the (abstract) isomorphism (3.7). Putting both maps together this gives an abstract isomorphism $\zeta(F[G]e(\eta)) \simeq F[G_\eta/N]$. Note that the quotient G_η/N identifies with the subgroup of Δ generated by $\bar{\delta}^{w_\chi}$, and an isomorphism is explicitly

given by $\delta_\chi \mapsto \bar{\delta}^{w_\chi}$. Finally, equation (3.8) shows that $\text{nr}(H) \in \mathcal{O}_F[G_\eta/N]$ under this identification. \square

Let p be a prime and let G be a finite group with a normal subgroup N such that $\Delta := G/N$ is cyclic. If F/\mathbb{Q}_p is a finite extension that is a splitting field for G and all of its subgroups, then we have canonical maps

$$(5.8) \quad K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \rightarrow K_0(\mathcal{O}_F[G], F) \simeq \zeta(F[G])^\times / \text{nr}(\mathcal{O}_F[G]^\times) \rightarrow \prod_{\chi/\sim} F[\Delta_\chi]^\times / \mathcal{O}_F[\Delta_\chi]^\times.$$

Here, the first map and the isomorphism are induced from extension of scalars and the reduced norm, respectively. The product on the right is over all irreducible characters χ of G modulo the relation $\chi \sim \chi'$ if and only if $\chi' = \chi \otimes \rho$ for a linear character ρ of G whose kernel contains N . Lemma 5.4 (i)(c) implies that

$$\zeta(F[G]) = \prod_{\chi/\sim} F[\Delta_\chi]$$

and the final claim of the same lemma implies that the rightmost arrow is a well-defined epimorphism. Note that if $(f_\chi)_\chi \in \zeta(F[G])$, then replacing a character χ by an equivalent character $\chi \otimes \rho$, changes f_χ to $f_{\chi \otimes \rho} = \rho^\sharp(f_\chi)$, where ρ^\sharp is the F -linear map on $F[\Delta_\chi]$ sending δ_χ^i to $(\rho(\delta)^{-w_\chi} \delta_\chi)^i = \delta_{\chi \otimes \rho}^i$, $i \in \mathbb{Z}$.

We now prove the following variant of Theorem 5.3 (iii).

Proposition 5.5. *Let L/K be a Galois CM extension of number fields with Galois group G . Let N be a normal subgroup of G such that $\Delta := G/N$ is a cyclic group. Let p be an odd prime and let $r \leq 0$ be an integer. Let S and T be two finite sets of places of K such that $\text{Hyp}(S, T)$ is satisfied. Let F/\mathbb{Q}_p be a finite extension that is a splitting field for G and all of its subgroups. Then the following holds.*

- (i) *The image of $T\Omega(L/K, r)$ in $\prod_{\chi/\sim} F[\Delta_\chi]^\times / \mathcal{O}_F[\Delta_\chi]^\times$ under (5.6) and (5.8) vanishes.*
- (ii) *We have that $\theta_S^T(L/K, r) \in (\prod_{\chi/\sim} \mathcal{O}_F[\Delta_\chi]) \cap \zeta(\mathbb{Q}_p[G])$ so that in particular $|N|\theta_S^T(L/K, r) \in \zeta(\mathbb{Z}_p[G])$.*

Proof. Let U be a subgroup of G containing τ . We set $V := U \cap N$ which is a normal subgroup of U . Then $\Sigma := U/V$ naturally identifies with a subgroup of Δ and is therefore cyclic. Consider the canonical restriction map $K_0(\mathcal{O}_F[G], F) \rightarrow K_0(\mathcal{O}_F[U], F)$. We view $K_0(\mathcal{O}_F[G], F)$ as a quotient of $\zeta(F[G])^\times \simeq \prod_{\chi/\sim} F[\Delta_\chi]^\times$ and likewise $K_0(\mathcal{O}_F[U], F)$ as a quotient of $\zeta(F[U])^\times \simeq \prod_{\psi/\sim} F[\Sigma_\psi]^\times$. We now describe the image $(g_\psi)_\psi$ of a tuple $(f_\chi)_\chi$ under restriction. For this let $j_\chi : F[\Delta_\chi] \rightarrow F[\Delta]$ be the injective map which sends δ_χ to $\bar{\delta}^{w_\chi}$. Define $j_\psi : F[\Sigma_\psi] \rightarrow F[\Sigma] \subseteq F[\Delta]$ similarly. Then one has the following equality in $F[\Delta]$ which determines g_ψ uniquely:

$$j_\psi(g_\psi) = \prod_{\chi} j_\chi(f_\chi)^{\langle \chi, \text{ind}_{U_i}^G \psi \rangle_G}$$

