A GENERALIZATION OF A THEOREM OF SWAN WITH APPLICATIONS TO IWASAWA THEORY

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Abstract. Let p be a prime and let G be a finite group. By a celebrated theorem of Swan, two finitely generated projective $\mathbb{Z}_p[G]$-modules P and P' are isomorphic if and only if $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P'$ are isomorphic as $\mathbb{Q}_p[G]$-modules. We prove an Iwasawa-theoretic analogue of this result and apply this to the Iwasawa theory of local and global fields. We thereby determine the structure of natural Iwasawa modules up to (pseudo-)isomorphism.

1. Introduction

Let p be a prime and let $\mathcal{G}$ be a profinite group. We denote the complete group algebra of $\mathcal{G}$ over $\mathbb{Z}_p$ by $\Lambda(\mathcal{G})$. In classical Iwasawa theory one studies modules over $\Lambda(\Gamma)$ with $\Gamma \simeq \mathbb{Z}_p$ up to pseudo-isomorphism. Jannsen [Jan89] has proposed a method for studying $\Lambda(\mathcal{G})$-modules up to isomorphism, which in fact works for more general $\mathcal{G}$.

In equivariant Iwasawa theory one is often concerned with the case where $\mathcal{G}$ is a one-dimensional p-adic Lie group. Then $\mathcal{G}$ may be written as a semi-direct product $H \rtimes \Gamma$ with a finite normal subgroup H and $\Gamma \simeq \mathbb{Z}_p$. Jannsen's theory works particularly nice if $\mathcal{G} = H \times \Gamma$ is a direct product and p does not divide the cardinality of H (see [NSW08, Chapter XI, §2 and §3]).

As a concrete example, let L/K be a finite Galois extension of p-adic fields with Galois group H, where a p-adic field shall always mean a finite extension of $\mathbb{Q}_p$ in this article. Let L_∞ be the cyclotomic $\mathbb{Z}_p$-extension of L. We denote the n-th layer of the $\mathbb{Z}_p$-extension L_∞/L by L_n as usual. Assume that p does not divide $|H|$ so that $\mathcal{G} := \mathrm{Gal}(L_\infty/K)$ decomposes into a direct product $H \times \Gamma$ with $\Gamma \simeq \mathbb{Z}_p$. Let us denote the group of principal units in a local field F by $U^1(F)$ and consider the inverse limit

$$U^1(L_\infty) := \varprojlim_n U^1(L_n),$$

where the transition maps are given by the norm maps. Moreover, we let X_{L_∞} be the Galois group over L_∞ of the maximal abelian pro-p-extension of L_∞. Then both $U^1(L_\infty)$ and X_{L_∞} are finitely generated $\Lambda(\mathcal{G})$-modules. If L contains a primitive p-th root of unity, then by [NSW08, Theorems 11.2.3 and 11.2.4] there are (non-canonical) isomorphisms of $\Lambda(\mathcal{G})$-modules

$$X_{L_\infty} \simeq U^1(L_\infty) \simeq \mathbb{Z}_p(1) \oplus \Lambda(\mathcal{G})^{[K:\mathbb{Q}_p]} \tag{1.1}$$

and without the $\mathbb{Z}_p(1)$-term otherwise. Similar statements in fact hold for more arbitrary $\mathbb{Z}_p$-extensions of L. However, this does not remain true if $\mathcal{G}$ contains an element of order p (this follows from the results recalled in §4.1 and in particular from sequence (4.1), since

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$\mathbb{Z}_p$ then has infinite projective dimension as a $\Lambda(\mathcal{G})$-module). In this case, the structure of these $\Lambda(\mathcal{G})$-modules has not yet been determined, although this is a very natural and important question of Iwasawa theory.

Now let H be arbitrary, but assume for simplicity in the introduction that $\mathcal{G} = H \times \Gamma$ is a direct product. Recall that a homomorphism of finitely generated $\Lambda(\Gamma)$-modules is a pseudo-isomorphism if and only if for every height 1 prime ideal $\mathfrak{p}$ of $\Lambda(\Gamma)$ it becomes an isomorphism after localization at $\mathfrak{p}$. We will show that (1.1) remains true after localization at such a prime ideal $\mathfrak{p}$. This is of independent interest, but we point out that our motivation originates from the equivariant Iwasawa main conjecture for local fields formulated by the author in [Nic]. The inverse limit of the principal units along the unramified $\mathbb{Z}_p$-tower naturally appears as a cohomology group of a certain perfect complex of $\Lambda(\mathcal{G})$-modules, which plays a key role in the formulation of this conjecture. It has been shown [Nic, Corollary 6.7] that it suffices to prove the conjecture after localization at the height 1 prime ideal (p). For this reason we are interested in the $\Lambda_{(p)}(\mathcal{G})$-module structure of the localization of $U^1(L_\infty)$ at (p), where for any height 1 prime ideal $\mathfrak{p}$ we denote the localization of $\Lambda(\mathcal{G})$ at $\mathfrak{p}$ by $\Lambda_{\mathfrak{p}}(\mathcal{G})$.

Our method is not restricted to the local case. We also consider finite Galois extensions L/K of number fields and the cyclotomic $\mathbb{Z}_p$-extension L_∞ of L. Then $\mathcal{G} := \mathrm{Gal}(L_\infty/K)$ is again a one-dimensional p-adic Lie group. Let S be a finite set of places of K containing all the archimedean places and all places that ramify in L_∞/K. We then determine the $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module structure of the inverse limit of the (p-completion of the) S-units, localized at $\mathfrak{p}$. We also consider the natural Iwasawa module X_S, the Galois group over L_∞ of the maximal abelian pro-p-extension unramified outside S.

Our method has two main ingredients: The homotopy theory of Iwasawa modules developed by Jannsen [Jan89] and, as a new ingredient, an Iwasawa-theoretic analogue of a theorem of Swan [Swa60]. The latter states that for a finite group G two projective $\mathbb{Z}_p[G]$-modules P and P' are isomorphic if and only if $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P'$ are isomorphic as $\mathbb{Q}_p[G]$-modules. Accordingly, we prove that two finitely generated projective $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules are isomorphic if and only if this is true after base change to $\mathcal{Q}(\mathcal{G})$, the total ring of fractions of $\Lambda(\mathcal{G})$ and thus also of $\Lambda_{\mathfrak{p}}(\mathcal{G})$. This then allows us to compute the projective summands of our Iwasawa modules.

