On the Equivariant Tamagawa Number Conjecture in tame CM-extensions

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Abstract Let L/K be a finite Galois CM-extension with Galois group G. The Equivariant Tamagawa Number Conjecture (ETNC) for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ naturally decomposes into p-parts, where p runs over all rational primes. If p is odd, these p-parts in turn decompose into a plus and a minus part. Let L/K be tame above p. We show that a certain ray class group of L defines an element in $K_0(\mathbb{Z}_p G_-, \mathbb{Q}_p)$ which is determined by a corresponding Stickelberger element if and only if the minus part of the ETNC at p holds. For this we use the Lifted Root Number Conjecture for small sets of places which is equivalent to the ETNC in the number field case. For abelian G, we show that the minus part of the ETNC at p implies the Strong Brumer-Stark Conjecture at p. We prove the minus part of the ETNC at p for almost all primes p.

Key words equivariant L-values, Tamagawa numbers, strong Brumer-Stark conjecture, CM-extensions
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Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. If r denotes the vanishing order at $s = 0$ of the Dedekind zeta function $\zeta_L(s)$ attached to the number field L, the integral Dirichlet class number formula states that

$$\lim_{s \to 0} \frac{1}{s^r} \zeta_L(s) = -\frac{h_L R_L}{w_L},$$

where h_L denotes the class number of L, w_L is the number of roots of unity in L and R_L is the regulator of L. Roughly speaking, the above equality connects an analytic object of L to arithmetic invariants of L. The ETNC for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ is a conjectural G-equivariant refinement of this formula, where the zeta function ζ_L is replaced by a Galois equivariant L-function with values in the center of the group ring $\mathbb{C}G$.
In the 1980s T. Chinburg [Ch1] defined an algebraic invariant $\Omega(L/K)$ of the extension L/K which lies in the K-group $K_0(\mathbb{Z}G)$. He conjectured that $\Omega(L/K)$ equals the root number class $W(L/K)$, an analytic invariant defined by Ph. Cassou-Noguès and A. Fröhlich in terms of Artin root numbers. In [Ch2] he introduced two further algebraic invariants in $K_0(\mathbb{Z}G)$, called $\Omega_i(L/K)$, $i = 1, 2, 3$, where $\Omega_3(L/K) = \Omega(L/K)$. These invariants are related by the equation

$$\Omega_2(L/K) = \Omega_1(L/K) \cdot \Omega_3(L/K).$$

Chinburg conjectured that $\Omega_1(L/K) = 1$, and hence that $\Omega_2(L/K)$ also equals the root number class. In addition, he proved his Ω_2-conjecture for at most tamely ramified extensions.
All these conjectures have meanwhile been lifted to corresponding conjectures in the relative K-group $K_0(\mathbb{Z}G, \mathbb{R})$; so D. Burns [B1] used complexes arising from étale cohomology of the constant sheaf $\mathbb{Z}$ to define a canonical element $T\Omega(L/K, 0)$ of $K_0(\mathbb{Z}G, \mathbb{R})$ which maps to zero under the connecting homomorphism $K_0(\mathbb{Z}G, \mathbb{R}) \to K_0(\mathbb{Z}G)$ of relative K-theory if and only if Chinburg's Ω_3-conjecture holds. So the vanishing of $T\Omega(L/K, 0)$ is a refinement of Chinburg's Ω_3-conjecture in $K_0(\mathbb{Z}G, \mathbb{R})$ rather than in $K_0(\mathbb{Z}G)$, whereas the conjectures in [BB] and [BrB] are the same concerning Chinburg's Ω_2 and Ω_1-conjecture, respectively. It was shown in [B1] that the Lifted Root Number Conjecture (LRNC) by K.W. Gruenberg, J. Ritter and A. Weiss [GRW] for the extension L/K is equivalent to the vanishing of $T\Omega(L/K, 0)$ and that this in turn is equivalent to the ETNC for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ (cf. loc.cit., Th. 2.3.3 and Th. 2.4.1). These conjectures make use of a finite G-invariant set S of places of L which is supposed to be large in the sense that all ramified and all archimedean primes lie in S and that the S-class group $\mathrm{cl}_S(L)$ vanishes. In [Ni2] a LRNC is formulated for small sets of places S which only need to contain the set S_∞ of all the archimedean primes. Since we will use a set S which indeed contains only totally decomposed (and thus unramified) primes, we decide for this variant of the conjecture.

Assume that L/K is a CM-extension which is tame above a fixed odd prime p (we actually permit a slightly more general class of extensions explained later). We denote the unique automorphism on L induced by complex conjugation by j and set $RG_- = RG/(1+j)$ for any ring R. For any G-module M and any integer i we write $H^i(G, M)$ for the Tate cohomology in degree i of M with respect to G. We say that M is a c.t. (short for cohomologically trivial) G-module if $H^i(U, M)$ vanishes for all $i \in \mathbb{Z}$ and each subgroup U of G. After a few preliminaries we prove in section two that the p-part of a certain ray class group of L is c.t. on minus parts. We give a definition of non-abelian Stickelberger elements which determine elements in the relative K-group $K_0(\mathbb{Z}_pG_-, \mathbb{Q}_p)$. We show that the minus part of the LRNC (resp. ETNC) at p holds if and only if these ray class groups are represented by corresponding Stickelberger elements.
In section three we assume the Galois group G to be abelian. In this case one can translate the minus part of the LRNC at p to the assertion that the Fitting ideal of the ray class group is generated by the corresponding Stickelberger element. We pass to the limit and get the respective statement at infinite level thanks to a result of C. Greither [Gr2] provided that the Iwasawa μ-invariant vanishes. We will remove this hypothesis for a special class of extensions (including the case $p \nmid |G|$) in the appendix. Note that the vanishing of μ is a long standing conjecture; the most general result is still due to B. Ferrero and L. Washington [FW] and says that $\mu = 0$ for absolutely abelian extensions.
For the descent we use a method which is due to A. Wiles [Wi2] in the extended version by C. Greither [Gr1]. For this we have to assume a slightly more restrictive hypothesis on the primes above p.
In the last section we prove that the minus part of the ETNC at p implies (for the case at hand) the Strong Brumer-Stark Conjecture at p as formulated in [Po]; we thus verify this conjecture for the same class of extensions. This conjecture states that the Fitting ideal of a certain ray class group of L contains a particular Stickelberger element. These are not the same ray class groups resp. Stickelberger elements as in the previous sections, but they are related to them closely enough. Note that this conjecture does not hold in general, as one can see from the results in [GK]. But all counterexamples in loc.cit. are wildly ramified. D. Burns [B3] has shown that the ETNC implies the Rubin-Stark conjecture which is implied by the Strong Brumer-Stark Conjecture, too ([Po], Th. 3.2.2.3). Thus, we reprove Burns' result for (almost) tame extensions.
We point out that this paper includes parts of the author's dissertation [Ni1].

1 Preliminaries

1.0.1 K-theory Let R be a left noetherian ring with 1 and $\mathrm{PMod}(R)$ the category of all finitely generated projective R-modules. We write $K_0(R)$ for the Grothendieck group of $\mathrm{PMod}(R)$, and $K_1(R)$ for the Whitehead group of R which is the abelianized infinite general linear group. If S is a multiplicatively closed subset of the center $Z(R)$ of R which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all S-torsion R-modules of finite projective dimension by $K_0S(R)$. Writing R_S for the ring of quotients of R with denominators in S we have the Localization Sequence (cf. [CR2], p. 65)

$$K_1(R) \to K_1(R_S) \xrightarrow{\partial} K_0S(R) \to K_0(R) \to K_0(R_S). \tag{1}$$

If T is a ring that contains R and M is an R-module, we will often write TM instead of $T \otimes_R M$. Moreover, if G is a group and $M = \Delta G$ is the kernel of the augmentation map $RG \twoheadrightarrow R$, we set $\Delta_T G := T \otimes_R \Delta G$. In the case $R = \mathbb{Z}$, $T = \mathbb{Z}_p$ for a prime p, we write $\Delta_p G$ instead of $\Delta_{\mathbb{Z}_p} G$.
Specializing to group rings $\mathbb{Z}G$ for finite groups G and $S = \mathbb{Z} \setminus \{0\}$ we write $K_0T(\mathbb{Z}G)$ instead of $K_0S(\mathbb{Z}G)$. So (1) reads

$$K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G) \xrightarrow{\partial} K_0T(\mathbb{Z}G) \to K_0(\mathbb{Z}G) \to K_0(\mathbb{Q}G). \tag{2}$$

Note that a finitely generated $\mathbb{Z}G$-module has finite projective dimension if and only if it is a G-c.t. module. Indeed, the projective dimension is lower or equal to 1 in this case. Further, recall that the relative K-group $K_0(\mathbb{Z}G, \mathbb{Q})$ is generated by elements of the form (P_1, ϕ, P_2) with finitely generated projective modules P_1 and P_2 and a $\mathbb{Q}G$-isomorphism $\phi : \mathbb{Q}P_1 \to \mathbb{Q}P_2$, and that there is an isomorphism

$$i_G : K_0T(\mathbb{Z}G) \simeq K_0(\mathbb{Z}G, \mathbb{Q}). \tag{3}$$

If a c.t. torsion $\mathbb{Z}G$-module T has projective resolution $P_1 \overset{\iota}{\rightarrowtail} P_0 \twoheadrightarrow T$, this isomorphism sends the corresponding element $[T] \in K_0T(\mathbb{Z}G)$ to $(P_1, \mathbb{Q} \otimes \iota, P_0) \in K_0(\mathbb{Z}G, \mathbb{Q})$.
If p is a finite rational prime, the local analogue of sequence (2) is

$$K_1(\mathbb{Z}_pG) \to K_1(\mathbb{Q}_pG) \xrightarrow{\partial_p} K_0T(\mathbb{Z}_pG) \to 0, \tag{4}$$

and we have an isomorphism

$$K_0T(\mathbb{Z}G) \simeq \bigoplus_{p \nmid \infty} K_0T(\mathbb{Z}_pG). \tag{5}$$

Moreover, there is a well defined map $\hat{\partial} : Z(\mathbb{Q}G)^\times \to K_0(\mathbb{Z}G, \mathbb{Q})$ such that $\hat{\partial} \circ \mathrm{nr} = \partial$, where nr is the reduced norm map on $K_1(\mathbb{Q}G)$. We denote the local analogue of $\hat{\partial}$ by $\hat{\partial}_p$.

Remark 1. If $\phi : \mathbb{Q}B \to \mathbb{Q}A$ is a $\mathbb{Q}G$-isomorphism, where B is projective and A is c.t., then there is a well defined object $(B, \phi, A) \in K_0(\mathbb{Z}G, \mathbb{Q})$ (cf. [Ni2], Def. 1.1). Assume that there is a projective module P such that ϕ can be written as the composition of two $\mathbb{Q}G$-isomorphisms $\phi_1 : \mathbb{Q}B \to \mathbb{Q}P$ and $\phi_2 : \mathbb{Q}P \to \mathbb{Q}A$, then we have an equality

$$(B, \phi, A) = (B, \phi_1, P) + (P, \phi_2, A) \tag{6}$$

in $K_0(\mathbb{Z}G, \mathbb{Q})$. We may replace $K_0(\mathbb{Z}G, \mathbb{Q})$ by $K_0(\mathbb{Z}_pG, \mathbb{Q}_p)$; everything remains the same except for the obvious modifications.

1.0.2 Complexes and refined Euler Characteristics For any ring R we write $\mathcal{D}(R)$ for the derived category of R-modules. Let $\mathcal{C}^b(\mathrm{PMod}(R))$ be the category of bounded complexes of finitely generated projective R-modules. A complex of R-modules is called perfect if it is isomorphic in $\mathcal{D}(R)$ to an element of $\mathcal{C}^b(\mathrm{PMod}(R))$. We denote the full triangulated subcategory of $\mathcal{D}(R)$ consisting of perfect complexes by $\mathcal{D}^{\mathrm{perf}}(R)$. For any $C^{\cdot} \in \mathcal{D}^{\mathrm{perf}}(R)$ we define R-modules

$$C^e := \bigoplus_{i \in \mathbb{Z}} C^{2i}, \; C^o := \bigoplus_{i \in \mathbb{Z}} C^{2i+1}.$$

For the following let R be a Dedekind domain of characteristic 0, K its field of fractions, A a finite dimensional K-algebra and Γ an R-order in A. A pair $(C^{\cdot}, t)$ consisting of a complex $C^{\cdot} \in \mathcal{D}^{\mathrm{perf}}(\Gamma)$ and an isomorphism $t : H^o(C^{\cdot}_K) \to H^e(C^{\cdot}_K)$ is called a trivialised complex, where $C^{\cdot}_K$ is the complex obtained by tensoring $C^{\cdot}$ with K. We refer to t as a trivialisation of $C^{\cdot}$.
One defines the refined Euler characteristic $\chi_{\Gamma,A}(C^{\cdot}, t) \in K_0(\Gamma, A)$ of a trivialised complex as follows: Choose a complex $P^{\cdot} \in \mathcal{C}^b(\mathrm{PMod}(R))$ which is quasi-isomorphic to $C^{\cdot}$. Let $B^i(P^{\cdot}_K)$ and $Z^i(P^{\cdot}_K)$ denote the i^{th} cobounderies and i^{th} cocycles of $P^{\cdot}_K$, respectively. We have the obvious exact sequences

$$B^i(P^{\cdot}_K) \rightarrowtail Z^i(P^{\cdot}_K) \twoheadrightarrow H^i(P^{\cdot}_K) \;, \quad Z^i(P^{\cdot}_K) \rightarrowtail P^i_K \twoheadrightarrow B^{i+1}(P^{\cdot}_K).$$

If we choose splittings of the above sequences we get an isomorphism

$$\phi_t : P^o_K \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P^{\cdot}_K) \oplus H^o(P^{\cdot}_K) \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P^{\cdot}_K) \oplus H^e(P^{\cdot}_K) \simeq P^e_K,$$

where the second map is induced by t. Then the refined Euler characteristic is defined to be

$$\chi_{\Gamma,A}(C^{\cdot}, t) := (P^o, \phi_t, P^e) \in K_0(\Gamma, A)$$

which indeed is independent of all choices made in the construction.
Now we specialize to group rings RG, where R is a finitely generated subring of $\mathbb{Q}$. Let H^i, $i = 0, 1$ be finitely generated RG-modules and

$$H^0 \rightarrowtail A \to B \twoheadrightarrow H^1$$

an exact sequence representing an extension class $\tau \in \mathrm{Ext}^2_{RG}(H^1, H^0)$. One obtains an associated complex $A \to B$, where A is placed in degree 0. If this complex is perfect, τ is called a perfect 2-extension. If there exists a $\mathbb{Q}G$-isomorphism $\phi : \mathbb{Q}H^1 \to \mathbb{Q}H^0$, the element

$$\chi_{RG,\mathbb{Q}G}(\tau, \phi) := \chi_{RG,\mathbb{Q}G}(A \to B, \phi)$$

only depends upon the class τ and the isomorphism ϕ. For further information concerning refined Euler characteristics we refer the reader to [B2].

1.0.3 Hom *description* Let G be a finite group, p a finite rational prime and $R(G)$ (resp. $R_p(G)$) the ring of virtual characters of G with values in $\mathbb{Q}^c$ (resp. $\mathbb{Q}_p^c$), an algebraic closure of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$). Choose a number field F, Galois over $\mathbb{Q}$ with Galois group Γ, which is large enough such that all representations of G can be realized over F. Let $\wp$ be a prime of F above p. Then there is an isomorphism (for this and the following cf. [GRW], Appendix A)

$$\begin{aligned} \mathrm{Det}\ : K_1(\mathbb{Q}_p G) &\xrightarrow{\simeq} \mathrm{Hom}_{\Gamma_\wp}(R_p(G), F_\wp^\times) \\ [X, g] &\mapsto [\chi \mapsto \det(g | \mathrm{Hom}_{F_\wp G}(V_\chi, F_\wp \otimes_{\mathbb{Q}_p} X))], \end{aligned}$$

where V_χ is a $F_\wp G$-module with character χ. Combined with the localization sequence (4) this gives the local Hom description

$$K_0 T(\mathbb{Z}_p G) \simeq \mathrm{Hom}_{\Gamma_\wp}(R_p(G), F_\wp^\times) / \mathrm{Det}\,(\mathbb{Z}_p G^\times). \tag{7}$$

One globally has

$$K_0 T(\mathbb{Z}G) \simeq \mathrm{Hom}^+_\Gamma(R(G), J_F) / \mathrm{Det}\, U(\mathbb{Z}G), \tag{8}$$

where J_F denotes the idèle group of F and $U(\mathbb{Z}G)$ the unit idèles of $\mathbb{Z}G$. The + indicates that a homomorphism in $\mathrm{Hom}^+_\Gamma(R(G), J_F)$ takes values in $\mathbb{R}^+$ for symplectic characters.

Remark 2. Define $\varepsilon_\chi := \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g$ for each $\chi \in \mathrm{Irr}\,(G)$, where $\mathrm{Irr}\,(G)$ denotes the set of irreducible characters of G. The ε_χ are orthogonal central idempotents of $\mathbb{C}G$. Each generates one of the minimal ideals of the center of $\mathbb{C}G$, hence

$$Z(\mathbb{C}G) = \bigoplus_{\chi\in\mathrm{Irr}\,(G)} \mathbb{C}\varepsilon_\chi.$$

If $z = \sum_\chi z_\chi \varepsilon_\chi \in Z(\mathbb{Q}G)^\times \subset Z(\mathbb{C}G)$, then $\hat{\partial}(z)$ has representing homomorphism $\chi \mapsto z_\chi$.