This can be shown as [RW04, Lemma 9]. Now fix $\chi \in \text{Irr}(G)$. By Brauer's induction theorem [CR81, Theorem 15.9] there are integers z_i and linear characters λ_i of subgroups U_i of G such that

$$\chi = \sum_i z_i \text{ind}_{U_i}^G \lambda_i.$$

If $\chi' = \chi \otimes \rho$ is a character equivalent to χ , then likewise

$$\chi \otimes \rho = \sum_i z_i (\text{ind}_{U_i}^G \lambda_i) \otimes \rho = \sum_i z_i \text{ind}_{U_i}^G (\lambda_i \otimes \text{res}_{U_i}^G \rho)$$

and each $\lambda_i \otimes \text{res}_{U_i}^G \rho$ is a character equivalent to λ_i . For (i) we now may conclude as in the proof of Theorem 5.3.

By Theorem 5.3 (i) the full p -part of the ETNC implies that the Stickelberger elements belong to

$$\mathcal{I}_p(G) \subseteq \prod_{\chi/\sim} \mathcal{O}_F[\Delta_\chi],$$

where the inclusion follows from Lemma 5.4 (iii). Since we know the ETNC up to a factor in the right hand side, we get (ii). \square

6. IWASAWA THEORY

6.1. Stickelberger elements in cyclotomic \mathbb{Z}_p -extensions. Fix an odd prime p . We write F_∞ for the cyclotomic \mathbb{Z}_p -extension of a number field F and F_n for its n -th layer which is the unique intermediate field of degree $[F_n : F] = p^n$. Let $\Gamma_F := \text{Gal}(F_\infty/F)$ which is (non-canonically) isomorphic to \mathbb{Z}_p . We choose a topological generator γ_F of Γ_F . For a profinite group \mathcal{G} we let $\Lambda(\mathcal{G}) := \mathbb{Z}_p[[\mathcal{G}]] = \varprojlim \mathbb{Z}_p[\mathcal{G}/\mathcal{N}]$, where the inverse limit runs over all open normal subgroups \mathcal{N} of \mathcal{G} .

Let L/K be a Galois CM extension of number fields with Galois group G . We set $G_n := \text{Gal}(L_n/K)$ and $\mathcal{G} := \text{Gal}(L_\infty/K) = \varprojlim_n G_n$. Then \mathcal{G} may be written as a semi-direct product $\mathcal{G} = H \rtimes \Gamma$, where H identifies with a subgroup of G and $\Gamma \simeq \mathbb{Z}_p$. The quotient \mathcal{G}/H naturally identifies with Γ_K . If n is large enough, one likewise has $G_n = H \rtimes \Gamma_n$, where $\Gamma_n \simeq G_n/H = \text{Gal}(L_n \cap K_\infty/K)$ is a finite cyclic p -group.

Now let S and T be two finite sets of places of K such that $\text{Hyp}(S, T)$ is satisfied for all extensions L_n/K , $n \in \mathbb{N}$. In particular, all p -adic places are contained in S . Let $r \leq 0$ be an integer. Then the Stickelberger element $\theta_S^T(L_{n+1}/K, r)$ maps to $\theta_S^T(L_n/K, r)$ under the canonical projection $\mathbb{Q}[G_{n+1}] \rightarrow \mathbb{Q}[G_n]$. Since \mathcal{G}/H is abelian, the commutator subgroup \mathcal{G}' of \mathcal{G} is indeed a subgroup of H and the canonical surjection $\mathcal{G} \rightarrow G_n$ induces an epimorphism $\mathcal{G}' \rightarrow G'_n$ which is an isomorphism for sufficiently large n . If the integrality conjectures hold along the cyclotomic tower, then the denominators of $\theta_S^T(L_n/K, r)$ are bounded by $|\mathcal{G}'|$ as a consequence of Theorem 1. We do not know this in general, but as a consequence of Proposition 5.5 (ii) the following holds.