If $\mathcal{G} = H \times \Gamma$ is a direct product, then our result is an easy consequence of Swan's original theorem. This is because then $\Lambda(\mathcal{G})$ is obtained from the group ring $\mathbb{Z}_p[H]$ by extension of scalars. However, the case of a semi-direct product is much harder, and in fact our result cannot be directly deduced from Swan's theorem or even from a more general result due to Hattori [Hat65] (see Remark 2.13 for details).

One method of proving Swan's theorem is via the Cartan–Brauer triangle, since the Cartan map is injective in this case by a theorem of Brauer. This method may be found in [CR81, §21] and we largely follow this approach. In fact, we construct a 'Cartan–Brauer square' in a rather abstract situation and show that the injectivity of the Cartan map always implies a result in the style of Swan's theorem. The case of localized Iwasawa algebras is then implied by a theorem of Ardakov and Wadsley [AW10] on the Cartan map of crossed product algebras. As a by-product we deduce the surjectivity of certain connecting homomorphisms that appear in relative K-theory of Iwasawa algebras.

This article is organized as follows. In §2 we first construct the Cartan–Brauer square, a generalization of the Cartan–Brauer triangle in the case of group rings. We then propose an abstract version of Swan's theorem (Corollary 2.7). Viewing the localized Iwasawa algebras as crossed products allows us to deduce our Iwasawa-theoretic analogue of Swan's theorem from the aforementioned result of Ardakov and Wadsley (see Corollary 2.12). In §3 we review the homotopy theory of Iwasawa modules and prove several auxiliary results for later use. In §4 we study the Iwasawa theory of local fields. In particular, our analogue of Swan's theorem allows us to show that (1.1) remains true for arbitrary one-dimensional p-adic Lie extensions of K after localization at an arbitrary height 1 prime ideal. Finally, we consider cyclotomic $\mathbb{Z}_p$-extensions of number fields in §5, where we prove analogues of [NSW08, Theorem 11.3.11] for arbitrary one-dimensional p-adic Lie extensions containing the cyclotomic $\mathbb{Z}_p$-extension.

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Notation and conventions. All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. If K is a field, we denote its absolute Galois group by G_K. If R is a ring and M is an R-module, we let $\mathrm{pd}_R(M)$ be the projective dimension of M over R.

2. A generalization of Swan's theorem

2.1. **Grothendieck groups.** For further details and background on Grothendieck groups and algebraic K-theory used in this section, we refer the reader to [CR87] and [Swa68]. Let Λ be a noetherian ring and $\mathrm{Mod}(\Lambda)$ be the category of all Λ-modules. We denote the full subcategories of all finitely generated and finitely generated projective Λ-modules by $\mathrm{Mod}^{fg}(\Lambda)$ and $\mathrm{PMod}(\Lambda)$, respectively. We let $G_0(\Lambda)$ and $K_0(\Lambda)$ be the Grothendieck groups of $\mathrm{Mod}^{fg}(\Lambda)$ and $\mathrm{PMod}(\Lambda)$, respectively (see [CR87, §38]). The natural inclusion functor $\mathrm{PMod}(\Lambda) \to \mathrm{Mod}^{fg}(\Lambda)$ induces a homomorphism

$$c : K_0(\Lambda) \longrightarrow G_0(\Lambda)$$

which is called the *Cartan map* or the *Cartan homomorphism*. We recall the following result (see [CR87, Proposition 38.22]).

Lemma 2.1. *Let $P, P' \in \mathrm{PMod}(\Lambda)$. Then we have $[P] = [P']$ in $K_0(\Lambda)$ if and only if $P \oplus Q \simeq P' \oplus Q$ for some $Q \in \mathrm{PMod}(\Lambda)$.*

We write $K_1(\Lambda)$ for the Whitehead group of Λ, which is the abelianized infinite general linear group (see [CR87, §40]). We denote the relative algebraic K-group corresponding to a ring homomorphism $\Lambda \to \Lambda'$ by $K_0(\Lambda, \Lambda')$. We recall that $K_0(\Lambda, \Lambda')$ is an abelian group with generators $[X, g, Y]$ where X and Y are finitely generated projective Λ-modules and $g : \Lambda' \otimes_\Lambda X \to \Lambda' \otimes_\Lambda Y$ is an isomorphism of Λ'-modules; for a full description in terms of generators and relations, we refer the reader to [Swa68, p. 215]. Furthermore, there is a long exact sequence of relative K-theory (see [Swa68, Chapter 15])

$$K_1(\Lambda) \longrightarrow K_1(\Lambda') \xrightarrow{\partial_{\Lambda,\Lambda'}} K_0(\Lambda, \Lambda') \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Lambda'). \tag{2.1}$$

2.2. **The decomposition map.** Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and uniformizer π. We denote the field of fractions of R by K and let $k := R/\mathfrak{m}$ be the residue field. Let A be a finite dimensional K-algebra and let Λ be an R-order in A. We put $\Omega := k \otimes_R \Lambda$, which is a finite dimensional k-algebra. Note that A and Ω are artinian (and thus noetherian) rings so that every finitely generated module has a composition series and satisfies the Jordan–Hölder theorem [CR81, Theorem 3.11].

We also observe that every finitely generated A-module V contains a full Λ-lattice. Indeed, if $v_1, \ldots, v_m$ is a K-basis of V, then $M := \sum_{i=1}^m \Lambda v_i$ is a Λ-submodule of V such that $K \otimes_R M = V$. As R is a discrete valuation ring, every finitely generated torsionfree R-module is in fact free, and so M is a full Λ-lattice in V. We put $\overline{M} := M/\mathfrak{m}M = k \otimes_R M$, which is a finitely generated Ω-module.

Proposition 2.2. *There is a unique homomorphism of abelian groups*

$$d : G_0(A) \longrightarrow G_0(\Omega)$$

such that for each finitely generated A-module V one has $d([V]) = [\overline{M}]$, where M is any full Λ-lattice in V.

Definition 2.3. The homomorphism $d : G_0(A) \to G_0(\Omega)$ in Proposition 2.2 is called the *decomposition map.*

Proof of Proposition 2.2. The proof is similar to that of [CR81, Proposition 16.17], where the case of group rings is considered. We repeat the argument for convenience of the reader.