1.0.4 The LRNC for small sets of places Let L/K be a Galois extension of number fields with Galois group G. For a prime $\mathfrak{P}$ of L we write $\mathfrak{p} = \mathfrak{P}\cap K$ for the prime below $\mathfrak{P}$, $G_\mathfrak{P}$ for the decomposition group attached to $\mathfrak{P}$ and $I_\mathfrak{P}$ for the inertia subgroup. We denote the Frobenius generator of the Galois group $\overline{G_\mathfrak{P}} = G_\mathfrak{P}/I_\mathfrak{P}$ of the corresponding residue field extension by $\phi_\mathfrak{P}$. The inertial lattice of the local extension $L_\mathfrak{P}/K_\mathfrak{p}$ is defined to be the $\mathbb{Z}G_\mathfrak{P}$-lattice (cf. [GW] or [We] p. 42)

$$W_\mathfrak{P} = \{(x,y) \in \Delta G_\mathfrak{P} \oplus \mathbb{Z}\overline{G_\mathfrak{P}} : \overline{x} = (\phi_\mathfrak{P} - 1)y\}. \tag{9}$$

Note that $W_\mathfrak{P} \simeq \mathbb{Z}G_\mathfrak{P}$ if the local extension $L_\mathfrak{P}/K_\mathfrak{p}$ is unramified. Projecting onto the first component yields an exact sequence of $G_\mathfrak{P}$-modules

$$\mathbb{Z} \rightarrowtail W_\mathfrak{P} \twoheadrightarrow \Delta G_\mathfrak{P}. \tag{10}$$

Let S_{ram} be the set of finite primes of L which ramify in L/K and $M^* = \mathrm{Hom}(M,\mathbb{Z})$ for any module M. The $\mathbb{Z}$-dual of sequence (10) induces a surjection $W_\mathfrak{P}^* \twoheadrightarrow \mathbb{Z}^* = \mathbb{Z}$. If we combine these surjections and the augmentation map $\mathbb{Z}S \twoheadrightarrow \mathbb{Z}$, we get an exact sequence

$$\overline{\nabla} \rightarrowtail \mathbb{Z}S \oplus \bigoplus_{\mathfrak{P}\in S_{\mathrm{ram}}^\diamond \setminus (S\cap S_{\mathrm{ram}})^\diamond} \mathrm{ind}\,_{G_\mathfrak{P}}^G (W_\mathfrak{P}^*) \twoheadrightarrow \mathbb{Z} \tag{11}$$

where the $\diamond$ indicates that the sum runs over a fixed set of representatives, one for each orbit of the action of G on the primes of L. Let $N_G = \sum_{g\in G} g$ for any finite group G. We can describe $W_\mathfrak{P}^*$ as the cokernel of the map (cf. [Gr3], p. 20)

$$\begin{aligned} \mathbb{Z}\overline{G_\mathfrak{P}} &\longrightarrow \mathbb{Z}G_\mathfrak{P}/N_{G_\mathfrak{P}} \times \mathbb{Z}\overline{G_\mathfrak{P}} \\ 1 &\mapsto (N_{I_\mathfrak{P}}, 1-\phi_\mathfrak{P}^{-1}). \end{aligned}$$

Therefore, we have a canonical epimorphism $\kappa : \mathbb{Z}G_\mathfrak{P}^2 \twoheadrightarrow W_\mathfrak{P}^*$ which fits into

$$W_\mathfrak{P} \overset{q}{\rightarrowtail} \mathbb{Z}G_\mathfrak{P}^2 \overset{\kappa}{\twoheadrightarrow} W_\mathfrak{P}^*, \tag{12}$$

where q maps $(x,y)\in W_\mathfrak{P}$ to $(N_{I_\mathfrak{P}}y, \phi_\mathfrak{P}^{-1}x)$ (cf. [GW], Lemma 4.1).
In [RW1] the authors derive an exact sequence of finitely generated $\mathbb{Z}G$-modules

$$E_S \rightarrowtail A \to B \twoheadrightarrow \nabla, \tag{13}$$

which has a uniquely determined extension class $\tau \in \mathrm{Ext}_G^2(\nabla, E_S)$. Note that the sequence itself is not unique. We will refer to a sequence (13) as a Tate-sequence for S. Here, E_S is the group of S-units of L, A is c.t., B projective and ∇ fits into an exact sequence of G-modules

$$\mathrm{cl}_S(L) \rightarrowtail \nabla \twoheadrightarrow \overline{\nabla},$$

where $\mathrm{cl}_S(L)$ is the S-class group of L. Hence τ is a perfect 2-extension. If S is sufficiently large, the modules ∇ and $\overline{\nabla}$ coincide and are just the kernel ΔS of the augmentation map $\mathbb{Z}S \twoheadrightarrow \mathbb{Z}$. In this case, the extension class of (13) is Tate's canonical class ([Ta1]).
We will use a set of places S which generates the ideal class group, but does not contain any ramified prime. So let us assume this for simplicity. We obviously have $\nabla = \overline{\nabla}$ in this case.

Let C be a free $\mathbb{Z}G$-module of rank s_{ram}, the number of finite primes of K which ramify in L/K. We add C to the left part of the Tate-sequence, namely

$$E_S \oplus C \rightarrowtail A \oplus C \to B \twoheadrightarrow \nabla,$$

which has a uniquely determined extension class $\tau_C \in \mathrm{Ext}^2_G(\nabla, E_S \oplus C)$. There exist trivialisations $\phi : \mathbb{Q}\nabla \to \mathbb{Q}(E_S \oplus C)$ and we may define

$$\Omega_\phi := \chi_{\mathbb{Z}G,\mathbb{Q}G}(\tau_C, \phi).$$

Let $1_{\mathfrak{P}}$, $\mathfrak{P} \in S^{\diamond}_{\mathrm{ram}}$ be a $\mathbb{Z}G$-basis of C. We choose a positive multiple h of h_L, the class number of L. Thus $\mathfrak{P}^h$ is principal generated by an element $u_{\mathfrak{P}} \in L$. We define (cf. [Ni2])

$$\begin{aligned} \lambda_C : C &\longrightarrow \mathbb{R} \otimes \nabla \\ 1_{\mathfrak{P}} &\mapsto \left(h \log N(\mathfrak{P}) \frac{1}{|G_{\mathfrak{P}}|} N_{G_{\mathfrak{P}}} + 1 - \frac{1}{|G_{\mathfrak{P}}|} N_{G_{\mathfrak{P}}} \right) d_{\mathfrak{P}} - \sum_{\mathfrak{Q}|\infty} \log |u_{\mathfrak{P}}|_{\mathfrak{Q}} \mathfrak{Q}, \end{aligned}$$

where $d_{\mathfrak{P}} = |G_{\mathfrak{P}}|^{-1} \kappa(|G_{\mathfrak{P}}|, N_{G_{\mathfrak{P}}})$ is a $\mathbb{Q}G_{\mathfrak{P}}$-generator of $\mathbb{Q}W^*_{\mathfrak{P}}$ by loc.cit., Lemma 4.5. Combined with the usual Dirichlet map

$$\begin{aligned} \lambda_S : E_S &\longrightarrow \Delta_{\mathbb{R}} S \\ e &\mapsto -\textstyle\sum_{\mathfrak{P} \in S} \log |e|_{\mathfrak{P}} \mathfrak{P} \end{aligned} \tag{14}$$

we get a modified Dirichlet map

$$\begin{aligned} \lambda^{\mathrm{mod}}_S : E_S \oplus C &\longrightarrow \mathbb{R}\nabla \\ (e, c) &\mapsto \lambda_S(e) + \lambda_C(c). \end{aligned} \tag{15}$$

If χ is a character of G, we write $\check{\chi}$ for the character contragredient to χ. We call the map $R^{\mathrm{mod}}_\phi : R(G) \to \mathbb{C}^\times$ defined by

$$R^{\mathrm{mod}}_\phi : \chi \mapsto \frac{\det(\lambda^{\mathrm{mod}}_S \phi | \mathrm{Hom}_G(V_{\check{\chi}}, \mathbb{C}\nabla))}{\prod_{\mathfrak{P} \in S^{\diamond}_{\mathrm{ram}}} (-h|G_{\mathfrak{P}}|)^{\dim V_{\check{\chi}}^{G_{\mathfrak{P}}}}}$$

the **modified Stark-Tate regulator** and set

$$A^{\mathrm{mod}}_\phi : \chi \mapsto \frac{R^{\mathrm{mod}}_\phi(\chi)}{c_{S \cup S_{\mathrm{ram}}}(\chi)},$$

where we write $c_S(\chi)$ for the leading coefficient of the Taylor expansion of the S-truncated L-function $L_S(L/K, \chi, s)$ at $s = 0$. By Theorem 4.8 (i) in [Ni2] A^{mod}_ϕ commutes with Galois action if and only if Stark's conjecture holds. Now we fix an embedding $F \rightarrowtail \mathbb{C}$ and denote the corresponding infinite prime by $\wp_\infty$. Define $W(L/K, \cdot) \in \mathrm{Hom}_\Gamma(R(G), J_F)$ by

$$W(L/K, \chi)_\wp = \begin{cases} W(\chi^{\gamma^{-1}}) & \text{if } \chi \text{ is symplectic and } \wp = \wp_\infty^\gamma \\ 1 & \text{otherwise} \end{cases}$$

The LRNC for small sets of places now states (assuming Stark's conjecture)

Conjecture 1 (LRNC for small S). The element $\Omega_\phi \in K_0(\mathbb{Z}G, \mathbb{Q})$ has representing homomorphism $\chi \mapsto A^{\mathrm{mod}}_\phi(\check{\chi}) W(L/K, \check{\chi})$.

The element Ω_ϕ decomposes into p-parts $\Omega^{(p)}_\phi$ via the isomorphism (5). If we choose a prime $\wp$ in F above p and an embedding $j_p : F \rightarrowtail F_\wp$ for each p, Stark's conjecture asserts that the map

$$(A^{\mathrm{mod}}_\phi)^{(p)} : \chi \mapsto j_p(A^{\mathrm{mod}}_\phi(j_p^{-1}(\chi)))$$

lies in $\mathrm{Hom}_{\Gamma_\wp}(R_p(G), F^\times_\wp)$. Conjecture 1 localizes to

Conjecture 2 (LRNC for small S at the prime p). The element $\Omega^{(p)}_\phi \in K_0(\mathbb{Z}_p G, \mathbb{Q}_p)$ has representing homomorphism $\chi \mapsto (A^{\mathrm{mod}}_\phi)^{(p)}(\check{\chi})$.

2 CM-extensions

Let L/K be a CM-extension, i.e. K is totally real and L is a totally imaginary quadratic extension of a totally real number field. Complex conjugation on $\mathbb{C}$ induces an automorphism on L which is independent of the embedding into $\mathbb{C}$ (cf. [Wa], p. 38). We denote this automorphism by j and refer to it as complex conjugation as well. For any number field E we write $\mathfrak{o}_E$ for the ring of integers in E; moreover, let $\mathfrak{o} := \mathfrak{o}_L$.
For any G-module M we define submodules

$$M^+ := \{m \in M : jm = m\}\,, \quad M^- := \{m \in M : jm = -m\}\,.$$

M^+ is a module over the ring $\mathbb{Z}G_+ := \mathbb{Z}G/(1-j) = \mathbb{Z}[G/\langle j\rangle]$, whereas M^- has a $\mathbb{Z}G_-$ action, but $\mathbb{Z}G_-$ is not a ring, since $\frac{1-j}{2} \notin \mathbb{Z}G_-$. The minus functor is left exact on exact sequences of RG-modules for any ring R, and even exact if 2 is invertible in R.
If a finitely generated G-module M decomposes into $M = M^+ \oplus M^-$, the natural maps

$$H^i(U, M^+) \to H^i(U, M)^+,$$

$$H^i(U, M^-) \to H^i(U, M)^-$$

are isomorphisms for all subgroups U of G of odd order, $i \in \mathbb{Z}$. Indeed, the composite map

$$H^i(U, M) \simeq H^i(U, M^+) \oplus H^i(U, M^-) \to H^i(U, M)^+ \oplus H^i(U, M)^- \simeq H^i(U, M)$$

is the identity. Here, the rightmost isomorphism exists, because $H^i(U, M)$ is finite of odd order and hence also decomposes into a plus and a minus part.
If $p \neq 2$ and M is a $\mathbb{Z}_pG$-module, there is a natural decomposition $M = M^+ \oplus M^-$ which induces an isomorphism

$$K_0T(\mathbb{Z}_pG) \simeq K_0T(\mathbb{Z}_pG_+) \oplus K_0T(\mathbb{Z}_pG_-). \tag{16}$$

A character χ is called even if $\chi(j) = \chi(1)$, and it is called odd if $\chi(j) = -\chi(1)$. Let us define $R_p^+(G)$ and $R_p^-(G)$ to be the subrings of $R_p(G)$ generated by even and odd characters, respectively. The Hom description and the above isomorphism now give

$$\frac{\mathrm{Hom}_{\Gamma_\wp}(R_p(G), F_\wp^\times)}{\mathrm{Det}\,(\mathbb{Z}_pG^\times)} \simeq \frac{\mathrm{Hom}_{\Gamma_\wp}(R_p^+(G), F_\wp^\times)}{\mathrm{Det}\,(\mathbb{Z}_pG_+^\times)} \oplus \frac{\mathrm{Hom}_{\Gamma_\wp}(R_p^-(G), F_\wp^\times)}{\mathrm{Det}\,(\mathbb{Z}_pG_-^\times)},$$

induced by the canonical restriction maps.
We denote the image of $\Omega_\phi^{(p)}$ in $K_0T(\mathbb{Z}_pG_+)$ and $K_0T(\mathbb{Z}_pG_-)$ by $\Omega_\phi^{(p),+}$ and $\Omega_\phi^{(p),-}$, respectively. Accordingly, the LRNC at p decomposes into a plus part and a minus part:

Proposition 1. *Let $p \neq 2$ be a rational prime and L/K a Galois CM-extension with Galois group G. The LRNC at p (Conjecture 2) is true if and only if the following two assertions hold*

(1) $\Omega_\phi^{(p),+}$ has representing homomorphism $[\chi \mapsto (A_\phi^{\mathrm{mod}})^{(p)}(\check{\chi})] \in \mathrm{Hom}_{\Gamma_\wp}(R_p^+(G), F_\wp^\times)$.

(2) $\Omega_\phi^{(p),-}$ has representing homomorphism $[\chi \mapsto (A_\phi^{\mathrm{mod}})^{(p)}(\check{\chi})] \in \mathrm{Hom}_{\Gamma_\wp}(R_p^-(G), F_\wp^\times)$.

2.1 Ray class groups

If T is a finite G-invariant set of non-archimedean places of L we write cl_L^T for the ray class group to the ray $\mathfrak{M}_T := \prod_{\mathfrak{P}\in T} \mathfrak{P}$. Let S be a second finite G-invariant set of places of L which contains all the archimedean primes and satisfies $S \cap T = \emptyset$. We write S_f for the set of all finite primes in S. There is a natural map $\mathbb{Z}S_f \to \mathrm{cl}_L^T$ which sends each prime $\mathfrak{P} \in S_f$ to the corresponding class $[\mathfrak{P}] \in \mathrm{cl}_L^T$. We denote the cokernel of this map by $\mathrm{cl}_S^T(L) =: \mathrm{cl}_S^T$. Further, define $E_S^T := \{x \in E_S : x \equiv 1 \bmod \mathfrak{M}_T\}$. Since the sets S and T are both G-invariant, all these modules are equipped with a natural G-action. We have the following exact sequences of G-modules

$$E^T_{S_\infty} \rightarrowtail E^T_S \xrightarrow{v} \mathbb{Z}S_f \to \mathrm{cl}^T_L \twoheadrightarrow \mathrm{cl}^T_S, \tag{17}$$

where $v(x) = \sum_{\mathfrak{P}\in S_f} v_{\mathfrak{P}}(x)\mathfrak{P}$ for $x \in E^T_S$, and

$$E^T_S \rightarrowtail E_S \to (\mathfrak{o}_S/\mathfrak{M}_T)^\times \xrightarrow{\nu} \mathrm{cl}^T_S \twoheadrightarrow \mathrm{cl}_S, \tag{18}$$

where the map ν lifts an element $\overline{x} \in (\mathfrak{o}_S/\mathfrak{M}_T)^\times$ to $x \in \mathfrak{o}_S$ and sends it to the ideal class $[(x)] \in \mathrm{cl}^T_S$ of the principal ideal (x). We define

$$A^T_S := (\mathrm{cl}^T_S)^-.$$

If $S = S_\infty$, we also write A^T_L and E^T_L instead of $A^T_{S_\infty}$ and $E^T_{S_\infty}$.

Since $E^-_L = \mu_L$, one can always find primes $\mathfrak{P}$ of L such that $(E^T_L)^- = 1$ for all sets of places T with $\mathfrak{P} \in T$. One only has to check that $1-\zeta \notin \prod_{g\in G/G_{\mathfrak{P}}} \mathfrak{P}^g$ for all $\zeta \in \mu_L$, $\zeta \neq 1$; this is true for all but finitely many primes of L.

Theorem 1. *Let L/K be a Galois CM-extension with Galois group G, $p \neq 2$ a rational prime and $S_p = \{\mathfrak{P} \subset L : \mathfrak{P} \mid p\}$. Assume that for all $\mathfrak{P} \in S_p \cap S_{\mathrm{ram}}$ the ramification is tame or $j \in G_{\mathfrak{P}}$. Choose a prime $\mathfrak{P}_0$ of L such that $1-\zeta \notin \prod_{g\in G/G_{\mathfrak{P}_0}} \mathfrak{P}_0^g$ for all $\zeta \in \mu_L$, $\zeta \neq 1$. Then $A^T_L \otimes \mathbb{Z}_p$ is a c.t. G-module for each finite G-invariant set T of places of L that contains $\mathfrak{P}_0$ and all the ramified primes which are not in S_p.*

Remark 3. If L/K is tame above p and G is abelian, the above theorem follows from the proof of Proposition 7 in [Gr2].

Proof. It suffices to show that $H^i(P, A^T_L \otimes \mathbb{Z}_p) = 1$ for $i \in \mathbb{Z}$ and all q-Sylow subgroups P of G. This is clear for $q \neq p$. So let P be a p-Sylow subgroup.
For any prime $\mathfrak{P}$ of L we write $U^0_{\mathfrak{P}}$ for the group of local units of the completion $L_{\mathfrak{P}}$ of L at $\mathfrak{P}$. Furthermore, we denote the group of local units congruent to 1 mod $\mathfrak{P}^n$ by $U^n_{\mathfrak{P}}$. Let us define an idèle subgroup

$$J^T_L := \prod_{\mathfrak{P}\in T} U^1_{\mathfrak{P}} \times \prod_{\mathfrak{P}\notin T} U^0_{\mathfrak{P}}.$$

Let C_L be the idèle class group of L. The following exact sequences define C^T_L:

$$E^T_L \rightarrowtail J^T_L \twoheadrightarrow C^T_L, \tag{19}$$

$$C^T_L \rightarrowtail C_L \twoheadrightarrow \mathrm{cl}^T_L. \tag{20}$$

For both sequences we take the long exact sequence in cohomology with respect to P. Thereafter, we apply the minus functor, which is exact in this case, since all the occurring cohomology groups are finite of odd order. The fact that $\mathfrak{P}_0 \in T$ forces

$$H^i(P, E^T_L)^- = H^i(P, (E^T_L)^-) = H^i(P, 1) = 1,$$

and hence sequence (19) implies $H^i(P, J^T_L)^- \simeq H^i(P, C^T_L)^-$. Global class field theory admits a similar argument for sequence (20):

$$H^i(P, C_L)^- \simeq H^{i-2}(P, \mathbb{Z})^- = H^{i-2}(P, \mathbb{Z}^-) = H^{i-2}(P, 0) = 1$$

and we therefore get isomorphisms

$$H^{i+1}(P, C^T_L)^- \simeq H^i(P, \mathrm{cl}^T_L)^- = H^i(P, \mathrm{cl}^T_L \otimes \mathbb{Z}_p)^- = H^i(P, A^T_L \otimes \mathbb{Z}_p).$$

Hence, it suffices to show that $H^i(P, J_L^T)^- = 1$ for all $i \in \mathbb{Z}$. The unit groups $U_{\mathfrak{P}}^n$ are c.t. $P_{\mathfrak{P}}$-modules if $\mathfrak{P}$ does not ramify in L/K, where we recall that $P_{\mathfrak{P}}$ denotes the decomposition subgroup of P at the prime $\mathfrak{P}$. Even before taking minus parts, we thus get an isomorphism

$$H^i(P, J_L^T) \simeq \prod_{\mathfrak{p} \in S_{\mathrm{ram}}(K)} H^i(P, \prod_{\mathfrak{P}|\mathfrak{p}} U_{\mathfrak{P}}^{n_{\mathfrak{P}}}),$$

where $n_{\mathfrak{P}}$ is equal to 1 or 0 depending on wether $\mathfrak{P} \in T$ or not. If $\mathfrak{p}$ lies over a rational prime $q \neq p$, we have $n_{\mathfrak{P}} = 1$ for all $\mathfrak{P} \mid \mathfrak{p}$ by assumption. But in this case the unit groups $U_{\mathfrak{P}}^1$ are pro-q-groups and thus $H^i(P, \prod_{\mathfrak{P}|\mathfrak{p}} U_{\mathfrak{P}}^1) = 1$.
We are left with the case $\mathfrak{P} \in S_{\mathrm{ram}} \cap S_p$. For this let F be the fixed field of P, and indicate the primes in F by a subscript F. We have

$$H^i(P, \prod_{\mathfrak{P}|\mathfrak{p}} U_{\mathfrak{P}}^{n_{\mathfrak{P}}}) \simeq \prod_{\mathfrak{p}_F|\mathfrak{p}} H^i(P, \prod_{\mathfrak{P}|\mathfrak{p}_F} U_{\mathfrak{P}}^{n_{\mathfrak{P}}}) = \prod_{\mathfrak{p}_F|\mathfrak{p}} H^i(P_{\mathfrak{P}}, U_{\mathfrak{P}}^{n_{\mathfrak{P}}}).$$