Theorem 6.1. *Let p be an odd prime and let $r \leq 0$ be an integer. Let L/K be a Galois CM extension of number fields and suppose that $\text{Hyp}(S, T)$ is satisfied for every extension L_n/K . Then the denominators of the Stickelberger elements $\theta_S^T(L_n/K, r) \in \mathbb{Q}_p[G_n]$ are bounded by a constant that does not depend on either n or r .*

This suffices to define

$$(6.1) \quad \theta_S^T(L_\infty/K, r) := \varprojlim_n \theta_S^T(L_n/K, r) \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}).$$

Note that we can use Theorem 5.3 (iii) if $\mathcal{G} = H \times \Gamma$ is a direct product.

6.2. The Iwasawa algebra as an order. Recall the $\mathcal{G} = H \rtimes \Gamma$ is a one-dimensional p -adic Lie group for an odd prime p . If F is a finite field extension of \mathbb{Q}_p with ring of integers $\mathcal{O} = \mathcal{O}_F$, we put $\Lambda^{\mathcal{O}}(\mathcal{G}) := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) = \mathcal{O}[[\mathcal{G}]]$. We choose a sufficiently large integer n such that Γ^{p^n} is central in \mathcal{G} and set $\Gamma_0 := \Gamma^{p^n}$ which is an open central subgroup of \mathcal{G} . Choose a topological generator γ of Γ . There is a ring isomorphism $R := \mathcal{O}[[\Gamma_0]] \simeq \mathcal{O}[[t]]$ induced by $\gamma^{p^n} \mapsto 1 + t$ where $\mathcal{O}[[t]]$ denotes the power series ring in one variable over \mathcal{O} . If we view $\Lambda^{\mathcal{O}}(\mathcal{G})$ as an R -module (or indeed as a left $R[H]$ -module), there is a decomposition

$$\Lambda^{\mathcal{O}}(\mathcal{G}) = \bigoplus_{i=0}^{p^n-1} R[H]\gamma^i.$$

Hence $\Lambda^{\mathcal{O}}(\mathcal{G})$ is finitely generated as an R -module and is an R -order in the separable $E := \text{Quot}(R)$ -algebra $\mathcal{Q}^F(\mathcal{G})$, the total ring of fractions of $\Lambda^{\mathcal{O}}(\mathcal{G})$, obtained from $\Lambda^{\mathcal{O}}(\mathcal{G})$ by adjoining inverses of all central regular elements. Note that $\mathcal{Q}^F(\mathcal{G}) = E \otimes_R \Lambda^{\mathcal{O}}(\mathcal{G})$ and that by [RW04, Lemma 1] we have $\mathcal{Q}^F(\mathcal{G}) = F \otimes_{\mathbb{Q}_p} \mathcal{Q}(\mathcal{G})$, where $\mathcal{Q}(\mathcal{G}) := \mathcal{Q}^{\mathbb{Q}_p}(\mathcal{G})$.

6.3. Characters and central primitive idempotents. Fix a character $\chi \in \text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G})$ (i.e. an irreducible \mathbb{Q}_p^c -valued character of \mathcal{G} with open kernel) and let η be an irreducible constituent of $\text{res}_H^{\mathcal{G}} \chi$. As in the case of finite groups, \mathcal{G} acts on η as ${}^g\eta(h) = \eta(g^{-1}hg)$ for $g \in \mathcal{G}$, $h \in H$, and we set

$$\mathcal{G}_\eta := \{g \in \mathcal{G} \mid {}^g\eta = \eta\}, \quad e_\eta := \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h, \quad \varepsilon_\chi := \sum_{\eta \mid \text{res}_H^{\mathcal{G}} \chi} e_\eta.$$

Note that ε_χ was denoted $e(\eta)$ in previous sections, but we now focus on its dependence on χ rather than η . By [RW04, Corollary to Proposition 6] ε_χ is a primitive central idempotent of $\mathcal{Q}^c(\mathcal{G}) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \mathcal{Q}(\mathcal{G})$. In fact, every primitive central idempotent of $\mathcal{Q}^c(\mathcal{G})$ is of this form and $\varepsilon_\chi = \varepsilon_{\chi'}$ if and only if $\chi = \chi' \otimes \rho$ for some character ρ of \mathcal{G} of type W (i.e. $\text{res}_H^{\mathcal{G}} \rho = 1$). Let $w_\chi = [\mathcal{G} : \mathcal{G}_\eta]$ and note that this is a power of p since H is a subgroup of \mathcal{G}_η .