Let V be a finitely generated A-module and choose a full Λ-lattice M in V. We first show that the class $[\overline{M}]$ in $G_0(\Omega)$ does not depend on the choice of M. For this let N be a second full Λ-lattice in V. By [CR81, Proposition 16.6] we have $[\overline{M}] = [\overline{N}]$ in $G_0(\Omega)$ if and only if $\overline{M}$ and $\overline{N}$ have the same composition factors. As $M + N$ is also a full lattice in V, we may assume that N is properly contained in M. Since M is noetherian, we may in addition assume that N is a maximal Λ-submodule of M. We claim that $\pi M \subseteq N$. Otherwise, the chain of inclusions $N \subsetneq N + \pi M \subseteq M$ gives $N + \pi M = M$ by maximality of N. Then Nakayama's Lemma implies $N = M$, contrary to our assumption. Now consider the chain of inclusions

$$\pi N \subseteq \pi M \subseteq N \subseteq M.$$

We see that $\overline{M}$ and $\overline{N}$ share the composition factors of $N/\pi M$. Thus it suffices to show that M/N and $\pi M/\pi N$ have the same composition factors; but this is clear as multiplication by π induces an isomorphism $M/N \simeq \pi M/\pi N$.

Now define d by $d([V]) = [\overline{M}]$. We have to show that d is additive on short exact sequences. Given a short exact sequence of finitely generated A-modules

$$0 \longrightarrow V_1 \longrightarrow V_2 \xrightarrow{\phi} V_3 \longrightarrow 0,$$

choose a full Λ-lattice M_2 in V_2 and define $M_3 := \phi(M_2)$ and $M_1 := M_2 \cap V_1$. Then we have a short exact sequence of Λ-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \xrightarrow{\phi} M_3 \longrightarrow 0, \tag{2.2}$$

and it is not hard to see that M_1 and M_3 are full Λ-lattices in V_1 and V_3, respectively. As M_3 is a free R-module, tensoring sequence (2.2) with k preserves exactness so that we obtain a short exact sequence of Ω-modules

$$0 \longrightarrow \overline{M_1} \longrightarrow \overline{M_2} \longrightarrow \overline{M_3} \longrightarrow 0.$$

Thus we get

$$d([V_2]) = [\overline{M_2}] = [\overline{M_1}] + [\overline{M_3}] = d([V_1]) + d([V_3])$$

as desired. ∎

2.3. **The Cartan–Brauer square.** We denote the radical of a ring S by $\mathrm{rad}(S)$. We put $\tilde{\Lambda} := \Lambda/\mathrm{rad}(\Lambda) = \Omega/\mathrm{rad}(\Omega)$. Then $\tilde{\Lambda}$ is a semisimple artinian ring, and Λ is *semiperfect* if and only if every idempotent in $\tilde{\Lambda}$ is the image of an idempotent in Λ. Note that Ω is always semiperfect by [CR81, Propositions 6.5 and 6.7].

Remark 2.4. The ring Λ is semiperfect whenever R is complete [CR81, Propositions 6.5 and 6.7] or A is a split semisimple K-algebra [CR81, Exercise 16].

Let us consider the following commutative square

$$\begin{tikzcd} K_0(\Lambda) \arrow[r, "b"] \arrow[d, "e"] & K_0(\Omega) \arrow[d, "c"] \\ G_0(A) \arrow[r, "d"] & G_0(\Omega), \end{tikzcd} \tag{2.3}$$

where for $P \in \mathrm{PMod}(\Lambda)$ we have $b([P]) = [\overline{P}]$ and $e([P]) = [K \otimes_R P]$. We call (2.3) the *Cartan–Brauer square*.

Proposition 2.5. *The homomorphism $b : K_0(\Lambda) \to K_0(\Omega)$ is injective. If Λ is semiperfect, then b is an isomorphism.*

Proof. Let $P, P' \in \mathrm{PMod}(\Lambda)$ and assume that $[\overline{P}] = [\overline{P'}]$ in $K_0(\Omega)$. By Lemma 2.1 there exists an $S \in \mathrm{PMod}(\Omega)$ such that $\overline{P} \oplus S \simeq \overline{P'} \oplus S$. We may assume that S is free and thus in particular that $S \simeq \overline{Q}$ for some $Q \in \mathrm{PMod}(\Lambda)$. We claim that $P \oplus Q \simeq P' \oplus Q$. Then clearly $[P] = [P']$ in $K_0(\Lambda)$ and thus b is injective. For the claim we may assume that R is complete by [CR81, Proposition 30.17] in which case it follows from [CR81, Proposition 6.17 (iv)].

Now suppose that Λ is semiperfect and let $Q \in \mathrm{PMod}(\Omega)$. In order to show that b is surjective, it suffices to find $P \in \mathrm{PMod}(\Lambda)$ such that $\overline{P} \simeq Q$. Let us put $\tilde{Q} := Q/\mathrm{rad}(\Omega)Q \in \mathrm{PMod}(\tilde{\Lambda})$. Then there is a $P \in \mathrm{PMod}(\Lambda)$ such that $P/\mathrm{rad}(\Lambda)P \simeq \tilde{Q}$ by [CR81, Theorem 6.23]. Then both $\overline{P}$ and Q are finitely generated projective Ω-modules and projective covers of $\tilde{Q}$ by [CR81, Corollary 6.22]. This implies $\overline{P} \simeq Q$ as projective covers are unique up to isomorphism [CR81, Proposition 6.20]. $\square$

Remark 2.6. If G is a finite group such that the group ring $R[G]$ is semiperfect, diagram (2.3) specializes to the *Cartan–Brauer triangle* (see [CR81, §18A])

$$\begin{tikzcd} & K_0(k[G]) \arrow[dl, "eb^{-1}"'] \arrow[dr, "c"] & \\ G_0(K[G]) \arrow[rr, "d"] & & G_0(k[G]). \end{tikzcd}$$

The following result might be seen as an abstract version of Swan's theorem [Swa60, §6] (see also [CR81, Theorem 32.1]).