If $\mathfrak{P}$ is tamely ramified, it cannot ramify in L/F, since $P_{\mathfrak{P}}$ is a p-group. Hence, we get $H^i(P_{\mathfrak{P}}, U_{\mathfrak{P}}^{n_{\mathfrak{P}}}) = 1$ in this case. If otherwise $j \in G_{\mathfrak{P}}$, the action of j commutes with the above isomorphism, and we have to show that $H^i(P_{\mathfrak{P}}, U_{\mathfrak{P}}^{n_{\mathfrak{P}}})^- = 1$, $n_{\mathfrak{P}} \in \{0,1\}$. By local class field theory

$$H^i(P_{\mathfrak{P}}, L_{\mathfrak{P}}^\times)^- \simeq H^{i-2}(P_{\mathfrak{P}}, \mathbb{Z})^- = H^{i-2}(P_{\mathfrak{P}}, \mathbb{Z}^-) = H^{i-2}(P_{\mathfrak{P}}, 0) = 1$$

and hence the short exact sequence $U_{\mathfrak{P}} \rightarrowtail L_{\mathfrak{P}}^\times \twoheadrightarrow \mathbb{Z}$ implies $H^i(P_{\mathfrak{P}}, U_{\mathfrak{P}})^- = 1$. Finally, the sequence $U_{\mathfrak{P}}^1 \rightarrowtail U_{\mathfrak{P}} \twoheadrightarrow (\mathfrak{o}/\mathfrak{P})^\times$ forces $H^i(P_{\mathfrak{P}}, U_{\mathfrak{P}}^1)^- = H^i(P_{\mathfrak{P}}, U_{\mathfrak{P}})^- = 1$, since the order of $(\mathfrak{o}/\mathfrak{P})^\times$ is relatively prime to p, and hence $H^i(P_{\mathfrak{P}}, (\mathfrak{o}/\mathfrak{P})^\times) = 1$. □

2.2 L-series and Stickelberger elements

In this section we fix, as before, a Galois CM-extension L/K of number fields with Galois group G, and denote the maximal real subfield of L by L^+. Let $w_L = |\mu_L|$ be the number of roots of unity in L and $Q := [E_L : \mu_L E_{L^+}]$ which is equal to 1 or 2 by [Wa], Theorem 4.12. By loc.cit. Theorem 4.10 the class number of L^+ divides the class number of L; the quotient h_L^- is called the relative class number.
The completed Artin L-series is defined to be

$$\Lambda(L/K, \chi, s) = c(L/K, \chi)^{s/2} \mathfrak{L}_\infty(L/K, \chi, s) L_{S_\infty}(L/K, \chi, s),$$

where

$$c(L/K, \chi) = |d_K|^{\chi(1)} N(\mathfrak{f}(\chi))$$
$$\mathfrak{L}_\infty(L/K, \chi, s) = \begin{cases} L_{\mathbb{R}}(s)^{|S_\infty(K)|\chi(1)} & \text{if } \chi \text{ is even} \\ L_{\mathbb{R}}(s+1)^{|S_\infty(K)|\chi(1)} & \text{if } \chi \text{ is odd} \end{cases}$$
$$L_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Here, d_K is the discriminant of the number field K, $\mathfrak{f}(\chi)$ the Artin conductor of the character χ and $\Gamma(s)$ the usual complex Gamma function. The completed Artin L-series satisfies the functional equation

$$\Lambda(L/K, \chi, s) = W(\chi) \Lambda(L/K, \check{\chi}, 1-s), \tag{21}$$

where $W(\chi)$ is the Artin root number of the character χ. We now prove the following result:

Proposition 2. *Let L/K be a Galois CM-extension of number fields with Galois group G. Keeping the above notation we have*

$$\prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi \,\mathrm{odd}}} L_{S_\infty}(L/K, \chi, 0)^{\chi(1)} = \pm \frac{2^{|S_\infty|} \cdot h_L^-}{Q \cdot w_L},$$

where the product runs through all the odd irreducible characters of G.

Proof. Let us denote the Dedekind zeta function of a number field F by $\zeta_F(s)$. We have (cf. [Neu], Kap. VII, Korollar (10.5))

$$\zeta_L(s) = \zeta_K(s) \prod_{1_G \neq \chi \in \mathrm{Irr}\,(G)} L_{S_\infty}(L/K, \chi, s)^{\chi(1)},$$

where we denote the trivial character by 1_G. Taking residues at $s = 1$ of both sides yields

$$\frac{(2\pi)^{|S_\infty|} \cdot h_L R_L}{w_L \sqrt{|d_L|}} = \mathrm{res}\,_{s=1} \zeta_K(s) \prod_{1_G \neq \chi \in \mathrm{Irr}\,(G)} L_{S_\infty}(L/K, \chi, 1)^{\chi(1)}$$

where R_L denotes the regulator of L. If we divide this equation by the corresponding equation for L^+, we get by [Wa], Proposition 4.16

$$\frac{(2\pi)^{|S_\infty|} \cdot h_L^-}{Q w_L \sqrt{|d_L/d_{L^+}|}} = \prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi \text{ odd}}} L_{S_\infty}(L/K, \chi, 1)^{\chi(1)}.$$

Specializing the functional equation (21) at $s = 1$ for odd characters χ gives

$$\begin{aligned} \frac{(2\pi)^{|S_\infty|} \cdot h_L^-}{Q w_L \sqrt{|d_L/d_{L^+}|}} &= \prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi \text{ odd}}} \left(L_{S_\infty}(L/K, \check{\chi}, 0) W(\chi) c(L/K, \chi)^{-1/2} \pi^{|S_\infty(K)| \chi(1)} \right)^{\chi(1)} \\ &\overset{(*)}{=} \pm \frac{\pi^{|S_\infty|}}{\sqrt{|d_K|^{|G|/2}}} \prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi \text{ odd}}} \left(L_{S_\infty}(L/K, \chi, 0) N(\mathfrak{f}(\chi))^{-1/2} \right)^{\chi(1)}, \end{aligned} \tag{22}$$

where (*) holds, since $\sum_{\chi \text{ odd}} \chi(1)^2 = |G|/2$ and $\prod_{\chi \text{ odd}} W(\chi) = \pm 1$, as the product is real and has absolute value 1 (cf. [Neu], Kap. VII, Theorem (12.6)).
Let us write $\delta_{E/F}$ for the relative discriminant of an extension E/F of number fields, in particular $\delta_{E/\mathbb{Q}} = (d_E)$. We now compute

$$\begin{aligned} \prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi \text{ odd}}} N(\mathfrak{f}(\chi))^{\chi(1)} &= \frac{\prod_{\chi \in \mathrm{Irr}\,(G)} N(\mathfrak{f}(\chi))^{\chi(1)}}{\prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi \text{ even}}} N(\mathfrak{f}(\chi))^{\chi(1)}} \overset{(1)}{=} \frac{N(\delta_{L/K})}{N(\delta_{L^+/K})} \\ &\overset{(2)}{=} N(\delta_{L^+/K}) N(\delta_{L/L^+}) \overset{(2)}{=} N(\delta_{L^+/K}) \frac{|d_L|}{|d_{L^+}|^2} \\ &\overset{(2)}{=} \frac{|d_L|}{|d_{L^+}| \cdot |d_K|^{|G|/2}}. \end{aligned} \tag{23}$$

Equality (1) follows from the "Führerdiskriminantenproduktformel" (cf. [Neu], Kap. VII, (11.9)). For the equalities (2) note that in any tower $F \subset E \subset M$ of number fields we have $\delta_{M/F} = \delta_{E/F}^{[M:E]} N_{E/F}(\delta_{M/E})$. If we put (23) in (22), we obtain the desired result. □

We define the following variant of a Stickelberger element which is closely related to the non-abelian Stickelberger-functions defined in [Ha]:

$$\omega := \sum_{\chi \in \mathrm{Irr}\,(G)} L_{S_\infty}(L/K, \check{\chi}, 0) \varepsilon_\chi \in Z(\mathbb{C}G) \tag{24}$$

Each $\mathbb{C}$-valued function on G extends to a $\mathbb{C}$-linear function on $\mathbb{C}G$. In particular, this applies to the irreducible characters of G, and obviously

$$\chi(\omega) = \chi(1) L_{S_\infty}(L/K, \check{\chi}, 0).$$

This property uniquely defines ω. If G is abelian, this element coincides with the element ω defined in [Gr3]. A priori, ω is an element of the group ring $\mathbb{C}G$, but we actually have

Proposition 3. *$\omega \in Z(\mathbb{Q}G)$, and even $\omega \in Z(\mathbb{Q}G_-)^\times$ if $|S_\infty| > 1$.*

Proof. Note that the vanishing order of $L_{S_\infty}(L/K, \chi, s)$ in $s = 0$ equals

$$r_{S_\infty}(\chi) = \sum_{\mathfrak{P} \in S_\infty} \dim V_\chi^{G_\mathfrak{P}} - \dim V_\chi^G$$

by [Ta2], Proposition 3.4, p. 24. Hence, $L_{S_\infty}(L/K, \chi, 0) \neq 0$ if and only if χ is odd or χ is the trivial character and $|S_\infty| = 1$. This shows $\omega \in Z(\mathbb{C}G_-)^\times$ if $|S_\infty| > 1$. The coefficient of ω at $g \in G$ equals

$$\sum_{\chi \in \mathrm{Irr}\,(G)} L_{S_\infty}(L/K, \check{\chi}, 0) \frac{\chi(1)}{|G|} \chi(g^{-1})$$

which is invariant under Galois action by Stark's conjecture, which is a theorem for odd characters and the trivial character (cf. [Ta2] Th. 1.2, p. 70 and Prop. 1.1, p. 44). □

Note that the proof also shows that in any case $\frac{1-j}{2}\omega \in Z(\mathbb{Q}G_-)^\times$.

Definition 1. *Let L/K be a Galois CM-extension with Galois group G and S, T be G-invariant sets of places of L. We define a Stickelberger element $\theta_S^T \in Z(\mathbb{C}G)$ by declaring*

$$\chi(\theta_S^T) = \chi(\omega) \prod_{\mathfrak{P} \in T^\diamond} \det(1 - \phi_\mathfrak{P}^{-1} q_\mathfrak{p} | V_\chi^{I_\mathfrak{P}}) \prod_{\mathfrak{P} \in S^\diamond} \det(1 - \phi_\mathfrak{P}^{-1} | V_\chi^{I_\mathfrak{P}} / V_\chi^{G_\mathfrak{P}}),$$

on irreducible characters χ, where $q_\mathfrak{p} = N(\mathfrak{p})$.

Since $\chi(\theta_S^T)$ differs from $\chi(\omega)$ by a factor which commutes with Galois action for each odd irreducible character χ, it follows from Proposition 3 that $\frac{1-j}{2}\theta_S^T \in Z(\mathbb{Q}G_-)^\times$. Let $R^-(G)$ be the free $\mathbb{Z}$-module generated by the odd irreducible characters of G. This enables us to make the following

Definition 2. *Let $F/\mathbb{Q}$ be a finite Galois extension with Galois group Γ such that each odd character of G can be realized over F. Then we define $\Theta_S^T \in \mathrm{Hom}_\Gamma(R^-(G), F^\times)$ by declaring*

$$\Theta_S^T(\chi) = \chi(1)^{-1} \chi(\theta_S^T)$$

on irreducible odd characters χ.

To afford an easier reading we will refer to the following setting as $(*)$:

- L/K is a Galois CM-extension with Galois group G.
- $p \neq 2$ is a rational prime.
- $S_p = \{\mathfrak{P} \subset L : \mathfrak{P} \mid p\}$
- Each $\mathfrak{P} \in S_p \cap S_{\mathrm{ram}}$ is at most tamely ramified or $j \in G_\mathfrak{P}$.
- $\mathfrak{P}_0$ is a prime of L, unramified in L/K such that $1 - \zeta \notin \prod_{g \in G/G_{\mathfrak{P}_0}} \mathfrak{P}_0^g$ for all $\zeta \in \mu_L$, $\zeta \neq 1$.
- T is a finite G-invariant set of places of L that contains $\mathfrak{P}_0$ and all the ramified primes which are not in S_p; $T \cap S_p = \emptyset$.
- S_1 is the set of all wildly ramified primes above p.

There is the following correspondence between the Stickelberger elements and the ray class groups $A_L^T \otimes \mathbb{Z}_p$.

Proposition 4. *Fix a setting $(*)$. Then there exists an $\alpha \in \mathbb{Z}_p^\times$ such that*

$$|A_L^T \otimes \mathbb{Z}_p| = \alpha \cdot \prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi\,\mathrm{odd}}} (\Theta_{S_1}^T(\chi))^{\chi(1)}.$$

Moreover, if G is abelian, we have $\frac{1-j}{2}\theta_{S_1}^T \in \mathbb{Z}_p G_-$ and

$$|A_L^T \otimes \mathbb{Z}_p| = |(\mathbb{Z}_p G)_- / \theta_{S_1}^T (\mathbb{Z}_p G)_-|.$$

Proof. For an integer $m \in \mathbb{Z}$ let $m_p := p^{v_p(m)}$. Then the minus part of sequence (18) for $S = S_\infty$ tensored with $\mathbb{Q}$ implies the equality

$$|A_L^T \otimes \mathbb{Z}_p| = |A_L^T|_p = \frac{h_{L,p}^-}{w_{L,p}} |(\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-}|_p. \tag{25}$$

Let us write $a \sim b$ if $ab^{-1} \in \mathbb{Z}_p^\times$. Then

$$\prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi\, \mathrm{odd}}} (\chi(1)^{-1}\chi(\omega))^{\chi(1)} = \prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi\, \mathrm{odd}}} L_{S_\infty}(L/K,\chi,0)^{\chi(1)} \sim \frac{h_{L,p}^-}{w_{L,p}}$$

by Proposition 2. For $\mathfrak{P} \in T$ we compute

$$\begin{aligned} \prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi\, \mathrm{odd}}} \det(1 - \phi_{\mathfrak{P}}^{-1} q_{\mathfrak{p}} | V_\chi^{I_{\mathfrak{P}}}) &= \det(1 - \phi_{\mathfrak{P}}^{-1} q_{\mathfrak{p}} | \bigoplus_{\chi\, \mathrm{odd}} \chi(1) V_\chi^{I_{\mathfrak{P}}}) \\ &\sim |(\mathrm{ind}\,_{G_{\mathfrak{P}}}^G (\mathbb{Z}_p\overline{G_{\mathfrak{P}}}/q_{\mathfrak{p}} - \phi_{\mathfrak{P}}))^-| \\ &\overset{(1)}{=} |(\mathrm{ind}\,_{G_{\mathfrak{P}}}^G (\mathfrak{o}_L/\mathfrak{P})^\times)^-|_p. \end{aligned}$$

Here, equation (1) derives from the exact sequence

$$\mathbb{Z}_p\overline{G_{\mathfrak{P}}} \rightarrowtail \mathbb{Z}_p\overline{G_{\mathfrak{P}}} \twoheadrightarrow (\mathfrak{o}_L/\mathfrak{P})^\times \otimes \mathbb{Z}_p,$$

where the first map is $1 \mapsto q_{\mathfrak{p}} - \phi_{\mathfrak{P}}$ and the second sends 1 to a generator of $(\mathfrak{o}_L/\mathfrak{P})^\times$. Since $j \in G_{\mathfrak{P}}$ for all primes $\mathfrak{P} \in S_1$, we have

$$\prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi\, \mathrm{odd}}} \det(1 - \phi_{\mathfrak{P}}^{-1} | V_\chi^{I_{\mathfrak{P}}}/V_\chi^{G_{\mathfrak{P}}}) \sim 1.$$

Indeed, if actually $j \in I_{\mathfrak{P}}$, the determinant equals 1. Otherwise it is a product of some $1 - \zeta_{2m}$, where ζ_{2m} are roots of unity of even order, and hence relatively prime to p. Thus, we get by (25)

$$\prod_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi\, \mathrm{odd}}} (\Theta_{S_1}^T(\chi))^{\chi(1)} \sim \frac{h_{L,p}^-}{w_{L,p}} \prod_{\mathfrak{P} \in T^\diamond} |(\mathrm{ind}\,_{G_{\mathfrak{P}}}^G (\mathfrak{o}_L/\mathfrak{P})^\times)^-|_p = |A_L^T|_p.$$

Now let G be abelian. If $\frac{1-j}{2}\theta_{S_1}^T \in \mathbb{Z}_p G_-$, the left-hand side of the above equation equals $|(\mathbb{Z}_p G)_-/\theta_{S_1}^T(\mathbb{Z}_p G)_-|$. Finally, the integrality of $\frac{1-j}{2}\theta_{S_1}^T$ follows from [Ca] p. 49. More precisely, define for each prime $\mathfrak{P}$ a local module $M_{\mathfrak{P}}$ by

$$M_{\mathfrak{P}} = \langle N_{I_{\mathfrak{P}}}, 1 - |I_{\mathfrak{P}}|^{-1} N_{I_{\mathfrak{P}}} \phi_{\mathfrak{P}}^{-1} \rangle_{\mathbb{Z} I_{\mathfrak{P}}} \subset \mathbb{Q} I_{\mathfrak{P}}. \tag{26}$$

Let $\mathfrak{A} = \mathrm{Ann}_{\mathbb{Z}G}(\mu_L)$ be the annihilator of the roots of unity in L. In [Gr3] the author defines the Sinnott-Kurihara ideal to be

$$SKu(L/K) = \mathfrak{A} \prod_{\mathfrak{P} \in S_{\mathrm{ram}}^\diamond} M_{\mathfrak{P}} \cdot \omega \mathbb{Z} G$$

which is actually contained in $\mathbb{Z}G$ (cf. loc.cit., end of §2). The proof of Proposition 4 gets completed by means of the following

Lemma 1. *Fix a setting (∗) and let G be abelian. Then*

$$\frac{1-j}{2}\theta_{S_1}^T \in SKu(L/K)^- \cdot \mathbb{Z}_p G.$$

Proof. Evaluating at odd irreducible characters χ of G shows that

$$\frac{1-j}{2}\theta_{S_1}^T = \frac{1-j}{2}\omega \prod_{\mathfrak{P}\in T^\diamond}(1-|I_\mathfrak{P}|^{-1}N_{I_\mathfrak{P}}\phi_\mathfrak{P}^{-1}q_\mathfrak{p}) \prod_{\mathfrak{P}\in S_1^\diamond}(1-|I_\mathfrak{P}|^{-1}N_{I_\mathfrak{P}}\phi_\mathfrak{P}^{-1}).$$

The condition on the prime $\mathfrak{P}_0 \in T$ causes $1-\phi_{\mathfrak{P}_0}^{-1}q_{\mathfrak{p}_0} \in \mathfrak{A}$. Let $\mathfrak{P} \in S_{\mathrm{ram}}^\diamond \cap T^\diamond$ and $q \in \mathbb{Z}$ the rational prime below $\mathfrak{P}$. If we denote the q-Sylow subgroup of the inertia group $I_\mathfrak{P}$ by $R_\mathfrak{P}$, the intermediate extension corresponding to $G_\mathfrak{P}/R_\mathfrak{P}$ is tame at $\mathfrak{P}$. Therefore, by [Ch2], p.369 the ramification index $e_\mathfrak{P} = |I_\mathfrak{P}|$ divides $q_\mathfrak{p}-1$ up to a power of q, since G is abelian. Hence

$$1-|I_\mathfrak{P}|^{-1}N_{I_\mathfrak{P}}\phi_\mathfrak{P}^{-1}q_\mathfrak{p} = 1-|I_\mathfrak{P}|^{-1}N_{I_\mathfrak{P}}\phi_\mathfrak{P}^{-1} - \phi_\mathfrak{P}^{-1}\frac{q_\mathfrak{p}-1}{e_\mathfrak{P}}N_{I_\mathfrak{P}} \in M_\mathfrak{P}\cdot\mathbb{Z}_pG.$$

For the tamely ramified primes above p the element

$$e_\mathfrak{P} = (e_\mathfrak{P}-N_{I_\mathfrak{P}})(1-|I_\mathfrak{P}|^{-1}N_{I_\mathfrak{P}}\phi_\mathfrak{P}^{-1}) + N_{I_\mathfrak{P}} \in M_\mathfrak{P}$$

lies in $\mathbb{Z}_pG^\times$, since $p \nmid e_\mathfrak{P}$. Therefore, we get $M_\mathfrak{P}\cdot\mathbb{Z}_pG = \mathbb{Z}_pG$ in this case. Finally, we obviously have $(1-|I_\mathfrak{P}|^{-1}N_{I_\mathfrak{P}}\phi_\mathfrak{P}^{-1}) \in M_\mathfrak{P}$ for the primes $\mathfrak{P}\in S_1$. □

We are going to show that the minus part of the LRNC for L/K at p can be restated in terms of a representing homomorphism for $A_L^T \otimes \mathbb{Z}_p$. The homomorphism involved is just the image of $\Theta_{S_1}^T$ in $\mathrm{Hom}_{\Gamma_\wp}(R_p^-(G), F_\wp^\times)$. Hence, Proposition 4 will give some evidence of the conjecture by means of the following

Proposition 5. *Let G be a finite group, p a finite rational prime and $R_p = \mathbb{Z}_pG$ (or $R_p = \mathbb{Z}_pG_+$, $\mathbb{Z}_pG_-$ if $p \neq 2$). If a finite c.t. R_p-module A has representing homomorphism $\chi \mapsto f(\chi)$, there exists an $\alpha \in \mathbb{Z}_p^\times$ such that*

$$|A| = \alpha \cdot \prod_{\chi\in\mathrm{Irr}\,(G)} f(\chi)^{\chi(1)},$$

where we set $f(\chi)=1$ if $R_p = \mathbb{Z}_pG_+$ and χ is odd or if $R_p = \mathbb{Z}_pG_-$ and χ is even.