Let F/\mathbb{Q}_p be a finite extension over which both characters χ and η have realisations. Let V_χ denote a realisation of χ over F . By [RW04, Propositions 5 and 6], there exists a unique element $\gamma_\chi \in \zeta(\mathcal{Q}^F(\mathcal{G})\varepsilon_\chi)$ such that γ_χ acts trivially on V_χ and $\gamma_\chi = g_\chi c_\chi$ with $g_\chi \in \mathcal{G}$ mapping to $\gamma_K^{w_\chi} \bmod H$ and with $c_\chi \in (F[H]\varepsilon_\chi)^\times$. Moreover, γ_χ generates a pro-cyclic p -subgroup Γ_χ of $\mathcal{Q}^F(\mathcal{G})\varepsilon_\chi$ and induces an isomorphism $\mathcal{Q}^F(\Gamma_\chi) \xrightarrow{\simeq} \zeta(\mathcal{Q}^F(\mathcal{G})\varepsilon_\chi)$. One should compare this to Lemma 5.4.

6.4. Determinants and reduced norms. Following [RW04, Proposition 6], we define a map

$$j_\chi : \zeta(\mathcal{Q}^F(\mathcal{G})) \twoheadrightarrow \zeta(\mathcal{Q}^F(\mathcal{G})\varepsilon_\chi) \simeq \mathcal{Q}^F(\Gamma_\chi) \rightarrow \mathcal{Q}^F(\Gamma_K),$$

where the last arrow is induced by mapping γ_χ to $\gamma_K^{w_\chi}$. It follows from op. cit. that j_χ is independent of the choice of γ_K and that for every matrix $\Theta \in M_{n \times n}(\mathcal{Q}(\mathcal{G}))$ we have

$$(6.2) \quad j_\chi(\text{nr}(\Theta)) = \det_{\mathcal{Q}^F(\Gamma_K)}(\Theta \mid \text{Hom}_{F[H]}(V_\chi, \mathcal{Q}^F(\mathcal{G})^n)).$$

Here, Θ acts on $f \in \text{Hom}_{F[H]}(V_\chi, \mathcal{Q}^F(\mathcal{G})^n)$ via right multiplication, and γ_K acts on the left via $(\gamma_K f)(v) = \gamma \cdot f(\gamma^{-1}v)$ for all $v \in V_\chi$ which is easily seen to be independent of the choice of γ . If ρ is a character of \mathcal{G} of type W , then we denote by ρ^\sharp the automorphism of the field $\mathcal{Q}^c(\Gamma_K)$ induced by $\rho^\sharp(\gamma_K) = \rho(\gamma_K)\gamma_K$. Moreover, we denote the additive group generated by all \mathbb{Q}_p^c -valued characters of \mathcal{G} with open kernel by $R_p(\mathcal{G})$. If \mathcal{Q} is

a subring of $\mathcal{Q}^c(\Gamma_K)$ which is closed under the actions of $G_{\mathbb{Q}_p}$ and ρ^\sharp for all ρ , we let $\text{Maps}^*(\text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G}), \mathcal{Q})$ (resp. $\text{Hom}^*(R_p(\mathcal{G}), \mathcal{Q}^\times)$) be the set of all maps $f : \text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G}) \rightarrow \mathcal{Q}$ (resp. the group of homomorphisms $f : R_p(\mathcal{G}) \rightarrow \mathcal{Q}^\times$) satisfying

$$\begin{aligned} f(\chi \otimes \rho) &= \rho^\sharp(f(\chi)) && \text{for all characters } \rho \text{ of type } W \text{ and} \\ f(\sigma\chi) &= \sigma(f(\chi)) && \text{for all Galois automorphisms } \sigma \in G_{\mathbb{Q}_p}. \end{aligned}$$