Corollary 2.7. *Let $P, P' \in \mathrm{PMod}(\Lambda)$ and suppose that the Cartan map c is injective. Then $P \simeq P'$ as Λ-modules if and only if $K \otimes_R P \simeq K \otimes_R P'$ as A-modules.*

Proof. As the map b is injective by Proposition 2.5 and the Cartan map c is injective by assumption, also the map e in diagram (2.3) has to be injective. Now assume that $K \otimes_R P \simeq K \otimes_R P'$ as A-modules. Then we have in particular that $e([P]) = e([P'])$ in $G_0(A)$ and thus $[P] = [P']$ in $K_0(\Lambda)$. By Lemma 2.1 there is a finitely generated projective Λ-module Q such that $P \oplus Q \simeq P' \oplus Q$. In order to deduce $P \simeq P'$ we may assume that R is complete by [CR81, Proposition 30.17]. Now the result follows from [CR81, Corollary 6.15]. □

Corollary 2.8 (Swan). *Let G be a finite group and let $P, P' \in \mathrm{PMod}(R[G])$. Then $P \simeq P'$ as $R[G]$-modules if and only if $K \otimes_R P \simeq K \otimes_R P'$ as $K[G]$-modules.*

Proof. It suffices to show that the Cartan map is injective. If k has positive characteristic, this follows from a theorem of Brauer (see [CR81, Theorem 21.22] or [Ser77, Corollary 1 of Theorem 35]). If k has characteristic zero (or if the characteristic is positive and does not divide the cardinality of G), then $k[G]$ is a semisimple ring by Maschke's theorem [CR81, Theorem 3.14]. Thus every finitely generated $k[G]$-module is indeed projective and the Cartan map becomes the identity morphism. □

In view of Corollary 2.7 it is an interesting question of study in which cases the Cartan map is injective. For this the following observation will be very useful.

Lemma 2.9. *Let s be the number of non-isomorphic simple (left) Ω-modules of an arbitrary (left) artinian ring Ω. Then the abelian groups $K_0(\Omega)$ and $G_0(\Omega)$ are free $\mathbb{Z}$-modules of rank s.*

Proof. Let s' be the number of non-isomorphic indecomposable left ideals in Ω. As Ω is an artinian ring, the groups $G_0(\Omega)$ and $K_0(\Omega)$ are free $\mathbb{Z}$-module of rank s and s' by [CR81, Propositions 16.6 and 16.7], respectively. However, if I is an indecomposable left ideal in Ω, then $\tilde{I} := I/\mathrm{rad}(\Omega)I$ is a simple left module by [CR81, Corollary 6.9], and I is the projective cover of $\tilde{I}$ by [CR81, Corollary 6.22]. This induces a one-to-one correspondence between the indecomposable left ideals and the simple left modules (see [CR81, §6B] and in particular [CR81, Proposition 6.17]). Thus we have $s = s'$ as desired. □

2.4. **Crossed products.** Let G be a finite group and let R be a ring. Recall from [MR01, 1.5.8] that a *crossed product* of R by G is an associative ring $R * G$ which contains R as a subring and a set of units $U_G = \{u_g \mid g \in G\}$ of cardinality $|G|$ such that

(i) $R * G$ is a free R-module with basis U_G;
(ii) for all $g, h \in G$ one has $u_g R = R u_g$ and $u_g \cdot u_h R = u_{gh} R$.

We need the following result which immediately follows from Lemma 2.9 and a theorem of Ardakov and Wadsley [AW10, §1.1] (where Brauer's theorem again appears as a key step in the proof).

Theorem 2.10. *Let G be a finite group and let k be a field. Then for every crossed product of k by G, the Cartan map*

$$c : K_0(k * G) \longrightarrow G_0(k * G)$$

is injective with finite cokernel.

2.5. **Iwasawa algebras.** Let p be a prime and $\mathcal{G}$ be a profinite group. The complete group algebra of $\mathcal{G}$ over $\mathbb{Z}_p$ is

$$\Lambda(\mathcal{G}) := \mathbb{Z}_p[\![\mathcal{G}]\!] = \varprojlim \mathbb{Z}_p[\mathcal{G}/\mathcal{N}],$$

where the inverse limit is taken over all open normal subgroups $\mathcal{N}$ of $\mathcal{G}$. Then $\Lambda(\mathcal{G})$ is a compact $\mathbb{Z}_p$-algebra and we denote the kernel of the natural augmentation map $\Lambda(\mathcal{G}) \twoheadrightarrow \mathbb{Z}_p$ by $\Delta(\mathcal{G})$. If M is a (left) $\Lambda(\mathcal{G})$-module we let $M_{\mathcal{G}} := M/\Delta(\mathcal{G})M$ be the module of coinvariants of M. This is the maximal quotient module of M with trivial $\mathcal{G}$-action. Similarly, we denote the maximal submodule of M upon which $\mathcal{G}$ acts trivially by $M^{\mathcal{G}}$.

Now suppose that $\mathcal{G}$ contains a finite normal subgroup H such that $\mathcal{G}/H \simeq \mathbb{Z}_p$. Then $\mathcal{G}$ may be written as a semi-direct product $\mathcal{G} = H \rtimes \Gamma$ where $\Gamma \simeq \mathbb{Z}_p$. In other words, $\mathcal{G}$ is a one-dimensional p-adic Lie group.

If F is a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O} = \mathcal{O}_F$, we put $\Lambda^{\mathcal{O}}(\mathcal{G}) := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) = \mathcal{O}[[\mathcal{G}]]$. We fix a topological generator γ of Γ. Since any homomorphism $\Gamma \to \mathrm{Aut}(H)$ must have open kernel, we may choose a natural number n such that γ^{p^n} is central in $\mathcal{G}$; we fix such an n. As $\Gamma_0 := \Gamma^{p^n} \simeq \mathbb{Z}_p$, there is a ring isomorphism $\Lambda^{\mathcal{O}}(\Gamma_0) \simeq \mathcal{O}[[T]]$ induced by $\gamma^{p^n} \mapsto 1+T$ where $\mathcal{O}[[T]]$ denotes the power series ring in one variable over $\mathcal{O}$. If we view $\Lambda^{\mathcal{O}}(\mathcal{G})$ as a $\Lambda^{\mathcal{O}}(\Gamma_0)$-module (or indeed as a left $\Lambda^{\mathcal{O}}(\Gamma_0)[H]$-module), there is a decomposition

$$\Lambda^{\mathcal{O}}(\mathcal{G}) = \bigoplus_{i=0}^{p^n-1} \Lambda^{\mathcal{O}}(\Gamma_0)[H]\gamma^i. \tag{2.4}$$