Proof. We only treat the case where $R_p = \mathbb{Z}_pG$; the others are similar. Since $|\cdot|$ is multiplicative on short exact sequences of finite modules, we get a well defined map $|\cdot| : K_0T(\mathbb{Z}_pG) \to \mathbb{Z}$. Since a c.t. $\mathbb{Z}_pG$-module has projective dimension at most 1, there is an injection $\phi : \mathbb{Z}_pG^n \rightarrowtail \mathbb{Z}_pG^n$ such that $A = \operatorname{cok}\phi$. Hence A has representing homomorphism

$$\chi \mapsto \det(\phi|\mathrm{Hom}_{\Gamma_\wp}(V_\chi, F_\wp G^n)).$$

We compute

$$\begin{aligned}\prod_{\chi\in\mathrm{Irr}\,(G)} \det(\phi|\mathrm{Hom}_{\Gamma_\wp}(V_\chi, F_\wp G^n))^{\chi(1)} &= \det(\phi|\mathrm{Hom}_{\Gamma_\wp}(\bigoplus_{\chi\in\mathrm{Irr}\,(G)}\chi(1)V_\chi, F_\wp G^n))\\ &= \det(\phi|\mathrm{Hom}_{\Gamma_\wp}(F_\wp G, F_\wp G^n))\\ &= \det(\phi|F_\wp G^n)\\ &= \alpha\cdot|\operatorname{cok}\phi|\end{aligned}$$

with an appropriate element $\alpha \in \mathbb{Z}_p^\times$. □

Remark 4. If G is abelian, the elements in $K_0T(R_p)$ can be described in terms of Fitting ideals. In this context Proposition 5 simply repeats the well known fact that $|A| = |R_p/\mathrm{Fitt}_{R_p}(A)|$ for each finite c.t. R_p-module A.

2.3 A restatement of the LRNC on minus parts

The aim of this section is to prove

Theorem 2. *Fix a setting* $(*)$, *where* $T = (S_{\mathrm{ram}} \setminus (S_{\mathrm{ram}} \cap S_p)) \cup \{\mathfrak{P}_0^g : g \in G\}$. *Then the minus part of the LRNC at* p *holds for* L/K *if and only if* $\Theta_{S_1}^T$ *is the representing homomorphism of the class of* $A_L^T \otimes \mathbb{Z}_p$ *in* $K_0T(\mathbb{Z}_pG_-)$.

Remark 5. In the above theorem we view $\Theta_{S_1}^T$ as an element of $\mathrm{Hom}_{\Gamma_\wp}(R_p^-(G), F_\wp^\times)$ via an embedding $F \rightarrowtail F_\wp$. Note that this homomorphism represents the class of $A_L^T \otimes \mathbb{Z}_p$ in $K_0T(\mathbb{Z}_pG_-)$ if and only if $\hat{\partial}_p(\theta_{S_1}^T) = i_G(A_L^T \otimes \mathbb{Z}_p)$.

Let us choose $S = S_f \cup S_\infty$, where S_f is a set of totally decomposed primes such that the ray class group cl_L^T is generated by these primes and $S_f \cap T = \emptyset$. In particular, the S-class group cl_S is trivial, and $\nabla = \overline{\nabla}$. Moreover, $\mathbb{Z}S_f$ is $\mathbb{Z}G$-free of rank $s_f = |S_f^\diamond|$. Tensoring with $\mathbb{Z}_p$ and taking minus parts of sequence (17) yields

$$E_S^{T,-} \otimes \mathbb{Z}_p \rightarrowtail \mathbb{Z}_pS^- \twoheadrightarrow A_L^T \otimes \mathbb{Z}_p. \tag{27}$$

Since $\mathbb{Z}_pS^- = \mathbb{Z}_pS_f^-$ is $\mathbb{Z}_pG_-$-free and $A_L^T \otimes \mathbb{Z}_p$ is c.t. by Theorem 1, we have proven

Lemma 2. *The* $\mathbb{Z}_pG_-$*-module* $E_S^{T,-} \otimes \mathbb{Z}_p$ *is cohomologically trivial.*

Since the cokernel of the injection $\iota_E : E_S^T \rightarrowtail E_S$ is finite, we may choose an equivariant injection $\phi_S^T : \Delta S \rightarrowtail E_S^T$ with finite cokernel and we define $\phi_S := \iota_E \circ \phi_S^T$. Due to the choice of the set S, we can fix an isomorphism

$$\rho_S : \Delta S^- \xrightarrow{\simeq} (\mathbb{Z}G_-)^{s_f}.$$

In particular, the minus part of $\mathrm{cok}\,\phi_S^T \otimes \mathbb{Z}_p$ is c.t. We build the following commutative diagram which defines a monomorphism ψ (note that we have invisibly tensored with $\mathbb{Z}[\frac{1}{2}]$):

$$\begin{array}{ccccc}
\Delta S^- & \xrightarrow[\simeq]{\rho_S} & (\mathbb{Z}G_-)^{s_f} & & \\
\big\downarrow{\scriptstyle \phi_S^T} & & \big\downarrow{\scriptstyle \psi} & & \\
E_S^{T,-} & \hookrightarrow & (\mathbb{Z}G_-)^{s_f} & \twoheadrightarrow & A_L^T \\
\big\downarrow & & \big\downarrow & & \big\| \\
\mathrm{cok}\,\phi_S^T & \hookrightarrow & \mathrm{cok}\,\psi & \twoheadrightarrow & A_L^T
\end{array}$$

Here, the middle row derives from sequence (17) and we obtain sequence (27) if we tensor with $\mathbb{Z}_p$. We obviously have an equality

$$i_G(\mathrm{cok}\,\phi_S^T \otimes \mathbb{Z}_p) = i_G(\mathrm{cok}\,\psi \otimes \mathbb{Z}_p) - i_G(A_L^T \otimes \mathbb{Z}_p). \tag{28}$$

Lemma 3. *The element* $i_G(\mathrm{cok}\,\psi) \in K_0(\mathbb{Z}[\frac{1}{2}]G_-, \mathbb{Q})$ *has representing homomorphism*

$$\chi \mapsto \frac{R_{\phi_S}(\check{\chi})}{\prod_{\mathfrak{p} \in S(K)} (-\log N(\mathfrak{p}))^{\dim V_\chi}},$$

where $S(K) := \{\mathfrak{P} \cap K | \mathfrak{P} \in S\}$.

Proof. Let us denote the inclusion $E_S^{T,-} \otimes \mathbb{Z}[\frac{1}{2}] \rightarrowtail (\mathbb{Z}[\frac{1}{2}]G_-)^{s_f}$ by μ. Define a map

$$\begin{aligned} \mathrm{Log} : (\mathbb{Z}[\tfrac{1}{2}]G_-)^{s_f} &\longrightarrow \mathbb{R} \otimes (\mathbb{Z}[\tfrac{1}{2}]G_-)^{s_f} \\ (x_1, \ldots, x_{s_f}) &\mapsto (-\log N(\mathfrak{p}_1) \otimes x_1, \ldots, -\log N(\mathfrak{p}_{s_f}) \otimes x_{s_f}), \end{aligned}$$

where we have numbered the primes in $S(K) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{s_f}\}$. Then

$$\psi = \mu \circ \phi_S \circ \rho_S^{-1}, \quad \lambda_S^- = \rho_S^{-1} \circ \mathrm{Log} \circ \mu,$$

where λ_S^- is the restriction of the Dirichlet map to minus parts. Hence, $\lambda_S^- \phi_S = \rho_S^{-1} \circ \mathrm{Log} \circ \psi \circ \rho_S$, and $i_G(\mathrm{cok}\,\psi)$ has representing homomorphism

$$\chi \mapsto \det(\psi | \mathrm{Hom}_{\mathbb{C}G}(V_\chi, (\mathbb{C}G_-)^{s_f})) = \frac{R_{\phi_S}(\check{\chi})}{\det(\mathrm{Log} | \mathrm{Hom}_{\mathbb{C}G}(V_\chi, (\mathbb{C}G_-)^{s_f}))}.$$

This completes the proof. □

Note that the Stark-Tate regulator occurring in the representing homomorphism of $i_G(\mathrm{cok}\,\psi)$ is closely related to the modified Stark-Tate regulator; more precisely, we have

$$\frac{R_\phi^{\mathrm{mod}}(\chi)}{R_{\phi_S}(\chi)} = \prod_{\mathfrak{P} \in S_{\mathrm{ram}}^\diamond} \left(-\frac{\log N(\mathfrak{P})}{|G_\mathfrak{P}|} \right)^{\dim V_\chi^{G_\mathfrak{P}}} \tag{29}$$

if we define the $\mathbb{Q}G$-isomorphism ϕ in the following way: Recall that for each finite prime $\mathfrak{P}$ of L the element $d_\mathfrak{P} = |G_\mathfrak{P}|^{-1} \kappa(|G_\mathfrak{P}|, N_{G_\mathfrak{P}})$ is a $\mathbb{Q}G_\mathfrak{P}$-generator of $\mathbb{Q}W_\mathfrak{P}^*$, where κ is the epimorphism of sequence (12). Hence, we can define isomorphisms

$$\delta_\mathfrak{P} : \mathbb{Q}W_\mathfrak{P}^* \longrightarrow \mathbb{Q}G_\mathfrak{P}, \quad d_\mathfrak{P} \mapsto 1,$$

and set $d := \sum_{\mathfrak{P} \in S_{\mathrm{ram}}^\diamond} \mathrm{ind}\,\delta_\mathfrak{P}$, where $\mathrm{ind}\,\delta_\mathfrak{P} : \mathrm{ind}_{G_\mathfrak{P}}^G \mathbb{Q}W_\mathfrak{P}^* \to \mathrm{ind}_{G_\mathfrak{P}}^G \mathbb{Q}G_\mathfrak{P} = \mathbb{Q}G$ denotes the map which is induced by $\delta_\mathfrak{P}$. Let C be a $\mathbb{Z}G$-free module of rank s_{ram} with basis $1_\mathfrak{P}$, $\mathfrak{P} \in S_{\mathrm{ram}}^\diamond$, and define ϕ to be the $\mathbb{Q}G$-isomorphism

$$\phi : \mathbb{Q}\nabla \simeq \mathbb{Q}(\Delta S \oplus \textstyle\bigoplus_{\mathfrak{P} \in S_{\mathrm{ram}}^\diamond} \mathrm{ind}_{G_\mathfrak{P}}^G W_\mathfrak{P}^*) \xrightarrow{\mathbb{Q} \otimes \phi_S \oplus d} \mathbb{Q}(E_S \oplus C).$$

Here, the first isomorphism is induced by the natural inclusion on minus parts, whereas we have to choose a splitting of sequence (11) on plus parts (after tensoring with $\mathbb{Q}$). But this choice will play no decisive role, since we are going to deal with minus parts only.

Since Lemma 3 gives the exact relation between the class of $A_L^T \otimes \mathbb{Z}_p$ and the class of $\mathrm{cok}\,\phi_S^T \otimes \mathbb{Z}_p$ in $K_0T(\mathbb{Z}_pG_-)$, and since the latter essentially determines $\Omega_\phi^{(p),-}$, it is now relatively clear that Theorem 2 can be proved. The remaining part of this section deals with the necessary explicit calculations and may be skipped while first reading.

Let $\mathfrak{P}$ be a finite prime of L. Take an exact sequence

$$L_\mathfrak{P}^\times \rightarrowtail V_\mathfrak{P} \twoheadrightarrow \Delta G_\mathfrak{P}$$

whose extension class in $\mathrm{Ext}_{G_\mathfrak{P}}^1(\Delta G_\mathfrak{P}, L_\mathfrak{P}^\times) \simeq H^2(G_\mathfrak{P}, L_\mathfrak{P}^\times)$ is the local fundamental class of $L_\mathfrak{P}/K_\mathfrak{p}$. By [We], Theorem 4 the inertial lattice $W_\mathfrak{P}$ is the push-out along the normalized valuation $v_\mathfrak{P} : L_\mathfrak{P}^\times \twoheadrightarrow \mathbb{Z}$. We are going to repeat this process. We have exact sequences

$$U_\mathfrak{P} \rightarrowtail V_\mathfrak{P} \twoheadrightarrow W_\mathfrak{P}, \quad U_\mathfrak{P}^1 \rightarrowtail U_\mathfrak{P} \twoheadrightarrow (\mathfrak{o}_L/\mathfrak{P})^\times$$

and define $T_{\mathfrak{P}}$ to be the push-out of the first sequence along the projection of the second as shown in the following commutative diagram

$$\begin{array}{ccccc} U^1_{\mathfrak{P}} & = & U^1_{\mathfrak{P}} & & \\ \downarrow & & \downarrow & & \\ U_{\mathfrak{P}} & \hookrightarrow & V_{\mathfrak{P}} & \twoheadrightarrow & W_{\mathfrak{P}} \\ \downarrow & & \downarrow & & \| \\ (\mathfrak{o}_L/\mathfrak{P})^\times & \hookrightarrow & T_{\mathfrak{P}} & \twoheadrightarrow & W_{\mathfrak{P}} \end{array} \tag{30}$$

Lemma 4. *(1) The G-module $\operatorname{ind}_{G_{\mathfrak{P}}}^G T_{\mathfrak{P}} \otimes \mathbb{Z}_p$ is cohomologically trivial for each finite prime $\mathfrak{P} \nmid p$ of L and for each finite prime $\mathfrak{P}$ which is at most tamely ramified in L/K.*
(2) The G-modules $(\operatorname{ind}_{G_{\mathfrak{P}}}^G T_{\mathfrak{P}})^- \otimes \mathbb{Z}_p$, $(\operatorname{ind}_{G_{\mathfrak{P}}}^G W_{\mathfrak{P}})^- \otimes \mathbb{Z}_p$ and $(\operatorname{ind}_{G_{\mathfrak{P}}}^G W^*_{\mathfrak{P}})^- \otimes \mathbb{Z}_p$ are $\mathbb{Z}_p G_-$-free of rank 1 for each finite prime $\mathfrak{P} \mid p$.