By [RW04, Proof of Theorem 8] we have isomorphisms

$$\begin{aligned} \zeta(\mathcal{Q}(\mathcal{G})) &\simeq \text{Maps}^*(\text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G}), \mathcal{Q}^c(\Gamma_K)) \\ \zeta(\mathcal{Q}(\mathcal{G}))^\times &\simeq \text{Hom}^*(R_p(\mathcal{G}), \mathcal{Q}^c(\Gamma_K)^\times) \\ x &\mapsto [\chi \mapsto j_\chi(x)]. \end{aligned}$$

Let \mathbb{Z}_p^c be the integral closure of \mathbb{Z}_p in \mathbb{Q}_p^c . Let $\mathcal{M}(\mathcal{G})$ be a maximal R -order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Then the first isomorphism restricts to

$$(6.3) \quad \zeta(\mathcal{M}(\mathcal{G})) \simeq \text{Maps}^*(\text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G}), \mathbb{Z}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_K))$$

by [RW04, Remark H]. Since $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}(\mathcal{G})$, we finally obtain an isomorphism

$$\zeta(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G})) \simeq \text{Maps}^*(\text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G}), \mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_K)).$$

For each totally odd irreducible character χ we set

$$g_{\chi,S}^T := j_\chi(\theta_S^T(L_\infty/K, 0)) \in \mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_K),$$

which is well-defined by (6.1). Let ρ be a character of type W . We may view $g_{\chi,S}^T$ as a power series in t via $\gamma_K \mapsto 1+t$. Evaluating at $\rho(\gamma_K) - 1$ gives

$$(6.4) \quad g_{\chi,S}^T(\rho(\gamma_K) - 1) = g_{\chi \otimes \rho, S}^T(0) = \delta_T(0, \chi \otimes \rho) L_S(0, \check{\chi} \otimes \check{\rho}).$$

If v is a place not in $S \cup T$ and $T' = T \cup \{v\}$, then

$$\theta_{S'}^T(L_\infty/K, 0) = \delta_v \cdot \theta_S^T(L_\infty/K, 0), \quad \delta_v := \text{nr}(1 - (Nv)\sigma_{w_\infty}^{-1}),$$

where σ_{w_∞} denotes the Frobenius automorphism at a place w_∞ in L_∞ above v . Therefore we may define

$$\theta_S(L_\infty/K, 0) := \left(\prod_{v \in T} \delta_v^{-1} \right) \theta_S^T(L_\infty/K, 0)$$

and likewise $g_{\chi,S} := j_\chi(\theta_S(L_\infty/K, 0))$. Then (6.4) implies that

$$(6.5) \quad g_{\chi,S}(\rho(\gamma_K) - 1) = g_{\chi \otimes \rho, S}(0) = L_S(0, \check{\chi} \otimes \check{\rho}).$$

If χ is a not necessarily irreducible character of \mathcal{G} with open kernel, we write χ as a finite sum $\chi = \sum_i n_i \chi_i$ with positive integers n_i and $\chi_i \in \text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G})$, and define $g_{\chi,S}^T := \prod_i (g_{\chi_i,S}^T)^{n_i}$ and similarly if $T = \emptyset$. Note that (6.4) and (6.5) remain valid.

Let N be a finite normal subgroup of \mathcal{G} such that $\tau \notin N$. Put $\bar{\mathcal{G}} := \mathcal{G}/N$. Suppose that $\chi = \text{infl}_{\bar{\mathcal{G}}}^{\mathcal{G}} \bar{\chi}$ for some character $\bar{\chi}$ of $\bar{\mathcal{G}}$. As Artin L -series behave well under inflation of characters, we have that

$$(6.6) \quad g_{\chi,S}^T = g_{\bar{\chi},S}^T \text{ and } g_{\chi,S} = g_{\bar{\chi},S} \text{ in } \mathcal{Q}^c(\Gamma_K).$$