Hence $\Lambda^{\mathcal{O}}(\mathcal{G})$ is finitely generated as an $\Lambda^{\mathcal{O}}(\Gamma_0)$-module and is an $\Lambda^{\mathcal{O}}(\Gamma_0)$-order in the separable $\mathcal{Q}^F(\Gamma_0) := Quot(\Lambda^{\mathcal{O}}(\Gamma_0))$-algebra $\mathcal{Q}^F(\mathcal{G})$, the total ring of fractions of $\Lambda^{\mathcal{O}}(\mathcal{G})$, obtained from $\Lambda^{\mathcal{O}}(\mathcal{G})$ by adjoining inverses of all central regular elements. It follows from (2.4) that $\Lambda^{\mathcal{O}}(\mathcal{G})$ is a crossed product of $\Lambda^{\mathcal{O}}(\Gamma_0)$ by $\mathcal{G}/\Gamma_0$ (see also [AB06, §2.3]):

$$\Lambda^{\mathcal{O}}(\mathcal{G}) \simeq \Lambda^{\mathcal{O}}(\Gamma_0) * (\mathcal{G}/\Gamma_0).$$

The commutative ring $\Lambda^{\mathcal{O}}(\Gamma_0)$ is a regular local ring of dimension 2. If $\mathfrak{p}$ is a prime ideal in $\Lambda^{\mathcal{O}}(\Gamma_0)$ of height 1, we denote the localization of $\Lambda^{\mathcal{O}}(\Gamma_0)$ at $\mathfrak{p}$ by $\Lambda^{\mathcal{O}}_{\mathfrak{p}}(\Gamma_0)$. This is a discrete valuation ring and we denote its residue field by $\Omega^{\mathcal{O}}_{\mathfrak{p}}(\Gamma_0)$. We also put

$$\begin{aligned} \Lambda^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G}) &:= \Lambda^{\mathcal{O}}_{\mathfrak{p}}(\Gamma_0) \otimes_{\Lambda^{\mathcal{O}}(\Gamma_0)} \Lambda^{\mathcal{O}}(\mathcal{G}) &&\simeq \Lambda^{\mathcal{O}}_{\mathfrak{p}}(\Gamma_0) * (\mathcal{G}/\Gamma_0) \\ \Omega^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G}) &:= \Omega^{\mathcal{O}}_{\mathfrak{p}}(\Gamma_0) \otimes_{\Lambda^{\mathcal{O}}(\Gamma_0)} \Lambda^{\mathcal{O}}(\mathcal{G}) &&\simeq \Omega^{\mathcal{O}}_{\mathfrak{p}}(\Gamma_0) * (\mathcal{G}/\Gamma_0). \end{aligned}$$

We therefore have the following special case of Theorem 2.10.

Proposition 2.11. *Let $\mathfrak{p}$ be a prime ideal in $\Lambda^{\mathcal{O}}(\Gamma_0)$ of height 1. Then the Cartan map*

$$c : K_0(\Omega^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G})) \longrightarrow G_0(\Omega^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G}))$$

is injective with finite cokernel.

The analogue of Swan's theorem for Iwasawa algebras is now easily established:

Corollary 2.12. *Let $\mathfrak{p}$ be a prime ideal in $\Lambda^{\mathcal{O}}(\Gamma_0)$ of height 1 and let $P, P' \in \mathrm{PMod}(\Lambda^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G}))$. Then $P \simeq P'$ as $\Lambda^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G})$-modules if and only if $\mathcal{Q}^F(\mathcal{G}) \otimes_{\Lambda^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G})} P \simeq \mathcal{Q}^F(\mathcal{G}) \otimes_{\Lambda^{\mathcal{O}}_{\mathfrak{p}}(\mathcal{G})} P'$ as $\mathcal{Q}^F(\mathcal{G})$-modules.*

Proof. This immediately follows from Corollary 2.7 and Proposition 2.11. □

Remark 2.13. If $\mathcal{G} = H \times \Gamma$ is a direct product, then Corollary 2.12 is a direct consequence of Swan's original theorem (Corollary 2.8). We now explain why even Hattori's more general approach [Hat65] (see [CR81, Theorem 32.5]) to Swan's theorem does not imply Corollary 2.12 if $\mathcal{G} = H \rtimes \Gamma$ is only a semi-direct product.

Assume for simplicity that $\mathcal{G}$ is a pro-p-group and that $\mathcal{O} = \mathbb{Z}_p$. If $\mathcal{G} = H \rtimes \Gamma$ is not a direct product, then any choice of Γ_0 will be a proper subgroup of Γ. Let $\Delta(H)$ be the (left) ideal of $\Omega_{(p)}(\mathcal{G})$ generated by the elements $h-1$, $h \in H$. Then $\Delta(H)$ is nilpotent and thus contained in the radical $\mathfrak{r} := \mathrm{rad}(\Omega_{(p)}(\mathcal{G}))$ by [CR81, Proposition 5.15]. However, we have that

$$\Omega_{(p)}(\mathcal{G})/\Delta(H) \simeq \Omega_{(p)}(\Gamma_0) * \Gamma/\Gamma_0 \simeq \Omega_{(p)}(\Gamma)$$

is an inseparable field extension of $\Omega_{(p)}(\Gamma_0)$. Hence $\mathfrak{r} = \Delta(H)$ and $\Omega_{(p)}(\mathcal{G})/\mathfrak{r}$ is not a separable $\Omega_{(p)}(\Gamma_0)$-algebra as it would be required for Hattori's theorem.

Corollary 2.14. *Let $\mathfrak{p}$ be a prime ideal in $\Lambda^{\mathcal{O}}(\Gamma_0)$ of height 1. Then the connecting homomorphism*

$$\partial_{\Lambda_{\mathfrak{p}}^{\mathcal{O}}(\mathcal{G}),\mathcal{Q}^F(\mathcal{G})} : K_1(\mathcal{Q}^F(\mathcal{G})) \longrightarrow K_0(\Lambda_{\mathfrak{p}}^{\mathcal{O}}(\mathcal{G}), \mathcal{Q}^F(\mathcal{G}))$$

is surjective.