Proof. Let P be a p-Sylow subgroup of G. We denote the p-completion of any module M by $\widehat{M}$; especially, if M is finitely generated as $\mathbb{Z}$-module, we have $\widehat{M} = M \otimes \mathbb{Z}_p$.
We start with the case $\mathfrak{P} \nmid p$. Then $\widehat{U^1_{\mathfrak{P}}}$ vanishes, since $U^1_{\mathfrak{P}}$ is a pro-q-group for a prime $q \neq p$. The exact sequence $U^1_{\mathfrak{P}} \rightarrowtail V_{\mathfrak{P}} \twoheadrightarrow T_{\mathfrak{P}}$ now implies that for all $i \in \mathbb{Z}$ we have

$$H^i(P, \operatorname{ind}_{G_{\mathfrak{P}}}^G T_{\mathfrak{P}} \otimes \mathbb{Z}_p) = H^i(P_{\mathfrak{P}}, T_{\mathfrak{P}} \otimes \mathbb{Z}_p) \simeq H^i(P_{\mathfrak{P}}, \widehat{V_{\mathfrak{P}}}) = 1,$$

since $\widehat{V_{\mathfrak{P}}}$ is c.t. by [GW], p. 282.
Now let $\mathfrak{P}$ be a prime above p. Then the bottom sequence of diagram (30) implies that $T_{\mathfrak{P}} \otimes \mathbb{Z}_p = W_{\mathfrak{P}} \otimes \mathbb{Z}_p$. The canonical projection $G_{\mathfrak{P}} \twoheadrightarrow \overline{G_{\mathfrak{P}}}$ induces an exact sequence

$$\Delta(G_{\mathfrak{P}}, I_{\mathfrak{P}}) \rightarrowtail \mathbb{Z}G_{\mathfrak{P}} \twoheadrightarrow \mathbb{Z}\overline{G_{\mathfrak{P}}}.$$

The projection onto the second component of $W_{\mathfrak{P}} \subset \Delta G_{\mathfrak{P}} \times \mathbb{Z}\overline{G_{\mathfrak{P}}}$ yields a quite similar exact sequence

$$\Delta(G_{\mathfrak{P}}, I_{\mathfrak{P}}) \rightarrowtail W_{\mathfrak{P}} \twoheadrightarrow \mathbb{Z}\overline{G_{\mathfrak{P}}}.$$

If $\mathfrak{P}$ is at most tamely ramified in L/K, the $G_{\mathfrak{P}}$-module $\mathbb{Z}_p\overline{G_{\mathfrak{P}}}$ is projective, since the corresponding idempotent lies in $\mathbb{Z}_p G_{\mathfrak{P}}$. Therefore, the p-completed versions of the above two sequences show that $W_{\mathfrak{P}} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p G_{\mathfrak{P}}$ which in particular has vanishing Tate cohomology.
We are left with the case $\mathfrak{P} \mid p$ and $j \in G_{\mathfrak{P}}$. Then j already acts on $G_{\mathfrak{P}}$-modules, and the two exact sequences

$$\mathbb{Z} \rightarrowtail W_{\mathfrak{P}} \twoheadrightarrow \Delta G_{\mathfrak{P}}, \quad \Delta G_{\mathfrak{P}} \rightarrowtail \mathbb{Z}G_{\mathfrak{P}} \twoheadrightarrow \mathbb{Z}$$

imply that $T^-_{\mathfrak{P}} \otimes \mathbb{Z}_p = W^-_{\mathfrak{P}} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p G^-_{\mathfrak{P}}$, since $\mathbb{Z}^-$ and likewise $\mathbb{Z}_p^-$ are zero. As the dual of a free module is free, we are done. □

Now we put

$$\nabla^{T,-} := (\Delta S \oplus \bigoplus_{\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^\diamond} \operatorname{ind}_{G_{\mathfrak{P}}}^G \mathbb{Z}G^2_{\mathfrak{P}} \oplus \bigoplus_{\mathfrak{P} \in (S_{\mathrm{ram}} \cap S_p)^\diamond} \operatorname{ind}_{G_{\mathfrak{P}}}^G W^*_{\mathfrak{P}})^-$$

such that the sequences (12) for the primes $\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^\diamond$ give an exact sequence

$$\mathcal{W}^- \otimes \mathbb{Z}[\tfrac{1}{2}] \rightarrowtail \nabla^{T,-} \otimes \mathbb{Z}[\tfrac{1}{2}] \twoheadrightarrow \nabla^- \otimes \mathbb{Z}[\tfrac{1}{2}], \tag{31}$$

where we have defined $\mathcal{W}$ to be

$$\mathcal{W} = \bigoplus_{\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^\diamond} \operatorname{ind}_{G_{\mathfrak{P}}}^G W_{\mathfrak{P}}.$$

Moreover we define

$$\mathcal{P}_0 = \operatorname{ind}_{G_{\mathfrak{P}_0}}^{G} (\mathfrak{o}_L/\mathfrak{P}_0)^\times, \quad \mathcal{T} = \bigoplus_{\mathfrak{P}\in(S_{\mathrm{ram}}\cap T)^\diamond} \operatorname{ind}_{G_{\mathfrak{P}}}^{G} T_{\mathfrak{P}},$$

such that the canonical surjections $t_{\mathfrak{P}} : T_{\mathfrak{P}} \twoheadrightarrow W_{\mathfrak{P}}$ induce an exact sequence

$$(\mathfrak{o}_L/\mathfrak{M}_T)^\times \rightarrowtail \mathcal{P}_0 \oplus \mathcal{T} \twoheadrightarrow \mathcal{W}. \tag{32}$$

Let us denote the kernel of the epimorphism $B \twoheadrightarrow \nabla$ of the Tate sequence by R. Since B is projective, we may build the following commutative diagram which we have invisibly tensored with $\mathbb{Z}[\frac{1}{2}]$:

$$\begin{array}{ccccc}
(\mathfrak{o}_L/\mathfrak{M}_T)^{\times,-} & \hookrightarrow & R^{T,-} & \overset{\pi_R}{\twoheadrightarrow} & R^- \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_0^- \oplus \mathcal{T}^- & \hookrightarrow & B^{T,-} & \twoheadrightarrow & B^- \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{W}^- & \hookrightarrow & \nabla^{T,-} & \twoheadrightarrow & \nabla^-
\end{array} \tag{33}$$

Lemma 5. *The G-modules $B^{T,-}\otimes\mathbb{Z}_p$, $\nabla^{T,-}\otimes\mathbb{Z}_p$ and $R^{T,-}\otimes\mathbb{Z}_p$ are cohomologically trivial.*

Proof. By Lemma 4 and the choice of the set S the modules $\nabla^{T,-}\otimes\mathbb{Z}_p$ and $\mathcal{T}^-\otimes\mathbb{Z}_p$ are c.t. Now the middle row of the above diagram shows that also $B^{T,-}\otimes\mathbb{Z}_p$ is c.t. Finally, the middle column implies the assertion for $R^{T,-}\otimes\mathbb{Z}_p$. □

In analogy to the elements $d_{\mathfrak{P}}$, we define $\mathbb{Q}G_{\mathfrak{P}}$-generators $c_{\mathfrak{P}}$ of $\mathbb{Q}W_{\mathfrak{P}}$ by

$$c_{\mathfrak{P}} := (1 - \frac{1}{|G_{\mathfrak{P}}|}N_{G_{\mathfrak{P}}}, N_{\overline{G_{\mathfrak{P}}}} + (\phi_{\mathfrak{P}}-1)^{-1}(1-\frac{1}{|\overline{G_{\mathfrak{P}}}|}N_{\overline{G_{\mathfrak{P}}}})), \tag{34}$$

where we write $(\phi_{\mathfrak{P}}-1)^{-1} = |\overline{G_{\mathfrak{P}}}|^{-1}\sum_{i=0}^{|\overline{G_{\mathfrak{P}}}|-1} i\phi_{\mathfrak{P}}^i$ in an intuitive notation. We establish a connection between the generators $c_{\mathfrak{P}}$ and $d_{\mathfrak{P}}$ by means of the commutative diagram

$$\begin{array}{ccccc}
\mathbb{Q}_pG_{\mathfrak{P}} & \overset{\iota_1}{\hookrightarrow} & \mathbb{Q}_pG_{\mathfrak{P}}^2 & \overset{\pi_2}{\twoheadrightarrow} & \mathbb{Q}_pG_{\mathfrak{P}} \\
\downarrow{\scriptstyle 1\mapsto c_{\mathfrak{P}}} & & \downarrow{\scriptstyle g_{\mathfrak{P}}} & & \downarrow{\scriptstyle 1\mapsto d_{\mathfrak{P}}} \\
\mathbb{Q}_pW_{\mathfrak{P}} & \overset{q}{\hookrightarrow} & \mathbb{Q}_pG_{\mathfrak{P}}^2 & \overset{\kappa}{\twoheadrightarrow} & \mathbb{Q}_pW_{\mathfrak{P}}^*
\end{array}$$

where the maps of the upper row are the natural inclusion into the first and the projection onto the second component. The isomorphism $g_{\mathfrak{P}}$ is defined to be

$$g_{\mathfrak{P}} : (1,0) \mapsto q(c_{\mathfrak{P}}) = (N_{G_{\mathfrak{P}}} + (\phi_{\mathfrak{P}}-1)^{-1}(N_{I_{\mathfrak{P}}} - \frac{1}{|\overline{G_{\mathfrak{P}}}|}N_{G_{\mathfrak{P}}}), \phi_{\mathfrak{P}}^{-1}(1-\frac{1}{|G_{\mathfrak{P}}|}N_{G_{\mathfrak{P}}}))$$
$$(0,1) \mapsto (1, \frac{1}{|G_{\mathfrak{P}}|}N_{G_{\mathfrak{P}}})$$

Let us split the free $\mathbb{Z}G$-module C into $C = C_{p'} \oplus C_p$, where C_p is free of rank $|(S_{\mathrm{ram}}\cap S_p)^\diamond|$. If we combine the above diagram for all primes $\mathfrak{P}\in S_{\mathrm{ram}}^\diamond$ which do not lie above p, we get the following commutative diagram on minus parts which is invisibly tensored with $\mathbb{Z}_p$:

$$\begin{array}{ccccc}
\mathcal{W}^- & \hookrightarrow & \nabla^{T,-} & \twoheadrightarrow & \nabla^- \\
\vdots{\scriptstyle c} & & \vdots & & \vdots{\scriptstyle \phi} \\
C_{p'}^- & \hookrightarrow & (E_S \oplus C_{p'}^2 \oplus C_p)^- & \twoheadrightarrow & (E_S \oplus C)^-
\end{array} \tag{35}$$

Here, the dotted maps only exist after tensoring with $\mathbb{Q}_p$, and c is induced by mapping $c_{\mathfrak{P}}$ to $1_{\mathfrak{P}}$. The map $g := \sum_{\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^{\diamond}} \mathrm{ind}\, g_{\mathfrak{P}}$ is incorporated in the middle dotted arrow.
We have $\Omega_{\phi}^{(p),-} = (B^- \otimes \mathbb{Z}_p, \tilde{\phi}, (A \oplus C)^- \otimes \mathbb{Z}_p)$, where we can write $\tilde{\phi}$ in the following way:

$$
\begin{aligned}
\tilde{\phi}:\ \mathbb{Q}_p B^- &\xrightarrow[\simeq]{\tilde{\beta}} \mathbb{Q}_p(R \oplus \nabla)^- \xrightarrow{\pi_R^{-1} \oplus \mathrm{id}} \mathbb{Q}_p(R^T \oplus \nabla)^- \\
&\xrightarrow{\simeq} \mathbb{Q}_p\Big(R^T \oplus \Delta S \oplus \bigoplus\nolimits_{\mathfrak{P} \in S_{\mathrm{ram}}^{\diamond}} \mathrm{ind}\,_{G_{\mathfrak{P}}}^{G} W_{\mathfrak{P}}^*\Big)^- \\
&\xrightarrow{\mathrm{id} \oplus d} \mathbb{Q}_p(R^T \oplus \Delta S \oplus C)^- \xrightarrow{\mathrm{id} \oplus \phi_S^T \oplus \mathrm{id}} \mathbb{Q}_p(R^T \oplus E_S^T \oplus C)^- \\
&\xrightarrow{\pi_R \oplus \iota_E \oplus \mathrm{id}} \mathbb{Q}_p(R \oplus E_S \oplus C)^- \xrightarrow[\simeq]{\tilde{\alpha} \oplus \mathrm{id}_C} \mathbb{Q}_p(A \oplus C)^-
\end{aligned}
$$

Here, $\tilde{\beta}$ and $\tilde{\alpha}$ are induced by sections of the exact sequences $\mathbb{Q}_p R \rightarrowtail \mathbb{Q}_p B \twoheadrightarrow \mathbb{Q}_p \nabla$ and $\mathbb{Q}_p E_S \rightarrowtail \mathbb{Q}_p A \twoheadrightarrow \mathbb{Q}_p R$, respectively. Since $(R^T \oplus \Delta S \oplus C)^- \otimes \mathbb{Z}_p$ and $(R^T \oplus E_S^T \oplus C)^- \otimes \mathbb{Z}_p$ are c.t. G-modules by Lemma 2, Lemma 5 and the choice of the set S, applying (6) yields

$$
\begin{aligned}
\Omega_{\phi}^{(p),-} = {} & (B^- \otimes \mathbb{Z}_p, (\mathrm{id} \oplus d)(\pi_R^{-1} \oplus \mathrm{id})\tilde{\beta}, (R^T \oplus \Delta S \oplus C)^- \otimes \mathbb{Z}_p) \\
& + i_G(\mathrm{cok}\, \phi_S^T \otimes \mathbb{Z}_p) + ((R^T \oplus E_S^T)^- \otimes \mathbb{Z}_p, \tilde{\alpha}(\pi_R \oplus \iota_E), A^- \otimes \mathbb{Z}_p).
\end{aligned}
\tag{36}
$$

Note that the G-module $\mathrm{cok}\, \phi_S^T \otimes \mathbb{Z}_p$ is c.t. and finite such that

$$((R^T \oplus \Delta S \oplus C)^- \otimes \mathbb{Z}_p, \mathrm{id} \oplus \phi_S^T \oplus \mathrm{id}, (R^T \oplus E_S^T \oplus C)^- \otimes \mathbb{Z}_p) = i_G(\mathrm{cok}\, \phi_S^T \otimes \mathbb{Z}_p).$$

Since $\tilde{\alpha}(\pi_R \oplus \iota_E)$ is a section of $\mathbb{Q}_p E_S^T \rightarrowtail \mathbb{Q}_p A \twoheadrightarrow \mathbb{Q}_p R^T$, the last term in (36) vanishes. Now let $\delta \in \mathrm{Ext}_G^2(\nabla^- \otimes \mathbb{Z}_p, (\Delta S \oplus C \oplus (\mathfrak{o}/\mathfrak{M}_T)^{\times})^- \otimes \mathbb{Z}_p)$ be the extension class whose associated complex is

$$(\Delta S \oplus C \oplus R^T)^- \otimes \mathbb{Z}_p \to B^- \otimes \mathbb{Z}_p,$$

where the map is induced by the epimorphism π_R followed by the inclusion $R^- \rightarrowtail B^-$ (cf. diagram (33)). Then the first term on the right hand side of equation (36) equals $\chi(\delta, \mathrm{id} \oplus d)$, where we drop the subscript $\mathbb{Z}_p G_-, \mathbb{Q}_p G_-$ of the refined Euler characteristic from the notation. We have the following short exact sequence of complexes (where we have invisibly taken minus parts and tensored with $\mathbb{Z}_p$):

$$
\begin{array}{ccc}
C_{p'} & \xrightarrow{\quad 0 \quad} & \mathcal{P}_0 \oplus \mathcal{T} \\
\downarrow & & \downarrow \\
\Delta S \oplus C_{p'}^2 \oplus C_p \oplus R^T & \longrightarrow & B^T \\
\downarrow & & \downarrow \\
\Delta S \oplus C \oplus R^T & \longrightarrow & B
\end{array}
$$

Here, the left column is the obvious exact sequence, whereas the right column is taken from diagram (33). The extension class of the bottom complex is δ and the middle horizontal map is induced by the inclusion $R^{T,-} \otimes \mathbb{Z}_p \rightarrowtail B^{T,-} \otimes \mathbb{Z}_p$. We denote the extension classes corresponding to the upper and middle complex by τ and δ^T, respectively. Hence we have

$$\tau \in \mathrm{Ext}^2_G((\mathcal{P}_0 \oplus \mathcal{T})^- \otimes \mathbb{Z}_p, C^-_{p'} \otimes \mathbb{Z}_p), \quad \delta^T \in \mathrm{Ext}^2_G(\nabla^{T,-} \otimes \mathbb{Z}_p, (\Delta S \oplus C^2_{p'} \oplus C_p)^- \otimes \mathbb{Z}_p).$$

Now we endow these complexes with trivialisations. We have already trivialised δ by $\mathrm{id} \oplus d$. Now let $t : \mathbb{Q}_p(\mathcal{P}_0 \oplus \mathcal{T})^- \simeq \mathbb{Q}_p\mathcal{W}^-$ be the isomorphism induced by the epimorphism of sequence (32). Then ct is a trivialisation of τ, where c was defined in diagram (35). If we trivialise δ^T by $\mathrm{id}_{\Delta S} \oplus g^{-1} \oplus d_{p'}$, where $d_{p'}$ is the restriction of d to $\bigoplus_{\mathfrak{P} \in (S_{\mathrm{ram}} \cap S_p)^\diamond} \mathrm{ind}\, \mathbb{Q}W^*_{\mathfrak{P}}$ (tensored with $\mathbb{Q}_p$ and on minus parts), then the above exact sequence of complexes is in fact well metrised (in the terminology of [Gr3]) such that we have an equality (cf. loc.cit., Prop. 7.1)

$$\begin{aligned}\chi(\delta, \mathrm{id} \oplus d) &= \chi(\delta^T, \mathrm{id} \oplus g^{-1} \oplus d_{p'}) - \chi(\tau, ct) \\ &= (\nabla^{T,-} \otimes \mathbb{Z}_p, \mathrm{id} \oplus g^{-1} \oplus d_{p'}, (\Delta S \oplus C^2_{p'} \oplus C_p)^- \otimes \mathbb{Z}_p) \\ &\quad -((\mathcal{P}_0 \oplus \mathcal{T})^- \otimes \mathbb{Z}_p, ct, C^-_{p'} \otimes \mathbb{Z}_p),\end{aligned}$$

where the second equality holds, since $H^i(\tau)$ and $H^i(\delta^T)$ are c.t. G-modules, $i = 0, 1$. Since $((\mathcal{P}_0 \oplus \mathcal{T})^- \otimes \mathbb{Z}_p, ct, C^-_{p'} \otimes \mathbb{Z}_p) = -i_G(\mathcal{P}^-_0 \otimes \mathbb{Z}_p) + (\mathcal{T}^- \otimes \mathbb{Z}_p, ct, C^-_{p'} \otimes \mathbb{Z}_p)$, we get the following description of $\Omega^{(p),-}_\phi$:

$$\begin{aligned}\Omega^{(p),-}_\phi &= i_G(\mathrm{cok}\, \phi^T_S \otimes \mathbb{Z}_p) + i_G(\mathcal{P}^-_0 \otimes \mathbb{Z}_p) \\ &+ \sum_{\mathfrak{P} \in (S_{\mathrm{tram}} \cap S_p)^\diamond} ((\mathrm{ind}\, W^*_{\mathfrak{P}})^- \otimes \mathbb{Z}_p, \mathrm{ind}\, \delta_{\mathfrak{P}}, (\mathrm{ind}\, \mathbb{Z}_pG_{\mathfrak{P}})^-) \\ &- \sum_{\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^\diamond} \partial[\mathrm{ind}\, (\mathbb{Q}_pG^2_{\mathfrak{P}})^-, \mathrm{ind}\, g_{\mathfrak{P}}] \\ &- \sum_{\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^\diamond} ((\mathrm{ind}\, T_{\mathfrak{P}})^- \otimes \mathbb{Z}_p, \mathrm{ind}\, (c_{\mathfrak{P}} \mapsto 1_{\mathfrak{P}})t_{\mathfrak{P}}, \mathrm{ind}\, \mathbb{Z}_pG^-_{\mathfrak{P}}),\end{aligned} \tag{37}$$

where we have defined $S_{\mathrm{tram}} \subset S_{\mathrm{ram}}$ to be the set of all primes of L which are tamely ramified in L/K. In fact $((\mathrm{ind}\, W^*_{\mathfrak{P}})^- \otimes \mathbb{Z}_p, \mathrm{ind}\, \delta_{\mathfrak{P}}, (\mathrm{ind}\, \mathbb{Z}_pG_{\mathfrak{P}})^-)$ vanishes for all wildly ramified primes above p for the following reason: Since by assumption $j \in G_{\mathfrak{P}}$ for these primes, [Ni2], Prop. 4.4 implies that we have an isomorphism $\mathbb{Z}_pG^-_{\mathfrak{P}} \simeq (W^*_{\mathfrak{P}})^- \otimes \mathbb{Z}_p$, which maps $(1-j)/2$ to $d_{\mathfrak{P}}$. Hence, the isomorphism $\delta_{\mathfrak{P}}$ derives, locally at p and on minus parts, from a $\mathbb{Z}_pG_{\mathfrak{P}}$-isomorphism.

Proposition 6. *Keeping the notation of the current paragraph the following holds:*

(1) $i_G(\mathcal{P}^-_0 \otimes \mathbb{Z}_p)$ *has representing homomorphism*

$$\chi \mapsto \det(q_0 - \phi_{\mathfrak{P}_0} | V_\chi),$$

where $q_0 = N(\mathfrak{p}_0)$ *and* $\mathfrak{p}_0 = \mathfrak{P}_0 \cap K$.