Let \mathcal{H} be an open subgroup of \mathcal{G} , which contains τ . Set $K' := L_\infty^{\mathcal{H}}$. Restricting $\gamma' \in \Gamma_{K'}$ to K_∞ induces a natural embedding $\Gamma_{K'} \hookrightarrow \Gamma_K$. We obtain an inclusion of $\mathcal{Q}^c(\Gamma_{K'})$ into $\mathcal{Q}^c(\Gamma_K)$. Now let χ' be a character of \mathcal{H} and set $\chi := \text{ind}_{\mathcal{H}}^{\mathcal{G}} \chi'$. Let $S' := S(K')$ and

$T' := T(K')$. As Artin L -series behave well under induction of characters, we have an equality

$$(6.7) \quad g_{\chi,S}^T = g_{\chi',S'}^{T'} \text{ and } g_{\chi,S} = g_{\chi',S'} \text{ in } \mathcal{Q}^c(\Gamma_K).$$

We refer the reader to [Ell21, Lemma 7.8 and Proposition 7.9] for more details.

6.5. The p -adic cyclotomic character and its projections. Let χ_{cyc} be the p -adic cyclotomic character

$$\chi_{\text{cyc}} : \text{Gal}(L_\infty(\zeta_p)/K) \longrightarrow \mathbb{Z}_p^\times,$$

defined by $\sigma(\zeta) = \zeta^{\chi_{\text{cyc}}(\sigma)}$ for any $\sigma \in \text{Gal}(L_\infty(\zeta_p)/K)$ and any p -power root of unity ζ . Let ω and κ denote the composition of χ_{cyc} with the projections onto the first and second factors of the canonical decomposition $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$, respectively; thus ω is the Teichmüller character. We note that κ factors through Γ_K (and thus also through \mathcal{G}) and by abuse of notation we also use κ to denote the associated maps with these domains. We put $u := \kappa(\gamma_K)$.

6.6. Power series and p -adic Artin L -functions. Fix a totally even character $\psi \in \text{Irr}_{\mathbb{Q}_p^c}(\mathcal{G})$. We set $\mathcal{G}^+ = \mathcal{G}/\langle \tau \rangle$ and view ψ as a character of \mathcal{G}^+ . Each topological generator γ_K of Γ_K permits the definition of a power series $G_{\psi,S}(t) \in \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \text{Quot}(\mathbb{Z}_p[[t]])$ by starting out from the Deligne-Ribet power series for one-dimensional characters of open subgroups of \mathcal{G} (see [DR80]; also see [Bar78, CN79]) and then extending to the general case by using Brauer induction (see [Gre83]). One then has an equality

$$L_{p,S}(1-s, \psi) = \frac{G_{\psi,S}(u^s - 1)}{H_\psi(u^s - 1)},$$

where $L_{p,S}(s, \psi)$ denotes the ‘ S -truncated p -adic Artin L -function’ attached to ψ constructed by Greenberg [Gre83], and where, for irreducible ψ , one has

$$H_\psi(t) = \begin{cases} \psi(\gamma_K)(1+t) - 1 & \text{if } H \subseteq \ker \psi \\ 1 & \text{otherwise.} \end{cases}$$

Recall that $L_{p,S}(s, \psi) : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ is the unique p -adic meromorphic function with the property that for each strictly negative integer r and each field isomorphism $\iota : \mathbb{C} \simeq \mathbb{C}_p$, we have

$$(6.8) \quad L_{p,S}(r, \psi) = \iota \left(L_S(r, \iota^{-1} \circ (\psi \otimes \omega^{r-1})) \right).$$

By Siegel’s result [Sie70], the right-hand side of (6.8) does not depend on the choice of ι . By a slight abuse of notation, we simply write $L_S(r, \psi \otimes \omega^{r-1})$ for the right hand side in (6.8) in the following. If ψ is linear, then (6.8) is also valid when $r = 0$.

Now [RW04, Proposition 11] implies that

$$L_{K,S} : \psi \mapsto \frac{G_{\psi,S}(\gamma_K - 1)}{H_\psi(\gamma_K - 1)}$$

is independent of the topological generator γ_K and lies in $\text{Hom}^*(R_p(\mathcal{G}^+), \mathcal{Q}^c(\Gamma_K)^\times)$.