Proof. This follows from the long exact sequence (2.1) and Corollary 2.12. □

Corollary 2.15. *Let $\mathfrak{p}$ be a prime ideal in $\Lambda^{\mathcal{O}}(\Gamma_0)$ of height 1. Then the connecting homomorphism*

$$\partial_{\Lambda^{\mathcal{O}}(\mathcal{G}),\Lambda_{\mathfrak{p}}^{\mathcal{O}}(\mathcal{G})} : K_1(\Lambda_{\mathfrak{p}}^{\mathcal{O}}(\mathcal{G})) \longrightarrow K_0(\Lambda^{\mathcal{O}}(\mathcal{G}), \Lambda_{\mathfrak{p}}^{\mathcal{O}}(\mathcal{G}))$$

is surjective. Moreover, we have a short exact sequence of abelian groups

$$0 \longrightarrow K_0(\Lambda^{\mathcal{O}}(\mathcal{G}), \Lambda_{\mathfrak{p}}^{\mathcal{O}}(\mathcal{G})) \longrightarrow K_0(\Lambda^{\mathcal{O}}(\mathcal{G}), \mathcal{Q}^F(\mathcal{G})) \longrightarrow K_0(\Lambda_{\mathfrak{p}}^{\mathcal{O}}(\mathcal{G}), \mathcal{Q}^F(\mathcal{G})) \longrightarrow 0.$$

Proof. Consider the long exact sequences (2.1) for the three occurring pairs. The connecting homomorphism

$$\partial_{\Lambda^{\mathcal{O}}(\mathcal{G}),\mathcal{Q}^F(\mathcal{G})} : K_1(\mathcal{Q}^F(\mathcal{G})) \longrightarrow K_0(\Lambda^{\mathcal{O}}(\mathcal{G}), \mathcal{Q}^F(\mathcal{G}))$$

is surjective by [Wit13, Corollary 3.8]. The result follows from Corollary 2.14 by an easy diagram chase. □

3. Homotopy theory

3.1. **Homotopy of modules.** We briefly recall basic material of homotopy theory of modules. The reader may also consult Jannsen [Jan89, §1] and [NSW08, Chapter V, §4].

Let Λ be a ring. If a homomorphism $f : M \to N$ of Λ-modules factors through a projective Λ-module, then we say that f is *homotopic to zero* and we write $f \sim 0$. Two homomorphisms $f, g : M \to N$ are *homotopic* ($f \sim g$) if $f - g$ is homotopic to zero. We let $\mathrm{Ho}(\Lambda)$ be the homotopy category of Λ-modules. This category has the same objects as $\mathrm{Mod}(\Lambda)$, but the homomorphism groups are given by $\mathrm{Hom}_\Lambda(M,N)/\{f \sim 0\}$. A homomorphism $f : M \to N$ of Λ-modules is a *homotopy equivalence* if it is an isomorphism in $\mathrm{Ho}(\Lambda)$. In this case, we say that M and N are *homotopy equivalent* and write $M \sim N$.

For any (left) Λ-module M and integer $i \geq 0$ we define (right) Λ-modules $M^+ := \mathrm{Hom}_\Lambda(M,\Lambda)$ and $E^i(M) := \mathrm{Ext}^i_\Lambda(M,\Lambda)$. In particular, we have $M^+ = E^0(M)$. We denote the full subcategory of $\mathrm{Ho}(\Lambda)$ whose objects are finitely presented Λ-modules by $\mathrm{Ho}^{fp}(\Lambda)$. The *transpose* is a contravariant functor

$$D : \mathrm{Ho}^{fp}(\Lambda) \longrightarrow \mathrm{Ho}^{fp}(\Lambda)$$

that on objects is given as follows. Let M be a finitely presented Λ-module and choose a presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

by finitely generated projective Λ-modules. Then DM is defined by the exact sequence

$$0 \longrightarrow M^+ \longrightarrow P_0^+ \longrightarrow P_1^+ \longrightarrow DM \longrightarrow 0.$$

The transpose is a contravariant autoduality of $\mathrm{Ho}^{fp}(\Lambda)$, i.e. $D \circ D \simeq \mathrm{id}$, by [NSW08, Proposition 5.4.9]. Moreover, for every finitely presented Λ-module M there is an exact sequence of Λ-modules

$$0 \longrightarrow E^1(DM) \longrightarrow M \xrightarrow{\phi_M} M^{++} \longrightarrow E^2(DM) \longrightarrow 0, \tag{3.1}$$

where ϕ_M is the canonical map of M to its bidual.

3.2. **Homotopy of Iwasawa modules.** Let $\mathcal{G}$ be a one-dimensional p-adic Lie group. As in subsection §2.5 we choose a central subgroup $\Gamma_0 \simeq \mathbb{Z}_p$ in $\mathcal{G}$ and view $\Lambda(\mathcal{G})$ as a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$. We denote the set of prime ideals in $\Lambda(\Gamma_0)$ of height 1 by $\boldsymbol{P}_0$. We let $^\sharp : \mathcal{Q}(\mathcal{G}) \to \mathcal{Q}(\mathcal{G})$ be the anti-involution that maps each group element $g \in \mathcal{G}$ to its inverse. For every $\mathfrak{p} \in \boldsymbol{P}_0$ we have $\mathfrak{p}^\sharp := \left\{x^\sharp \mid x \in \mathfrak{p}\right\} \in \boldsymbol{P}_0$ and in particular an equality $(\Lambda_\mathfrak{p}(\mathcal{G}))^\sharp = \Lambda_{\mathfrak{p}^\sharp}(\mathcal{G})$.

The functors D and E^i interchange left and right Λ-action. If $\Lambda = \Lambda(\mathcal{G})$ is the Iwasawa algebra, then we have a natural equivalence between left and right modules, induced by the anti-involution $^\sharp$. We then endow DM and $E^i(M)$ with this left module structure. Namely, for $\lambda \in \Lambda(\mathcal{G})$ and $x \in DM$ or $x \in E^i(M)$ we let $\lambda \cdot x := x \cdot \lambda^\sharp$. Similarly, if $\Lambda = \Lambda_\mathfrak{p}(\mathcal{G})$ for some $\mathfrak{p} \in \boldsymbol{P}_0$, then for every finitely presented left $\Lambda_\mathfrak{p}(\mathcal{G})$-module M, the transpose DM and $E^i(M)$ are natural left $\Lambda_{\mathfrak{p}^\sharp}(\mathcal{G})$-modules.