(2) Let $\mathfrak{P} \in (S_{\mathrm{ram}} \cap S_p)^\diamond$ *be at most tamely ramified in* L/K. *Then* $((\mathrm{ind}\, W^*_{\mathfrak{P}})^- \otimes \mathbb{Z}_p, \mathrm{ind}\, \delta_{\mathfrak{P}}, (\mathrm{ind}\, \mathbb{Z}_pG_{\mathfrak{P}})^-)$ *has representing homomorphism*

$$\chi \mapsto (-e_{\mathfrak{P}})^{-\dim V_\chi^{G_{\mathfrak{P}}}} \cdot \det(1 - \phi^{-1}_{\mathfrak{P}} | V_\chi^{I_{\mathfrak{P}}} / V_\chi^{G_{\mathfrak{P}}})^{-1},$$

where $e_{\mathfrak{P}} = |I_{\mathfrak{P}}|$ *is the ramification index of the prime* $\mathfrak{P}$ *in* L/K.

(3) Let $\mathfrak{P}$ *be any finite prime of* L. *Then* $\partial[\mathrm{ind}\, (\mathbb{Q}_pG^2_{\mathfrak{P}})^-, \mathrm{ind}\, g_{\mathfrak{P}}]$ *has representing homomorphism*

$$\chi \mapsto (-|G_{\mathfrak{P}}|)^{\dim V_\chi^{G_{\mathfrak{P}}}}.$$

(4) Let $\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^\diamond$. *Then* $((\mathrm{ind}\, T_{\mathfrak{P}})^- \otimes \mathbb{Z}_p, \mathrm{ind}\, (c_{\mathfrak{P}} \mapsto 1_{\mathfrak{P}})t_{\mathfrak{P}}, \mathrm{ind}\, \mathbb{Z}_pG^-_{\mathfrak{P}})$ *has representing homomorphism*

$$\chi \mapsto (f_{\mathfrak{P}}(1 - q_{\mathfrak{p}}))^{-\dim V_\chi^{G_{\mathfrak{P}}}} \cdot \det\Big(\frac{1 - \phi_{\mathfrak{P}}}{q_{\mathfrak{p}} - \phi_{\mathfrak{P}}} | V_\chi^{I_{\mathfrak{P}}} / V_\chi^{G_{\mathfrak{P}}}\Big),$$

where $f_{\mathfrak{P}} = |\overline{G_{\mathfrak{P}}}|$ *is the degree of the corresponding residue field extension and* $q_{\mathfrak{p}} = N(\mathfrak{p})$.

Proof. Recall that $\mathcal{P}_0 = \mathrm{ind}_{G_{\mathfrak{P}_0}}^G (\mathfrak{o}_L/\mathfrak{P}_0)^\times$. Since $\mathfrak{P}_0$ is unramified in L/K, the decomposition group $G_{\mathfrak{P}_0}$ is cyclic with generator $\phi_{\mathfrak{P}_0}$, which acts as q_0 on $(\mathfrak{o}_L/\mathfrak{P}_0)^\times$. So (1) is clear. For (2) let $\mathfrak{P} \in (S_{\mathrm{ram}} \cap S_p)^\diamond$ be tamely ramified. Then the idempotent $\varepsilon_\mathfrak{P} = e_\mathfrak{P}^{-1} N_{I_\mathfrak{P}}$ lies in $\mathbb{Z}_p G_\mathfrak{P}$, and we claim that we have an isomorphism

$$w_\mathfrak{P} : \mathbb{Z}_p G_\mathfrak{P} \xrightarrow{\simeq} W_\mathfrak{P}^* \otimes \mathbb{Z}_p, \quad 1 \mapsto \kappa(1 - \varepsilon_\mathfrak{P}, 1).$$

Indeed $w_\mathfrak{P}(\varepsilon_\mathfrak{P}) = \kappa(0,1)$ and $w_\mathfrak{P}(1 - \varepsilon_\mathfrak{P} + e_\mathfrak{P}^{-1}(\phi_\mathfrak{P}^{-1} - 1)\varepsilon_\mathfrak{P}) = \kappa(1,0)$. Therefore, $w_\mathfrak{P}$ is surjective and hence bijective, since both modules are torsion free of the same rank. We have

$$((W_\mathfrak{P}^*)^- \otimes \mathbb{Z}_p, \delta_\mathfrak{P}, (\mathbb{Z}_p G_\mathfrak{P})^-) = -((\mathbb{Z}_p G_\mathfrak{P})^-, \delta_\mathfrak{P}^{-1} w_\mathfrak{P}, (\mathbb{Z}_p G_\mathfrak{P})^-).$$

Since $w_\mathfrak{P}(1 - \varepsilon_\mathfrak{P} + e_\mathfrak{P}^{-1}(\phi_\mathfrak{P}^{-1} - 1)\varepsilon_\mathfrak{P} + |G_\mathfrak{P}|^{-1} N_{G_\mathfrak{P}}) = d_\mathfrak{P}$, the representing homomorphism in demand is

$$\chi \mapsto \det(e_\mathfrak{P}^{-1}(\phi_\mathfrak{P}^{-1} - 1) | V_\chi^{I_\mathfrak{P}} / V_\chi^{G_\mathfrak{P}})^{-1}.$$

We have proved (2), since the desired homomorphism differs from this by

$$[\chi \mapsto \det((-e_\mathfrak{P})\varepsilon_\mathfrak{P} + 1 - \varepsilon_\mathfrak{P} | V_\chi)] \in \mathrm{Det}\,((\mathbb{Z}_p G_-)^\times).$$

(3) is an easy computation. Finally, let $\mathfrak{P} \in (S_{\mathrm{ram}} \cap T)^\diamond$, i.e. $\mathfrak{P}$ is a ramified prime not above p. It directly follows from the definition that $T_\mathfrak{P}$ is the push-out of the local fundamental class along the canonical projection $L_\mathfrak{P}^\times \twoheadrightarrow L_\mathfrak{P}^\times / U_\mathfrak{P}^1$. Actually before taking minus parts,

$$-(T_\mathfrak{P} \otimes \mathbb{Z}_p, (c_\mathfrak{P} \mapsto 1_\mathfrak{P}) t_\mathfrak{P}, \mathbb{Z}_p G_\mathfrak{P}) = \chi_{\mathbb{Z}_p G_\mathfrak{P}, \mathbb{Q}_p G_\mathfrak{P}}(\hat{u}_\mathfrak{P}, v_\mathfrak{P}^{-1}),$$

where $\hat{u}_\mathfrak{P} \in \mathrm{Ext}_{G_\mathfrak{P}}^2(\mathbb{Z}_p, \widehat{L_\mathfrak{P}^\times})$ derives from the local fundamental class. Hence (4) follows from Theorem D in [RW3]. □

If we now combine the equations (37) and (28) with Lemma 3, equation (29) and Proposition 6, we get Theorem 2 by an easy computation. □

3 Iwasawa theory

As an application of Theorem 2 we are going to prove the minus part of the LRNC at a prime $p \neq 2$ if L/K is an abelian CM-extension fulfilling the assumptions of Theorem 2 ; actually, we need to work under a slightly more restrictive hypothesis on the primes above p. We additionally require the vanishing of the μ-invariant of the standard Iwasawa module (all this will be made explicit below). The main ingredient of the proof turns out to be the validity of the Iwasawa main conjecture for abelian extensions.

3.1 Passing to the limit

Let L/K be an abelian CM-extension with Galois group G and $p \neq 2$ a finite rational prime such that all primes $\mathfrak{p} \subset K$ above p are tamely ramified in L/K or $j \in G_\mathfrak{p}$. Here, we write $G_\mathfrak{p}$ instead of $G_\mathfrak{P}$, since the decomposition group only depends on the prime $\mathfrak{p}$ in K if G is abelian. We will accordingly write $I_\mathfrak{p}$, $\phi_\mathfrak{p}$ etc. As it is required for the use of Theorem 2, we choose a finite prime $\mathfrak{P}_0$ of L such that $1 - \zeta \notin \prod_{g \in G/G_{\mathfrak{p}_0}} \mathfrak{P}_0^g$ for all roots of unity $\zeta \neq 1$ in L. We may assume that $\mathfrak{P}_0$ is unramified in L/K and does not divide p. Indeed, it would suffice to ask for a corresponding condition on $\mathfrak{P}_0$ for all p-power roots of unity in L, since we tensor with $\mathbb{Z}_p$. Hence, any prime which lies not above p will do.
As before we define a finite set of places of L

$$T = (S_{\mathrm{ram}} \setminus (S_{\mathrm{ram}} \cap S_p)) \cup \{\mathfrak{P}_0^g | g \in G\}, \tag{38}$$

and set $A_L^T = \mathrm{cl}_L^{T,-}$. Then $A_L^T \otimes \mathbb{Z}_p$ is c.t. by Theorem 1.
Let L_∞ and K_∞ be the cyclotomic $\mathbb{Z}_p$-extensions of L and K, respectively. We denote the Galois group of K_∞/K by Γ_K. Hence Γ_K is isomorphic to $\mathbb{Z}_p$, and we fix a topological generator γ_K. Furthermore, we denote the n-th layer in the cyclotomic extension K_∞/K by K_n such that K_n/K is cyclic of order p^n. Accordingly, we set $\Gamma_L = \mathrm{Gal}(L_\infty/L)$ with a topological generator γ_L whose restriction to K_∞ is γ_K^a for an appropriate integer a. We enumerate the intermediate fields starting with $L = L_a$ such that L_n/L is cyclic of order p^{n-a}. This is because then L_n is the smallest intermediate field of L_∞/L which lies above K_n. It may also be convenient to define $L_n = L$ if $n \le a$. Let $T_n := \{\mathfrak{P}_n \subset L_n | \mathfrak{P}_n \cap L \in T\}$, so $T_0 = T$ and $A_{L_n}^{T_n} \otimes \mathbb{Z}_p$ is $\mathrm{Gal}(L_n/K_n)$-c.t., since each of the extensions L_n/K_n inherits the required properties from the extension L/K. We define

$$X_T^- := \varprojlim A_{L_n}^{T_n} \otimes \mathbb{Z}_p.$$

We denote the Galois group of L_∞/K by $\mathcal{G}$, hence $\mathcal{G} = \tilde{G} \times \Gamma_K$, where $\tilde{G}$ is a subgroup of G. The completed group ring $\mathbb{Z}_p[[\mathcal{G}]]$ is isomorphic to $\Lambda[\tilde{G}]$, where Λ is the Iwasawa algebra $\mathbb{Z}_p[[T]]$. Since we are going to use some of the results in [Gr2], we set $\gamma_K = 1 - T$ as in loc.cit. We have an exact sequence (cf. [Gr2], Proposition 6)

$$\mathbb{Z}_p(1) \rightarrowtail \bigoplus_{\mathfrak{p} \in T(K)} Z_{\mathfrak{p}}(1)^- \to X_T^- \twoheadrightarrow X_{\mathrm{std}}^- \tag{39}$$

if $\zeta_p \in L$, and without the $\mathbb{Z}_p(1)$ term if $\zeta_p \notin L$. Here, X_{std} is the standard Iwasawa module which is the projective limit of the p-parts of the class groups in the cyclotomic tower over L, and $Z_{\mathfrak{p}}(1)$ is the first Tate twist of

$$Z_{\mathfrak{p}} = \mathrm{ind}_{\mathcal{G}_{\mathfrak{p}}}^{\mathcal{G}} \mathbb{Z}_p = \mathbb{Z}_p[[\Gamma_K \times \tilde{G}/\tilde{I}_{\mathfrak{p}}]]/(1 - \phi_{\mathfrak{p}}),$$

where we now write $\phi_{\mathfrak{p}}$ for the Frobenius automorphism at $\mathfrak{p}$ in the Galois group $\mathcal{G}$. The basic facts about the Iwasawa module X_T^- are summarized in the following Proposition.

Proposition 7. *The Iwasawa module X_T^- is a finitely generated, torsion $\mathbb{Z}_p[[\mathcal{G}]]_-$-module, which has no non-trivial finite submodules and*

$$\mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}]]_-}(X_T^-) \le 1.$$

Proof. This is Proposition 7 in [Gr2], where the ramification above p is assumed to be tame. But what is needed is just the cohomological triviality of the ray class groups $A_{L_n}^{T_n} \otimes \mathbb{Z}_p$. □

The Fitting ideal of X_T^- is described in terms of p-adic L-functions. To make this explicit we have to introduce some further notation. Let $\kappa : \mathcal{G} \to \mathbb{Z}_p^\times$ denote the cyclotomic character and define $u = \kappa(\gamma_K)$. Any character ψ of $\mathcal{G}$ with open kernel can be written as $\psi = \chi \otimes \rho$, where χ is a character of $\tilde{G}$ and ρ is trivial on $\tilde{G}$ (so χ is of type S and ρ is of type W in the terminology of [Wi1]). If χ is an odd character and S a set of places of K containing all the primes above p, there exists a well-defined element $f_{\chi,S}(T) \in Quot(\mathbb{Z}_p(\chi)[[T]])$ determined by

$$f_{\chi,S}(u^s - 1) = L_{p,S}(s, \omega\chi^{-1}),\ s = 1, 2, 3, \dots$$

where ω is the Teichmüller character[1] on $L(\zeta_p)/K$. This definition of $f_{\chi,S}$ follows the convention of Washington's book [Wa], and is used in [Gr2]. It is also common to replace the argument s on the right-hand side by $1 - s$, but this makes no essential difference.
For all χ of type S and ρ of type W we have (cf. [Gr2], Lemma 7)

$$f_{\chi\otimes\rho,S}(T) = f_{\chi,S}(\rho(\gamma_K)(1 + T) - 1). \tag{40}$$

[1] Do not confuse with the group ring element ω occurring in Proposition 3. ω will always denote the Teichmüller character in what follows.

For this, note that in the notation of [Wi1] we have an equality

$$f_{\chi\otimes\rho,S}(T)=\frac{G_{\omega\chi^{-1}\otimes\rho,S}(u(1+T)^{-1}-1)}{H_{\omega\chi^{-1}\otimes\rho,S}(u(1+T)^{-1}-1)}$$

and a similar formula holds for the right-hand side. The Iwasawa series $f_{\chi\otimes\rho,S}(T)$ glue together for varying characters, i.e. there exists a unique element $\Phi_S\in Quot(\mathbb{Z}_p[[\mathcal{G}]])^-$ such that for all odd characters $\psi=\chi\otimes\rho$ of $\mathcal{G}$ we have (cf. [Gr2], Proposition 11)

$$\psi(\Phi_S)=f_{\chi,S}(\rho(\gamma_K)-1).$$

Let $\mathfrak{p}\nmid p$ be a finite prime of K. Put

$$\xi_{\mathfrak{p}}=\frac{\kappa(\phi_{\mathfrak{p}})-\phi_{\mathfrak{p}}}{1-\phi_{\mathfrak{p}}}\ \varepsilon_{\mathfrak{p}}+1-\varepsilon_{\mathfrak{p}}\in Quot(\mathbb{Z}_p[[\mathcal{G}]]),\tag{41}$$

where $\varepsilon_{\mathfrak{p}}=|I_{\mathfrak{p}}|^{-1}N_{I_{\mathfrak{p}}}\in\mathbb{Q}_p\tilde{G}\subset\mathbb{Q}_p[[\mathcal{G}]]$. If T is a finite set of primes of L which contains no prime above p, define

$$\Psi_T=\left(\prod_{\mathfrak{p}\in T(K)}\xi_{\mathfrak{p}}\right)\cdot\Phi_{T(K)\cup S_p}.$$

If T is the set of places defined in (38), we have $\frac{1-j}{2}\Psi_T\in\mathbb{Z}_p[[\mathcal{G}]]^-$ (cf. [Gr2], Proposition 9). The Iwasawa main conjecture is the main ingredient in proving

Theorem 3. *Let T be the set of places of L defined in (38) and μ_- the μ-invariant of the standard Iwasawa module X_{std}^-. Then it holds:*

(1) The Fitting ideal of $\mathbb{Q}_pX_T^-$ is generated by Ψ_T.
(2) If $\mu_-=0$, we actually have

$$\mathrm{Fitt}_{\mathbb{Z}_p[[\mathcal{G}]]_-}(X_T^-)=(\Psi_T).$$

Proof. If the ramification above p is almost tame, this is Proposition 8 and Theorem 6 in [Gr2]. But once more the condition on the ramification is only needed to guarantee the cohomological triviality of $A_L^T\otimes\mathbb{Z}_p$. □

Lemma 6. *Let ψ be a character of $\mathcal{G}$ with open kernel and S a set of places of K that contains all the p-adic places. Put $S_\psi=\{\mathfrak{p}\in S|I_{\mathfrak{p}}\not\subset\ker(\psi)\}\cup S_p$ and write the Frobenius automorphism at a prime $\mathfrak{p}$ as $\phi_{\mathfrak{p}}=\sigma_{\mathfrak{p}}\gamma_K^{c_{\mathfrak{p}}}$, where $\sigma_{\mathfrak{p}}\in\tilde{G}$ and $c_{\mathfrak{p}}\in\mathbb{Z}_p$.*

(1) Let χ be a character of $\tilde{G}$. Then

$$L_{p,S}(s,\omega\chi^{-1})=L_{p,S_\chi}(s,\omega\chi^{-1})\prod_{\mathfrak{p}\in S\setminus S_\chi}(1-\chi^{-1}(\sigma_{\mathfrak{p}})u^{-s\cdot c_{\mathfrak{p}}}).$$

(2) We have an equality

$$f_{\psi,S}(T)=f_{\psi,S_\psi}(T)\prod_{\mathfrak{p}\in S\setminus S_\psi}(1-\psi^{-1}(\phi_{\mathfrak{p}})(1+T)^{-c_{\mathfrak{p}}}).$$

Proof. (1) is well known and follows by evaluating both sides of the equation at $s=1-n$, where $n\equiv 0 \bmod (p-1)$. (2) is an easy consequence of (1) using formula (40) for the character $\psi=\chi\otimes\rho$ with a $\tilde{G}$-character χ. □

Corollary 1. *Let T be the set of places of L defined in (38) and S_1 be the set of places of L which are wildly ramified in L/K. Each character χ of G can be viewed as a character of $\mathcal{G}$ and, if χ is odd, we have*

$$\chi(\Psi_T)=\chi(\theta_{S_1}^T)\cdot\prod_{\mathfrak{p}\in S_p\cap S_{\mathrm{tram}}}(1-\chi(\varepsilon_{\mathfrak{p}}\phi_{\mathfrak{p}}^{-1})).$$

Proof. Write $\chi = \chi' \otimes \rho$, where χ' is a character of $\tilde{G}$ and ρ is of type W. Since only p-adic primes ramify in the cyclotomic towers over K and L, we have $\Sigma_\chi = \Sigma_{\chi'}$, where $\Sigma = T(K) \cup S_p$. At first, we determine $\chi'(\Psi_T) \in \mathbb{Z}_p(\chi')[[T]]$. With the notation of Lemma 6 we have

$$\begin{aligned}
\chi'(\Psi_T) &= f_{\chi',\Sigma}(-T) \prod_{\mathfrak{p}\in T(K)} \frac{\kappa(\phi_\mathfrak{p}) - \chi'(\sigma_\mathfrak{p})\gamma_K^{c_\mathfrak{p}}}{1-\chi'(\sigma_\mathfrak{p})\gamma_K^{c_\mathfrak{p}}} \\
&\overset{(*)}{=} f_{\chi',\Sigma_{\chi'}}(-T) \prod_{\mathfrak{p}\in T(K)} \frac{\kappa(\phi_\mathfrak{p}) - \chi'(\sigma_\mathfrak{p})\gamma_K^{c_\mathfrak{p}}}{1-\chi'(\sigma_\mathfrak{p})\gamma_K^{c_\mathfrak{p}}} (1-\chi'(\sigma_\mathfrak{p})^{-1}\gamma_K^{-c_\mathfrak{p}}) \\
&= f_{\chi',\Sigma_{\chi'}}(-T) \prod_{\mathfrak{p}\in T(K)} (1-\chi'(\sigma_\mathfrak{p})^{-1}\gamma_K^{-c_\mathfrak{p}}\kappa(\phi_\mathfrak{p})),
\end{aligned}$$

where (*) holds by means of (2) of Lemma 6. Since $\rho(f_{\chi',\Sigma_{\chi'}}(-T)) = f_{\chi',\Sigma_{\chi'}}(\rho(\gamma_K)-1) = f_{\chi,\Sigma_\chi}(0) = L_{S_\chi}(0,\chi^{-1})$, we get

$$\begin{aligned}
\chi(\Psi_T) &= \rho(\chi'(\Psi_T)) \\
&= L_{S_\chi}(0,\chi^{-1}) \prod_{\mathfrak{p}\in T(K)} (1-\chi(\phi_\mathfrak{p})^{-1}\kappa(\phi_\mathfrak{p})) \\
&= L_{S_\infty}(0,\chi^{-1}) \left(\prod_{\mathfrak{p}\in S_p} (1-\chi(\varepsilon_\mathfrak{p}\phi_\mathfrak{p}^{-1}))\right) \left(\prod_{\mathfrak{p}\in T(K)} (1-\chi(\phi_\mathfrak{p})^{-1}q_\mathfrak{p})\right) \\
&= \chi(\theta_{S_1}^T) \cdot \prod_{\mathfrak{p}\in S_p\cap S_{\mathrm{tram}}} (1-\chi(\varepsilon_\mathfrak{p}\phi_\mathfrak{p}^{-1})),
\end{aligned}$$

where as before $q_\mathfrak{p} = N(\mathfrak{p})$. □

3.2 The descent

We are going to use an idea, which originates from [Wi2], in the extended version of [Gr1], where the author proves Brumer's conjecture for a special class of CM-extensions. Note that the class of CM-extensions treated here includes the class of loc. cit. The same approach is also used in [Ku] to compute the Fitting ideals of minus class groups of absolute abelian CM-fields. But before we go for this, we look at a special case, where a rather restrictive condition forces the Euler factors at p to become units in $\mathbb{Z}_pG_-$.