6.7. Comparing the power series. We now relate the power series $g_{\chi,S}$ of a totally odd character χ to the power series of the previous subsection associated to the even character $\check{\chi}\omega$.

Theorem 6.2. *Let p be an odd prime. Let L/K be a Galois CM extension and let L_∞ be the cyclotomic \mathbb{Z}_p -extension of L . Let χ be a totally odd irreducible character of $\mathcal{G} = \text{Gal}(L_\infty/K)$ with open kernel. Then for each finite set of places S of K containing all archimedean places and all places that ramify in L_∞/K we have an equality*

$$(6.9) \quad g_{\chi,S} = \frac{G_{\check{\chi}\omega,S}(u(1+t)^{-1} - 1)}{H_{\check{\chi}\omega}(u(1+t)^{-1} - 1)}.$$

Proof. Let us denote the right hand side of equation (6.9) by $f_{\chi,S}$. Let ρ be a character of type W . Since $L_{K,S}$ lies in $\text{Hom}^*(R_p(\mathcal{G}^+), \mathcal{Q}^c(\Gamma_K)^\times)$, an easy computation shows that also $\chi \mapsto f_{\chi,S}$ is compatible with ρ -twists. See [Ell21, Lemma 7.13] for details.

Now assume that χ is a linear character. Then we compute

$$\begin{aligned} f_{\chi,S}(\rho(\gamma_K) - 1) &= f_{\chi \otimes \rho,S}(0) = \frac{G_{(\check{\chi} \otimes \check{\rho})\omega,S}(u - 1)}{H_{(\check{\chi} \otimes \check{\rho})\omega}(u - 1)} \\ &= L_{p,S}(0, (\check{\chi} \otimes \check{\rho})\omega) = L_S(0, \check{\chi} \otimes \check{\rho}) \\ &= g_{\chi,S}(\rho(\gamma_K) - 1), \end{aligned}$$

where the last equality is (6.5). Then an application of [Was97, Corollary 7.4] shows that $f_{\chi,S} = g_{\chi,S}$. The equation (6.9) is actually well-known for linear characters and we refer the reader to [GP15, §5] and [Ell21, §7] for more details. Note that both $f_{\chi,S}$ and $g_{\chi,S}$ behave well under induction and inflation of characters. For $f_{\chi,S}$ this actually holds by Greenberg's construction [Gre83, §2] (also see [RW04, Proposition 12]). For $g_{\chi,S}$ this holds by (6.7) and (6.6), respectively. Hence the general case follows from Brauer's induction theorem. \square

We now draw a few important consequences. Our first observation is that we obtain a direct construction of p -adic L -series of totally even characters.

Corollary 6.3. *Let $\psi \in \text{Irr}(\mathcal{G})$ be a totally even character. Then one has*

$$L_{p,S}(s, \psi) = g_{\check{\psi}\omega,S}(u^s - 1).$$

As a second application we give an affirmative answer to a question raised by Greenberg [Gre83, §4], which is also a special case of a conjecture of Gross [Gro81].

Corollary 6.4. *Let $\psi \in \text{Irr}(\mathcal{G})$ be a totally even character. Then one has an equality*

$$L_{p,S}(0, \psi) = L_S(0, \psi\omega^{-1}).$$

Proof. We apply Theorem 6.2 to the character $\chi = \check{\psi}\omega$. Evaluating $g_{\chi,S}$ at $t = 0$ yields $L_S(0, \check{\chi}) = L_S(0, \psi\omega^{-1})$ by (6.5) with ρ equal to the trivial character. Evaluating the right hand side of (6.9) at $t = 0$ gives $L_{p,S}(0, \psi)$ as desired. \square

Remark 6.5. In [JN19] Johnston and the second named author deduced cases of the non-abelian Brumer–Stark conjecture from the equivariant Iwasawa main conjecture. In many of these results it had been assumed that the underlying group is monomial. However, this was only required to guarantee the conclusion of Corollary 6.4. Since this now holds unconditionally for an arbitrary totally even character, one can drop the hypothesis of being monomial in the respective results.