The functors D and E^i then commute with localization in the sense that for every prime ideal $\mathfrak{p}$ of $\Lambda(\Gamma_0)$ we have $DM_\mathfrak{p} = (DM)_{\mathfrak{p}^\sharp}$ and $E^i(M_\mathfrak{p}) = E^i(M)_{\mathfrak{p}^\sharp}$; here and in the following the notation $DM_\mathfrak{p}$ always means the transpose of $M_\mathfrak{p}$ and *not* the localization of DM at $\mathfrak{p}$. In particular, for every finitely generated $\Lambda(\mathcal{G})$-module M and every $\mathfrak{p} \in \boldsymbol{P}_0$ we have $E^2(M_\mathfrak{p}) = E^2(M)_{\mathfrak{p}^\sharp} = 0$ by [NSW08, Proposition 5.5.3]. In fact, we have the following result which will often be used without reference.

Lemma 3.1. *Let $\mathcal{G}$ be a one-dimensional p-adic Lie group and let $\mathfrak{p} \in \boldsymbol{P}_0$. Then $E^2(M)$ vanishes for every finitely generated $\Lambda_\mathfrak{p}(\mathcal{G})$-module M. In particular, there is an exact sequence*

$$0 \longrightarrow E^1(DM) \longrightarrow M \longrightarrow M^{++} \longrightarrow 0.$$

Proof. The map $M/E^1(DM) \to M^{++}$ induced by ϕ_M is an injective pseudo-isomorphism by (the proof of) [NSW08, Proposition 5.1.8]. Then sequence (3.1) implies that $E^2(DM)$ is pseudo-null as a $\Lambda_\mathfrak{p}(\Gamma_0)$-module and thus vanishes, since $\Lambda_\mathfrak{p}(\Gamma_0)$ is a discrete valuation ring. Applying this argument to DM, we obtain $E^2(M) \simeq E^2(DDM) = 0$. □

Lemma 3.2. *Let $\mathcal{G}$ be a one-dimensional p-adic Lie group and let Λ be either the Iwasawa algebra $\Lambda(\mathcal{G})$ or $\Lambda_\mathfrak{p}(\mathcal{G})$ for some prime ideal $\mathfrak{p} \in \boldsymbol{P}_0$. Let M be a finitely generated Λ-module such that M^{++} has finite projective dimension. Then the $\Lambda^\sharp$-module M^+ and the Λ-module M^{++} are indeed projective.*

Proof. We assume that $\Lambda = \Lambda(\mathcal{G})$; the other case can be treated similarly. We put $d := \mathrm{pd}_{\Lambda(\mathcal{G})}(M^{++})$ and choose a projective resolution

$$0 \longrightarrow P_d \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M^{++} \longrightarrow 0.$$

As M^+ and M^{++} are reflexive and thus free as a $\Lambda(\Gamma_0)$-modules by [NSW08, Propositions 5.1.9 and 5.4.17], this induces an exact sequence of $\Lambda(\mathcal{G})$-modules

$$0 \longrightarrow M^+ \longrightarrow P_0^+ \longrightarrow P_1^+ \longrightarrow \cdots \longrightarrow P_d^+ \longrightarrow 0.$$

As each P_i^+, $0 \leq i \leq d$ is a projective $\Lambda(\mathcal{G})^\sharp$-module, so is M^+. The result follows. □

The next result shows that Lemma 3.2 is only interesting if $\Lambda = \Lambda(\mathcal{G})$ or $\Lambda = \Lambda_{(p)}(\mathcal{G})$.

Lemma 3.3. *Let $\mathfrak{p} \in \boldsymbol{P}_0$ be a prime ideal and assume that $\mathfrak{p} \neq (p)$. Let M be a finitely generated $\Lambda_\mathfrak{p}(\mathcal{G})$-module. Then M is a projective $\Lambda_\mathfrak{p}(\mathcal{G})$-module if and only if M is (torsion-)free as a $\Lambda_\mathfrak{p}(\Gamma_0)$-module. In particular, every reflexive $\Lambda_\mathfrak{p}(\mathcal{G})$-module is projective.*

Proof. We first recall that every torsionfree (and in particular every projective) $\Lambda_{\mathfrak{p}}(\Gamma_0)$-module is in fact free, since $\Lambda_{\mathfrak{p}}(\Gamma_0)$ is a discrete valuation ring. Now suppose that M is a projective $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module. As $\Lambda_{\mathfrak{p}}(\mathcal{G})$ is free as a $\Lambda_{\mathfrak{p}}(\Gamma_0)$-module, the module M is a submodule of a free $\Lambda_{\mathfrak{p}}(\Gamma_0)$-module and thus free. For the converse we put $G := \mathcal{G}/\Gamma_0$. Then G is a finite group and $|G|$ is invertible in $\Lambda_{\mathfrak{p}}(\Gamma_0)$ since $\mathfrak{p} \neq (p)$. Thus $\mathrm{Hom}_{\Lambda_{\mathfrak{p}}(\Gamma_0)}(M,N)$ is a $\mathbb{Q}_p[G]$-module for any two $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules M and N. Since taking G-invariants is an exact functor on $\mathbb{Q}_p[G]$-modules, the equality

$$\mathrm{Hom}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}(M,N) = \mathrm{Hom}_{\Lambda_{\mathfrak{p}}(\Gamma_0)}(M,N)^G$$

implies isomorphisms

$$\mathrm{Ext}^i_{\Lambda_{\mathfrak{p}}(\mathcal{G})}(M,N) \simeq \mathrm{Ext}^i_{\Lambda_{\mathfrak{p}}(\Gamma_0)}(M,N)^G$$

for all $i \geq 0$. This gives the converse implication. □

Remark 3.4. Suppose that $\mathcal{G} \simeq H \times \Gamma$ and that p does not divide the cardinality of H. Then we can take $\Gamma_0 = \Gamma$ and Lemma 3.3 remains true for $\mathfrak{p} = (p)$ and the Iwasawa algebra $\Lambda(\mathcal{G})$ by [NSW08, Lemma 5.4.16].

Corollary 3.5. *Let $\mathfrak{p} \in \boldsymbol{P}_0$ be a prime ideal and assume that $\mathfrak{p} \neq (p)$. Then every finitely generated $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module has projective dimension at most* 1.