Proposition 8. *Let L/K be an abelian CM-extension with Galois group G and $p \neq 2$ a rational prime. Let T be the set of places of L defined in (38) and S_1 be the set of all wildly ramified primes. Suppose that $\mu_- = 0$ and $j \in G_\mathfrak{p}$ for all primes $\mathfrak{p}$ of K above p.*
Then $\theta_{S_1}^T$ generates the Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T \otimes \mathbb{Z}_p)$. In particular, the minus part of the LRNC at p is true.

Proof. The canonical restriction map $X_T^- \to A_L^T \otimes \mathbb{Z}_p$ is an epimorphism, since the cokernel is a quotient of Γ_L which has trivial j-action. By general properties of Fitting ideals we have

$$\mathrm{Fitt}_{\mathbb{Z}_pG_-}(X_T^-/\gamma_L - 1) \subset \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T \otimes \mathbb{Z}_p),$$

and the Fitting ideal on the left-hand side is generated by $\Psi_T \bmod (\gamma_L - 1)$ by Theorem 3. Corollary 1 now implies that

$$\Psi_T \bmod (\gamma_L - 1) = \theta_{S_1}^T \prod_{\mathfrak{p}\in S_p\cap S_{\mathrm{tram}}} (1-\varepsilon_\mathfrak{p}\phi_\mathfrak{p}^{-1}).$$

But the product on the right-hand side is a unit in $\mathbb{Z}_pG_-$, since $j \in G_\mathfrak{p}$ for these primes. Hence $\theta_{S_1}^T \in \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T \otimes \mathbb{Z}_p)$. Finally, Proposition 4 and 5 imply that $\theta_{S_1}^T$ has to be a generator of the Fitting ideal. The minus part of the LRNC at p follows from Theorem 2. □

Now we use the method in [Gr1] to prove the minus part of the LRNC at p without the additional assumption of Proposition 8. But this works only for primes p such that $L^{\mathrm{cl}} \not\subset (L^{\mathrm{cl}})^+(\zeta_p)$, where L^{cl} denotes the normal closure of L over $\mathbb{Q}$, which is again a CM-field. This condition particularly forces $\zeta_p \notin L$. But note that this condition holds for almost all primes p, since each prime for which it fails has to ramify in $L^{\mathrm{cl}}/\mathbb{Q}$. Our main result is

Theorem 4. *Let L/K be an abelian CM-extension with Galois group G and $p \neq 2$ a rational prime. Let T be the set of places of L defined in (38) and S_1 be the set of all wildly ramified primes. Suppose that $\mu_- = 0$ and that each prime $\mathfrak{p}$ above p ramifies at most tame or $j \in G_{\mathfrak{p}}$. Moreover, assume that $j \in G_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of K above p whenever $L^{\mathrm{cl}} \subset (L^{\mathrm{cl}})^+(\zeta_p)$. Then $\theta_{S_1}^T$ generates the Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T \otimes \mathbb{Z}_p)$. In particular, the minus part of the LRNC at p is true.*

Remark 6. The vanishing of μ_- is only required for computing the Fitting ideal of X_T^- (cf. Theorem 3). As already mentioned, we will show in the appendix that we can remove this hypothesis for some special cases, including the case $p \nmid |G|$.

Corollary 2. [2] *Let L/K be any Galois CM-extension with group G and $p \neq 2$ a rational prime such that $j \in G_{\mathfrak{P}}$ for all primes $\mathfrak{P}$ of L above p whenever $L^{\mathrm{cl}} \subset (L^{\mathrm{cl}})^+(\zeta_p)$. Then the Strong Stark Conjecture at p holds for each odd character χ of G.*

Proof. By Artin's induction theorem we may write $\chi = \sum_C a_C \mathrm{ind}_C^G \tilde{\phi}_C$, where the sum runs over all cyclic subgroups of G, $a_C \in \mathbb{Q}$ and the $\tilde{\phi}_C$ are linear characters. Define $U_C := C$ if C contains j, and $U_C := C \times \langle j \rangle$ if $j \notin C$. Moreover, let $\phi_C = \mathrm{ind}_C^{U_C} \tilde{\phi}_C$. Then $\chi = \sum_C a_C \mathrm{ind}_{U_C}^G \phi_C$, where each ϕ_C belongs to a CM-subextension of L/K. Since even resp. odd characters remain even resp. odd after induction, we may assume the ϕ_C to be odd. Now we adjust the proof of Proposition 11 in [RW2] to show that the Strong Stark Conjecture at p for odd characters is implied by the respective statement for all CM-subextensions of L/K which have a Galois group of type U_C of degree prime to p. In this case, the Strong Stark Conjecture at p is equivalent to the LRNC at p. □

Proof (of Theorem 4). The assertion follows from Proposition 8 if $L^{\mathrm{cl}} \subset (L^{\mathrm{cl}})^+(\zeta_p)$. Hence, we may assume that this is not the case in the following. We state the following result, which is Proposition 4.1 in [Gr1].

Proposition 9. *Let p be a prime such that $L^{\mathrm{cl}} \not\subset (L^{\mathrm{cl}})^+(\zeta_p)$ and $N \in \mathbb{N}$. Then there exist infinitely many primes r such that*

- $r \equiv 1 \bmod p^N$
- *$j \in G_{\mathfrak{r}}$ for each prime $\mathfrak{r}$ in K above r*
- *the Frobenius automorphism at p in the extension $\mathbb{Q}(\zeta_r)/\mathbb{Q}$ generates $\mathrm{Gal}(E/\mathbb{Q})$, where E is the subfield of $\mathbb{Q}(\zeta_r)$ such that $[E : \mathbb{Q}] = p^N$.*

Let N be a large integer, and choose a prime r as in the Proposition which does not ramify in $L^{\mathrm{cl}}/\mathbb{Q}$. The extension $E/\mathbb{Q}$ is cyclic of degree p^N, and we denote the corresponding Galois group by C_N. It is generated by the Frobenius automorphism $\mathrm{Frob}_p \in C_N$. Let $L' = LE$ and $K' = KE$. Then L'/K is an abelian extension with Galois group $G' = G \times C_N$, and the only new ramification occurs above r. Moreover, the primes $\mathfrak{r}$ above r satisfy both of our standard conditions: They are tamely ramified and $j \in G_{\mathfrak{r}}$.
Set $T' = \{\mathfrak{P}' \subset L' : \mathfrak{P}' \cap L \in T\}$ and $T_0' = T' \cup \{\mathfrak{R}' \in L' : \mathfrak{R}' \mid r\}$. There is an exact sequence

$$\left(\mathfrak{o}_{L'} / \prod_{\mathfrak{R}' \mid r} \mathfrak{R}' \right)^{\times,-} \otimes \mathbb{Z}_p \rightarrowtail A_{L'}^{T_0'} \otimes \mathbb{Z}_p \twoheadrightarrow A_{L'}^{T'} \otimes \mathbb{Z}_p.$$

We claim that the leftmost term is trivial, and hence $A_{L'}^{T'} \otimes \mathbb{Z}_p \simeq A_{L'}^{T_0'} \otimes \mathbb{Z}_p$ is c.t. by Theorem 1. To see this let $\mathfrak{r}$ be a prime in K above r, and $\mathfrak{R}'$ a prime in L' above $\mathfrak{r}$. Since $j \in G_{\mathfrak{r}}$, it

[2] added in proof - I would like to thank David Burns for his useful hint.

acts on the corresponding residue field extension of degree $f_{\mathfrak{r}}$, say. Therefore, $(\mathfrak{o}_{L'}/\mathfrak{R}')^{\times,-}$ has exactly $q_{\mathfrak{r}}^{f_{\mathfrak{r}}/2}+1$ elements, where $q_{\mathfrak{r}}=N(\mathfrak{r})$ is a power of r. But thanks to the first condition on r we have $q_{\mathfrak{r}}^{f_{\mathfrak{r}}/2}+1\equiv 2\not\equiv 0 \bmod p$. Hence, the leftmost term vanishes, since we are only dealing with p-parts.
For the same reasons as in Proposition 8 the natural restriction map $A_{L'}^{T'}\otimes\mathbb{Z}_p\twoheadrightarrow A_L^T\otimes\mathbb{Z}_p$ is surjective. The composite map

$$A_{L'}^{T'}\otimes\mathbb{Z}_p\xrightarrow{\text{res}}A_L^T\otimes\mathbb{Z}_p\to A_{L'}^{T'}\otimes\mathbb{Z}_p$$

is given by the norm N_{C_N}, and the kernel of the norm is just $\Delta C_N\cdot A_{L'}^{T'}\otimes\mathbb{Z}_p$. Therefore, the restriction map induces an isomorphism

$$(A_{L'}^{T'}\otimes\mathbb{Z}_p)_{C_N}\xrightarrow{\simeq}A_L^T\otimes\mathbb{Z}_p.$$

Now we adjust the method in [Gr1] to show that (see [Ni1] for a complete proof)

$$\mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T\otimes\mathbb{Z}_p)\subset(\theta_{S_1}^T).$$

As in the proof of Proposition 8, $\theta_{S_1}^T$ has to be a generator of the Fitting ideal by Proposition 4 and 5. The minus part of the LRNC at p again follows from Theorem 2. □

4 On the Strong Brumer-Stark Conjecture

4.1 The conjecture

Let L/K be a finite abelian extension of number fields with Galois group G. Let S be a finite G-invariant set of primes of L, containing all the infinite primes and all the primes which ramify in L/K. If T is a second G-invariant, finite, nonempty set of primes of L, disjoint from S, we define for each character χ of G a complex-analytic function $\delta_T(\chi,s)=\prod_{\mathfrak{p}\in T(K)}(1-N(\mathfrak{p})^{1-s}\check{\chi}(\phi_{\mathfrak{p}}))$. The (S,T)-modified L-function associated to χ is defined to be

$$L_{S,T}(L/K,\chi,s)=\delta_T(\chi,s)\cdot L_S(L/K,\chi,s).$$

Set $\delta_T(s)=\sum_{\chi\in\mathrm{Irr}\,(G)}\delta_T(\check{\chi},s)\varepsilon_\chi$ for all $s\in\mathbb{C}$. The S-Stickelberger and respectively (S,T)-Stickelberger functions[3] are defined by

$$\Theta_S(s)=\Theta_S(L/K,s):=\sum_{\chi\in\mathrm{Irr}\,(G)}L_S(L/K,\check{\chi},s)\varepsilon_\chi,$$

$$\Theta_{S,T}(s)=\Theta_{S,T}(L/K,s):=\delta_T(s)\cdot\Theta_S(s)=\sum_{\chi\in\mathrm{Irr}\,(G)}L_{S,T}(L/K,\check{\chi},s)\varepsilon_\chi.$$

We now fix a set of data $(L/K,S,T)$ which satisfies the following hypotheses (H):

- S contains all the infinite primes of L and all primes of L which ramify in L/K.
- $T\neq\emptyset$, $S\cap T=\emptyset$, $E_S^T\cap\mu_L=1$.

The Strong Brumer-Stark Conjecture ([Po], following Theorem 3.2.2.3) now states

Conjecture 3. The image of $\frac{1}{2}\Theta_{S,T}(0)$ in $\mathbb{Z}G_-$ lies in $\mathrm{Fitt}_{\mathbb{Z}G_-}(A_L^T)$.

We will refer to this conjecture as $SBrSt(L/K,S,T)$. Replacing $\mathbb{Z}G$ by $\mathbb{Z}_pG$ and A_L^T by $A_L^T\otimes\mathbb{Z}_p$ for a prime p one gets localized versions $\mathbb{Z}_{(p)}SBrSt(L/K,S,T)$ of the above conjecture. Of course, we have

$$SBrSt(L/K,S,T)\iff\mathbb{Z}_{(p)}SBrSt(L/K,S,T)\ \forall p.$$

Note that $SBrSt(L/K,S,T)$ for all sets S, T satisfying (H) implies the Rubin-Stark Conjecture (cf. [Po], Theorem 3.2.2.3).

[3] Do not confuse with the representing homomorphism Θ_S^T defined in 2.

4.2 The tamely ramified case

Theorem 5. *Let L/K be an abelian Galois CM-extension with Galois group G and $p \neq 2$ a prime. Assume that for each prime $\mathfrak{p}$ above p the ramification is almost tame or $j \in G_{\mathfrak{p}}$. Then the minus part of the LRNC at p implies the Strong Brumer-Stark conjecture $\mathbb{Z}_{(p)}SBrSt(L/K,S,T)$ for each sets of places S, T such that $(L/K,S,T)$ satisfies (H).*

Corollary 3. *(1) Assume that L/K additionally satisfies $j \in G_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ above p, whenever $L^{\mathrm{cl}} \subset (L^{\mathrm{cl}})^+(\zeta_p)$, and that $\mu_- = 0$. Then $\mathbb{Z}_{(p)}SBrSt(L/K,S,T)$ holds whenever $(L/K,S,T)$ satisfies (H).*
(2) If L is abelian over $\mathbb{Q}$, then $\mathbb{Z}_{(p)}SBrSt(L/K,S,T)$ holds whenever $(L/K,S,T)$ satisfies (H) and all $\mathfrak{p}$ above p are almost tamely ramified or $j \in G_{\mathfrak{p}}$.

Here, (1) immediately follows from Theorem 4. Since the ETNC is known to be true for absolutely abelian extensions [BG1], we get (2). Note that we can again remove the condition $\mu_- = 0$ if $p \nmid |G|$.

Proof (of Theorem 5). It follows from the behavior of Fitting ideals that it suffices to verify the conjecture for minimal sets S, T. Hence, let $S = S_{\mathrm{ram}}$ and $T_0 = \{\mathfrak{P}_0^g | g \in G\}$ for an unramified prime $\mathfrak{P}_0$ such that $E_{S_{\mathrm{ram}}}^{T_0} \cap \mu_L = 1$. This is equivalent to the statement on earlier occasions that $1 - \zeta \notin \prod_{g \in G/G_{\mathfrak{p}_0}} \mathfrak{P}_0^g$ for all $1 \neq \zeta \in \mu_L$. As before, set $T = T_0 \cup (S_{\mathrm{ram}} \setminus (S_{\mathrm{ram}} \cap S_p))$. By Theorem 2 the minus part of the LRNC at p implies (and is indeed equivalent to)

$$\mathrm{Fitt}_{\mathbb{Z}_p G_-}(A_L^T \otimes \mathbb{Z}_p) = (\theta_{S_1}^T) = (\Theta_{S_1,T}(0)). \tag{42}$$

We have two exact sequences

$$\left(\mathfrak{o}_L / \prod_{\mathfrak{P} \in T \setminus T_0} \mathfrak{P}\right)^{\times,-} \otimes \mathbb{Z}_p \rightarrowtail A_L^T \otimes \mathbb{Z}_p \twoheadrightarrow A_L^{T_0} \otimes \mathbb{Z}_p, \tag{43}$$

$$\left(\mathfrak{o}_L / \prod_{\mathfrak{P} \in T \setminus T_0} \mathfrak{P}\right)^{\times} \otimes \mathbb{Z}_p \rightarrowtail \bigoplus_{\mathfrak{P} \in T^\diamond \setminus T_0^\diamond} \mathrm{ind}_{G_{\mathfrak{p}}}^G T_{\mathfrak{P}} \otimes \mathbb{Z}_p \twoheadrightarrow \bigoplus_{\mathfrak{P} \in T^\diamond \setminus T_0^\diamond} \mathrm{ind}_{G_{\mathfrak{p}}}^G W_{\mathfrak{P}} \otimes \mathbb{Z}_p.$$

Here, the lower sequence derives from diagram (30). We want to apply the following Lemma, which is a special case of Lemma 5 in [BG2].

Lemma 7. *Let $M_1 \rightarrowtail P_1 \to P_2 \twoheadrightarrow M_2$ be an exact sequence of finite $\mathbb{Z}_p G_-$-modules, where P_1 and P_2 are c.t. Then* $\mathrm{Fitt}(P_i)$ *is invertible for $i = 1, 2$ and*

$$\mathrm{Fitt}(M_2) = \mathrm{Fitt}(M_1^\vee) \cdot \mathrm{Fitt}(P_1)^{-1} \cdot \mathrm{Fitt}(P_2),$$

where $M_1^\vee = \mathrm{Hom}(M_1, \mathbb{Q}/\mathbb{Z})$ denotes the Pontryagin dual of M_1.