Finally, we obtain a new proof of the p -adic Artin conjecture [Gre83, p. 82], which does not rely on the main conjecture proved by Wiles [Wil90a].

Corollary 6.6 (The p -adic Artin conjecture). *Let p be an odd prime and let K be a totally real number field. Let ψ be an irreducible totally even character of G_K . Then for each finite set S of places of K containing $S_\infty \cup S_p$ and all places where ψ is ramified, we have*

$$G_{\psi,S}(t) \in \mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[t]].$$

Proof. By Theorem 6.2 we have $G_{\psi,S}(t) = H_\psi(t) \cdot g_{\chi,S}(u(1+t)^{-1} - 1)$, where we set $\chi := \check{\psi}\omega$. The main obstacle now is that we only know that $g_{\chi,S}^T(t) \in \mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[t]]$ for any set T of places of K such that $\text{Hyp}(S, T)$ is satisfied for all layers L_n/K . We may choose $T = T_v := \{v\}$, where v is a place of K of sufficiently large norm. For each such v we then have an equality

$$g_{\chi,S}^{T_v} = j_\chi(\delta_v)g_{\chi,S}.$$

Moreover, there is a short exact sequence of $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G})$ -modules

$$(6.10) \quad 0 \rightarrow \mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) \rightarrow \mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) \rightarrow \text{ind}_{\mathcal{G}_{w_\infty}}^{\mathcal{G}} \mathbb{Q}_p^c(1) \rightarrow 0,$$

where the first non-trivial arrow is multiplication by $(Nv)\sigma_{w_\infty}^{-1} - 1$. Here σ_{w_∞} denotes again the Frobenius automorphism at a place w_∞ in L_∞ above v , which is also a topological generator of the decomposition group \mathcal{G}_{w_∞} .

We now consider the case where H is not contained in the kernel of ψ . Then the Teichmüller character does not occur as an irreducible constituent of $\text{res}_H^{\mathcal{G}} \chi$, and there is an $h \in H$ such that $\psi(h) \neq \psi(1)$. By Chebotarev's density theorem we may choose v such that $\sigma_{w_\infty} = h\gamma \in H \rtimes \Gamma$ for some $\gamma \in \Gamma$. The primitive central idempotents ε_χ lie in $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G})$ and taking ' χ -parts' of sequence (6.10) shows that $((Nv)\sigma_{w_\infty}^{-1} - 1)\varepsilon_\chi$ is invertible in $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G})_{\varepsilon_\chi}$. Hence $j_\chi(\delta_v) \in (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_K))^\times$ and we are done in this case.

Now assume that H is contained in the kernel of ψ so that ψ is a character of type W . As $G_{\psi \otimes \rho, S} = \rho^\sharp(G_{\psi, S})$ for any totally even ψ and any ρ of type W , we may assume that ψ is the trivial character. Now choose v such that σ_{w_∞} is a topological generator of Γ . By replacing γ_K if necessary, we may assume in addition that σ_{w_∞} maps to γ_K under $\mathcal{G} \twoheadrightarrow \Gamma_K$. Then we get $j_\chi(\delta_v)(t) = u(1+t)^{-1} - 1$ and thus $j_\chi(\delta_v)(u(1+t)^{-1} - 1) = t = H_\psi(t)$ as desired. \square

The main conjecture actually implies the stronger statement that $G_{\psi,S} \in \mathbb{Z}_p^c \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[t]]$, see [RW04, Remark G]. The proof of Corollary 6.6 and (6.3) then imply the following result in the case $r = 0$. The case $r < 0$ can be shown similarly (indeed the proof simplifies as (6.8) was already known to hold for non-linear χ).

Corollary 6.7. *Let p be an odd prime. Let L/K be a Galois CM extension and let L_∞ be the cyclotomic \mathbb{Z}_p -extension of L . Let $\mathcal{G} = \text{Gal}(L_\infty/K)$ and choose a maximal R -order $\mathcal{M}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Let S and T be finite sets of places of K such that $\text{Hyp}(S, T)$ is satisfied for all layers L_n/K . Then for each integer $r \leq 0$ one has*

$$\theta_S^T(L_\infty/K, r) \in \zeta(\mathcal{M}(\mathcal{G})).$$

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