We have to modify the above two exact sequences slightly. For each prime $\mathfrak{P}$ we have an exact sequence

$$\mathcal{K}_{\mathfrak{P}} \rightarrowtail (\mathrm{ind}_{G_{\mathfrak{p}}}^G \mathbb{Z}_p G_{\mathfrak{p}} / (N(\mathfrak{P}) - 1))^- \twoheadrightarrow \left(\mathrm{ind}_{G_{\mathfrak{p}}}^G (\mathfrak{o}_L / \mathfrak{P})\right)^{\times,-} \otimes \mathbb{Z}_p,$$

where the second map is induced by mapping 1 to a generator of $(\mathfrak{o}_L/\mathfrak{P})^\times$. These sequences glue together and give

$$\mathcal{K} \rightarrowtail P \twoheadrightarrow \left(\mathfrak{o}_L / \prod_{\mathfrak{P} \in T \setminus T_0} \mathfrak{P}\right)^{\times,-} \otimes \mathbb{Z}_p, \tag{44}$$

where $\mathcal{K}$ and P are the direct sums of the $\mathcal{K}_{\mathfrak{P}}$ and the middle terms in the above sequence, respectively. Note that $\mathcal{K}$ and P are finite, and P is c.t. Define

$$c'_{\mathfrak{P}} := (|G_{\mathfrak{p}}|(1 - \frac{1}{|G_{\mathfrak{p}}|}N_{G_{\mathfrak{p}}}) + \frac{1}{|G_{\mathfrak{p}}|}N_{G_{\mathfrak{p}}}) \cdot c_{\mathfrak{P}} \in W_{\mathfrak{P}},$$

where $c_{\mathfrak{P}}$ was defined in (34). Moreover, let $t'_{\mathfrak{P}}$ be a preimage of $c'_{\mathfrak{P}}$ in $T_{\mathfrak{P}}$. The maps $\mathbb{Z}_p G_{\mathfrak{p}} \to W_{\mathfrak{P}} \otimes \mathbb{Z}_p$, $1 \mapsto c'_{\mathfrak{P}}$ and $\mathbb{Z}_p G_{\mathfrak{p}} \to T_{\mathfrak{P}} \otimes \mathbb{Z}_p$, $1 \mapsto t'_{\mathfrak{P}}$ are injective and become isomorphisms after tensoring with $\mathbb{Q}_p$. Hence, the direct sum

$$\mathcal{T} := \bigoplus_{\mathfrak{P} \in T^{\diamond} \setminus T_0^{\diamond}} \operatorname{ind}_{G_{\mathfrak{p}}}^{G} T_{\mathfrak{P}}/t'_{\mathfrak{P}} \otimes \mathbb{Z}_p$$

is finite and c.t. by Lemma 4. Therefore, the sequences (43) and (44) give two exact sequences

$$\mathcal{K} \rightarrowtail P \to A_L^T \otimes \mathbb{Z}_p \twoheadrightarrow A_L^{T_0} \otimes \mathbb{Z}_p,$$
$$\mathcal{K} \rightarrowtail P \to \mathcal{T}^- \twoheadrightarrow \mathcal{W}^-,$$

where $\mathcal{W}$ is the direct sum of the $\operatorname{ind}_{G_{\mathfrak{p}}}^{G} W_{\mathfrak{P}}/c'_{\mathfrak{P}} \otimes \mathbb{Z}_p$. We can apply Lemma 7 to these sequences and get

$$\operatorname{Fitt}(A_L^{T_0} \otimes \mathbb{Z}_p) = \operatorname{Fitt}(A_L^T \otimes \mathbb{Z}_p) \cdot \operatorname{Fitt}(\mathcal{T}^-)^{-1} \cdot \operatorname{Fitt}(\mathcal{W}^-). \tag{45}$$

Proposition 6 (4) implies

$$\operatorname{Fitt}(\mathcal{T}^-) = \prod_{\mathfrak{P} \in T^{\diamond} \setminus T_0^{\diamond}} (\tau_{\mathfrak{P}}), \tag{46}$$

$$\tau_{\mathfrak{P}} = f_{\mathfrak{p}}(1 - q_{\mathfrak{p}})\frac{1}{|G_{\mathfrak{p}}|}N_{G_{\mathfrak{p}}} + (|G_{\mathfrak{p}}| - N_{G_{\mathfrak{p}}})(\frac{q_{\mathfrak{p}} - \phi_{\mathfrak{p}}}{1 - \phi_{\mathfrak{p}}}\varepsilon_{\mathfrak{p}} + 1 - \varepsilon_{\mathfrak{p}}),$$

where $\varepsilon_{\mathfrak{p}} = |I_{\mathfrak{p}}|^{-1}N_{I_{\mathfrak{p}}}$ and $f_{\mathfrak{p}}$ is the degree of the corresponding residue field extension.

Lemma 8. *Let $\mathfrak{P} \notin S_p$ be a finite prime of L. Then*

$$\operatorname{Fitt}_{\mathbb{Z}_p G_{\mathfrak{p}}}(W_{\mathfrak{P}}/c'_{\mathfrak{P}} \otimes \mathbb{Z}_p) = \langle N_{G_{\mathfrak{p}}} - |G_{\mathfrak{p}}|, N_{G_{\mathfrak{p}}} + e_{\mathfrak{p}}(f_{\mathfrak{p}}N_{I_{\mathfrak{p}}} - N_{G_{\mathfrak{p}}})(\phi_{\mathfrak{p}} - 1)^{-1}\rangle_{\mathbb{Z}_p G_{\mathfrak{p}}}.$$

Proof. Since $\mathfrak{P}$ lies not above p, we may assume that $\mathfrak{P}$ is at most tamely ramified. We keep the notation of [Ch2], Lemma 6.2. So choose a generator a of $I_{\mathfrak{p}}$ and let $b \in G_{\mathfrak{p}}$ be a lift of $\phi_{\mathfrak{p}}^{-1}$ which is of maximal order $|b|$ among all such elements. Set $e_{\mathfrak{p}} = |I_{\mathfrak{p}}|$; then $b^{-f_{\mathfrak{p}}} = a^{c_{\mathfrak{p}}}$ for a divisor $c_{\mathfrak{p}}$ of $e_{\mathfrak{p}}$. Define a map

$$\pi : \mathbb{Z}G_{\mathfrak{p}}e_1 \oplus \mathbb{Z}G_{\mathfrak{p}}e_2 \twoheadrightarrow W_{\mathfrak{P}}$$

by $\pi(e_1) = (b^{-1} - 1, 1)$ and $\pi(e_2) = (a - 1, 0)$. We claim that the kernel is generated by $N_{I_{\mathfrak{p}}}e_2$ and $(a - 1)e_1 + (1 - b^{-1})e_2$. For this, assume that

$$\pi(x_1e_1 + x_2e_2) = (x_1(b^{-1} - 1) + x_2(a - 1), \overline{x}_1) = 0 \in W_{\mathfrak{P}}.$$

By Lemma 6.6 in [Ch2] $x_1 = (a-1)x'_1$ for an appropriate $x'_1 \in \mathbb{Z}G_{\mathfrak{p}}$. By the same Lemma in loc.cit. we get $x'_1(b^{-1}-1)+x_2 = y \cdot N_{I_{\mathfrak{p}}}$ for a $y \in \mathbb{Z}G_{\mathfrak{p}}$, since the left-hand side is annihilated by $(a - 1)$. This proves the claim. Define two group ring elements

$$\delta_1 := \sum_{i=0}^{f_{\mathfrak{p}}-1} b^{-i} + (f_{\mathfrak{p}}N_{I_{\mathfrak{p}}} - N_{G_{\mathfrak{p}}})(b^{-1} - 1)^{-1} \in \mathbb{Z}_p G_{\mathfrak{p}},$$

$$\delta_2 := \sum_{i=0}^{c_{\mathfrak{p}}-1} a^i + f_{\mathfrak{p}} \cdot \sum_{i=1}^{e_{\mathfrak{p}}-1}\sum_{j=0}^{i-1} a^j \in \mathbb{Z}_p G_{\mathfrak{p}}.$$

An easy computation shows that $\pi(\delta_1e_1 - \delta_2e_2) = c'_{\mathfrak{P}}$. Hence, the kernel of the epimorphism

$$\mathbb{Z}_pG_{\mathfrak{p}}e_1 \oplus \mathbb{Z}_pG_{\mathfrak{p}}e_2 \twoheadrightarrow W_{\mathfrak{P}}/c'_{\mathfrak{P}} \otimes \mathbb{Z}_p$$

induced by π is generated by the kernel of π and $\delta_1e_1 - \delta_2e_2$. From this one can compute the desired Fitting ideal. ☐

Recall the definitions (24) and (26) of ω and the modules $M_{\mathfrak{P}}$. The above Lemma together with (45), (42), (46) now yields

Corollary 4. *Assume that the minus part of the LRNC at p holds. Then:*

$$\mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^{T_0} \otimes \mathbb{Z}_p) = (q_{\mathfrak{p}_0} - \phi_{\mathfrak{p}_0})\omega \prod_{\mathfrak{P} \in S_{\mathrm{ram}}^{\diamond}} M_{\mathfrak{P}} \subset SKu(L/K)^- \cdot \mathbb{Z}_pG.$$

In particular, this implies

$$\Theta_{S_{\mathrm{ram}},T_0}(0) = (q_{\mathfrak{p}_0} - \phi_{\mathfrak{p}_0}) \cdot \omega \prod_{\mathfrak{P} \in S_{\mathrm{ram}}^{\diamond}} (1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1}) \in \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^{T_0} \otimes \mathbb{Z}_p),$$

which proves Theorem 5. □

A Removing $\mu_- = 0$

We combine methods used by J. Ritter and A. Weiss [RW5], A. Wiles [Wi1] and C. Greither [Gr1] to remove the hypothesis $\mu_- = 0$ in Theorem 3 (2) for a special class of cases, including the case $p \nmid |G|$. More precisely, we prove

Theorem 6. *Let T be the set of places of L defined in (38). Suppose that for each prime $\mathfrak{p} \in T(K)$ at least one of the following conditions is satisfied:*

- $j \in I_{\mathfrak{p}}$
- $j \notin I_{\mathfrak{p}}$, *but* $j \in G_{\mathfrak{p}}$ *and* $N(\mathfrak{p})^{f_{\mathfrak{p}}/2} \not\equiv -1 \mod p$
- $p \nmid |I_{\mathfrak{p}}|$

Then we have

$$\mathrm{Fitt}_{\mathbb{Z}_p[[\mathcal{G}]]_-}(X_T^-) = (\Psi_T).$$

Remark 7. In the proof of Theorem 4 we have enlarged the extension L/K to L'/K. But if L/K satisfies the hypotheses of the above theorem, then so does L'/K.

Proof. Since the projective dimension of X_T^- as a $\mathbb{Z}_p[[\mathcal{G}]]_-$-module is at most 1 by Proposition 7, the Fitting ideal in demand is principal, generated by $\tilde{\Psi}_T$, say. The integral closure of $\mathbb{Z}_p[[\mathcal{G}]]_-$ is $R := \sum_\chi \mathbb{Z}_p[\chi][[T]]$, where the sum runs over all odd irreducible characters of $\tilde{G}$. Since $\mathbb{Z}_p[[\mathcal{G}]]_- \cap R^\times = (\mathbb{Z}_p[[\mathcal{G}]]_-)^\times$, it suffices to show

(1) $R\tilde{\Psi}_T = R\Psi_T$
(2) $(\tilde{\Psi}_T) \subset (\Psi_T)$.

If χ is an odd irreducible character of $\tilde{G}$ and X is any $\mathbb{Z}_p[[\mathcal{G}]]_-$-module, we define $\mathbb{Z}_p[\chi][[T]]$-modules

$$X_\chi := X \otimes_{\mathbb{Z}_p[[\mathcal{G}]]_-} \mathbb{Z}_p[\chi][[T]],$$

$$\begin{aligned} X^\chi &:= \left\{ x \in \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} X | gx = \chi(g)x \ \forall g \in \tilde{G} \right\} \\ &\simeq \mathrm{Hom}_{\mathbb{Z}_p[\chi]\tilde{G}}(\mathbb{Z}_p[\chi], \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} X). \end{aligned}$$

To prove (1) we have to show that $\mathrm{Fitt}_{\mathbb{Z}_p[\chi][[T]]}((X_T^-)_\chi)$ is generated by $\chi(\Psi_T)$. By (1) of Theorem 3 this holds apart from the μ-invariants. By Lemma 3.3 in [Gr1] there is an isomorphism $(X_T^-)_\chi \simeq X_T^\chi$, since X_T^- is c.t. over $\tilde{G}$. Moreover, the epimorphism $X_T^- \twoheadrightarrow X_{\mathrm{std}}^-$ has a kernel C which is finitely generated as $\mathbb{Z}_p$-module (cf. (39)), and thus it induces an exact sequence

$$C^\chi \rightarrowtail X_T^\chi \to X_{\mathrm{std}}^\chi \twoheadrightarrow H^1(\tilde{G}, \mathrm{Hom}_{\mathbb{Z}_p[\chi]}(\mathbb{Z}_p[\chi], \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} C)),$$

where the rightmost term is finite. Hence, $\mu(X_T^\chi) = \mu(X_{\mathrm{std}}^\chi)$, and the latter equals the μ-invariant of $\chi(\Psi_T)$ by Theorem 1.4 in [Wi1] if χ is of order prime to p. For arbitrary χ one has to adjust the (second part of the) proof of Theorem 16 in [RW5]. Note that one should think of the claim of Theorem 6 as a reformulation of the equivariant Iwasawa main conjecture; hence equation (1) states that the conjecture is true over the maximal order R, which is Theorem 6 in [RW4].

It remains to prove (2). Write $\tilde{G} = G' \times \tilde{G}_p$, where $\tilde{G}_p$ is the p-Sylow subgroup of $\tilde{G}$, and thus $p \nmid |G'|$. We have a natural decomposition

$$\mathbb{Z}_p[[\mathcal{G}]]_- = \bigoplus_{\substack{\chi' \in \mathrm{Irr}\,(G') \\ \chi' \text{ odd}}} R(\chi'),$$

where $R(\chi') = \mathbb{Z}_p[\chi'][[\tilde{G}_p \times \Gamma_K]]$ is a local ring. Its maximal ideal $\mathfrak{m}_{\chi'}$ is generated by p and the augmentation ideal $\Delta[[\tilde{G}_p \times \Gamma_K]]$. We define a prime ideal $P_{\chi'} := (p, \Delta\tilde{G}_p) \subsetneq \mathfrak{m}_{\chi'}$.

Lemma 9. *For each $\mathfrak{p} \in T(K)$ the element $\xi_\mathfrak{p}$ defined in (41) becomes a unit in $R(\chi')_{P_{\chi'}}$.*

Proof. Recall the definition $Z_\mathfrak{p} = \mathrm{ind}_{\mathcal{G}_\mathfrak{p}}^{\mathcal{G}} \mathbb{Z}_p$. As one can learn from the proof of Proposition 8 in [Gr2], we have

$$(\xi_\mathfrak{p}) = \mathrm{Fitt}_{\mathbb{Q}_p[[\mathcal{G}]]_-}(\mathbb{Q}_p Z_\mathfrak{p}(1)^-)\mathrm{Fitt}_{\mathbb{Q}_p[[\mathcal{G}]]_-}(\mathbb{Q}_p Z_\mathfrak{p}^-)^{-1}.$$

But $Z_\mathfrak{p}^- = 0$ if $j \in G_\mathfrak{p}$. Moreover, $Z_\mathfrak{p}(1) = \mathbb{Z}_p[[\mathcal{G}]]/\langle q_\mathfrak{p} - \phi_\mathfrak{p}, \tau - 1, \tau \in I_\mathfrak{p}\rangle$. Hence, $Z_\mathfrak{p}(1)^- = 0$ if $j \in I_\mathfrak{p}$. Now assume that $j \notin I_\mathfrak{p}$, but $j \in G_\mathfrak{p}$ and $q_\mathfrak{p}^{f_\mathfrak{p}/2} \not\equiv -1 \bmod p$. Then $\phi_\mathfrak{p}^{f_\mathfrak{p}/2} - q_\mathfrak{p}^{f_\mathfrak{p}/2} \equiv j - q_\mathfrak{p}^{f_\mathfrak{p}/2} \bmod T$, and $j - q_\mathfrak{p}^{f_\mathfrak{p}/2}$ becomes a unit on minus parts. This means that $\phi_\mathfrak{p}^{f_\mathfrak{p}/2} - q_\mathfrak{p}^{f_\mathfrak{p}/2}$ is a unit in $\mathbb{Z}_p[[\mathcal{G}]]_-$, and hence $Z_\mathfrak{p}(1)^- = 0$ in this case, too. We have proven so far that $\xi_\mathfrak{p}$ is actually a unit in $\mathbb{Z}_p[[\mathcal{G}]]_-$ if $\mathfrak{p}$ satisfies the first or the second condition of the theorem. We are left with the case $p \nmid |I_\mathfrak{p}|$.

It suffices to show that $(1 - \phi_\mathfrak{p}^{-1} q_\mathfrak{p})\varepsilon_\mathfrak{p} + 1 - \varepsilon_\mathfrak{p}$ and $(1 - \phi_\mathfrak{p})\varepsilon_\mathfrak{p} + 1 - \varepsilon_\mathfrak{p}$ become units at $P_{\chi'}$. We only treat the first element, the other case is similar.

For this we have to prove that $\chi'((1-\phi_\mathfrak{p}^{-1} q_\mathfrak{p})\varepsilon_\mathfrak{p} + 1 - \varepsilon_\mathfrak{p}) \notin P_{\chi'}$. Assume that this is false. Since $1 \notin P_{\chi'}$, we must have $\chi'(\varepsilon_\mathfrak{p}) = 1$. Let us write $\phi_\mathfrak{p}^{-1} = \sigma' \cdot \sigma_p \cdot \gamma_K^c$, where $\sigma' \in G'$, $\sigma_p \in \tilde{G}_p$, $0 \neq c \in \mathbb{Z}_p$. Since $\sigma_p - 1 \in P_{\chi'}$, we have $1 - \chi'(\sigma')\gamma_K^c q_\mathfrak{p} = 1 - \chi'(\sigma') q_\mathfrak{p} (1-T)^c \in P_{\chi'}$. Since $P_{\chi'}$ contains no unit, we must have $p | (1 - \chi'(\sigma') q_\mathfrak{p})$, and hence $1 - (1-T)^c \in P_{\chi'}$. If we write $c = p^n \cdot \alpha$, $\alpha \in \mathbb{Z}_p^\times$, we find out that $1 - (1 - T^{p^n})^\alpha \in P_{\chi'}$. Finally, $1 - (1-T^{p^n})^\alpha = T^{p^n} \cdot g(T)$ with a power series $g(T)$ with $g(0) = -\alpha$, hence $g(T)$ is a unit. This implies $T \in P_{\chi'}$, a contradiction. □

We now return to the proof of Theorem 6. The epimorphism $X_T^- \twoheadrightarrow X_{\mathrm{std}}^-$ implies the first inclusion in

$$\mathrm{Fitt}_{R(\chi')}((X_T^-)_{\chi'}) \subset \mathrm{Fitt}_{R(\chi')}((X_{\mathrm{std}}^-)_{\chi'}) \subset (G_{(\chi')^{-1}\omega, S_{\mathrm{ram}} \cup S_p}(T)),$$

whereas the second inclusion is (10), p. 562 in [Wi2]. Localizing at $P_{\chi'}$ gives

$$(\chi'(\tilde{\Psi}_T))_{P_{\chi'}} \subset (G_{(\chi')^{-1}\omega, S_{\mathrm{ram}} \cup S_p}(T))_{P_{\chi'}} = (\chi'(\Psi_T))_{P_{\chi'}},$$

since all the $\xi_\mathfrak{p}$ become units at $P_{\chi'}$. Therefore, there is an element $r' \in R(\chi') \setminus P_{\chi'}$ such that $r' \cdot \chi'(\tilde{\Psi}_T) \in (\chi'(\Psi_T))$. We already know from Theorem 3 that one can find a positive integer i such that $p^i \cdot \chi'(\tilde{\Psi}_T) \in (\chi'(\Psi_T))$. Hence

$$(p^i, r')(\chi'(\tilde{\Psi}_T)) \subset (\chi'(\Psi_T))$$

and the ideal (p^i, r') has finite index in $R(\chi')$.

Thus, $(\chi'(\tilde{\Psi}_T)) + (\chi'(\Psi_T))/(\chi'(\Psi_T))$ is a submodule of $R(\chi')/(\chi'(\Psi_T))$ of finite cardinality. Now the proof following (10.5) in [Wi1] shows that the only such module is trivial. We obtain $(\chi'(\tilde{\Psi}_T)) \subset (\chi'(\Psi_T))$, and thus we get (2). This completes the proof of the theorem. □

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