Corollary 3.6. *Let $\mathfrak{p} \in \boldsymbol{P}_0$ be a prime ideal and assume that $\mathfrak{p} \neq (p)$. Let M be a finitely generated $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module. Then there is an isomorphism*

$$M \simeq E^1(DM) \oplus M^{++}.$$

Proof. This follows from Lemma 3.1 and Lemma 3.3. □

Corollary 3.7. *For every $\mathfrak{p} \in \boldsymbol{P}_0$ the $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module $\Delta(\mathcal{G})_{\mathfrak{p}}$ is free of rank* 1.

Proof. We identify $\Lambda(\Gamma_0)$ with the power series ring $\mathbb{Z}_p[[T]]$ as usual. If $\mathfrak{p} \neq (T)$ then $(\mathbb{Z}_p)_{\mathfrak{p}}$ vanishes so that the exact sequence

$$0 \longrightarrow \Delta(\mathcal{G}) \longrightarrow \Lambda(\mathcal{G}) \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

induces an isomorphism $\Delta(\mathcal{G})_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}(\mathcal{G})$. If $\mathfrak{p} = (T)$ or more generally if $\mathfrak{p} \neq (p)$ then $\Delta(\mathcal{G})_{\mathfrak{p}}$ is a projective $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module by Lemma 3.3. Then Corollary 2.12 implies that it is free of rank 1 (in fact, an isomorphism $\Lambda_{\mathfrak{p}}(\mathcal{G}) \simeq \Delta(\mathcal{G})_{\mathfrak{p}}$ is explicitly given by $1 \mapsto (1-\gamma)e_H + (1-e_H)$, where $e_H := |H|^{-1}\sum_{h \in H} h$). □

4. Iwasawa theory of local fields

4.1. **Galois cohomology.** If F is a field and M is a topological G_F-module, we write $R\Gamma(F,M)$ for the complex of continuous cochains of G_F with coefficients in M and $H^i(F,M)$ for its cohomology in degree i. We likewise write $H_i(F,M)$ for the i-th homology group of G_F with coefficients in M. If F is an algebraic extension of $\mathbb{Q}_p$ or $\mathbb{Q}$ and M is a discrete or compact G_F-module, then for $r \in \mathbb{Z}$ we denote the r-th Tate twist of M by $M(r)$. For an abelian group A we write $\widehat{A}$ for its p-completion, that is $\widehat{A} = \varprojlim_n A/p^nA$.

Let L/K be a finite Galois extension of p-adic fields with Galois group G. Let L_∞ be an arbitrary $\mathbb{Z}_p$-extension of L with Galois group Γ_L and for each $n \in \mathbb{N}$ let L_n be its n-th layer. We assume that L_∞/K is again a Galois extension with Galois group $\mathcal{G} := \mathrm{Gal}(L_\infty/K)$. We let X_{L_∞} denote the Galois group over L_∞ of the maximal abelian pro-p-extension of L_∞. We put

$$Y_{L_\infty} := \Delta(G_K)_{G_{L_\infty}} = \mathbb{Z}_p \widehat{\otimes}_{\Lambda(G_{L_\infty})} \Delta(G_K)$$

and observe that $\mathrm{pd}_{\Lambda(\mathcal{G})}(Y_{L_\infty}) \leq 1$ by [NSW08, Theorem 7.4.2]. As $H_1(L_\infty, \mathbb{Z}_p)$ canonically identifies with X_{L_∞}, taking G_{L_∞}-coinvariants of the obvious short exact sequence

$$0 \longrightarrow \Delta(G_K) \longrightarrow \Lambda(G_K) \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow X_{L_\infty} \longrightarrow Y_{L_\infty} \longrightarrow \Lambda(\mathcal{G}) \longrightarrow \mathbb{Z}_p \longrightarrow 0 \tag{4.1}$$

of $\Lambda(\mathcal{G})$-modules (this should be compared to the sequence constructed by Ritter and Weiss [RW02, §1]; see also [NSW08, Proposition 5.6.7]). This sequence will be crucial in the following.

Remark 4.1. The middle arrow in (4.1) defines a (perfect) complex of $\Lambda(\mathcal{G})$-modules

$$\cdots \longrightarrow 0 \longrightarrow Y_{L_\infty} \longrightarrow \Lambda(\mathcal{G}) \longrightarrow 0 \longrightarrow \cdots$$

If we place Y_{L_∞} in degree 1, then this complex and $R\mathrm{Hom}(R\Gamma(L_\infty, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)[-2]$ become isomorphic in the derived category of $\Lambda(\mathcal{G})$-modules by [Nic, Proposition 4.1]. If L_∞ is the unramified $\mathbb{Z}_p$-extension of L, then this complex plays a key role in the equivariant Iwasawa main conjecture for local fields as formulated by the author [Nic, Conjecture 5.1]. In order to verify this conjecture, one may localize at the height 1 prime ideal (p) by [Nic, Corollary 6.7]. This has motivated our interest in the $\Lambda_{(p)}(\mathcal{G})$-module structure of $(X_{L_\infty})_{(p)}$.

For any p-adic field F, we denote the group of principal units in F by $U^1(F)$. We put $U^1(L_\infty) := \varprojlim_n U^1(L_n)$ where the transition maps are given by the norm maps. We note that $\varprojlim_n \widehat{L_n^\times} \simeq X_{L_\infty}$ by local class field theory. For each $n \geq 0$ the valuation map $L_n^\times \to \mathbb{Z}$ induces an exact sequence

$$0 \longrightarrow U^1(L_n) \longrightarrow \widehat{L_n^\times} \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

Taking inverse limits over all n induces an exact sequence of $\Lambda(\mathcal{G})$-modules

$$0 \longrightarrow U^1(L_\infty) \longrightarrow X_{L_\infty} \longrightarrow \mathbb{Z}_p \longrightarrow 0 \tag{4.2}$$

if L_∞/L is ramified and an isomorphism $U^1(L_\infty) \simeq X_{L_\infty}$ otherwise (also see the proof of [NSW08, Theorem 11.2.4]).

4.2. **Local Iwasawa modules.** In this subsection we prove analogues of [NSW08, Theorems 11.2.3 and 11.2.4] for arbitrary one-dimensional p-adic Lie extensions. As in subsection §2.5 we choose a central subgroup $\Gamma_0 \simeq \mathbb{Z}_p$ in $\mathcal{G}$ and view $\Lambda(\mathcal{G})$ as a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$.

Lemma 4.2. *For every $\mathfrak{p} \in \boldsymbol{P}_0$ the following hold.*

(i) *We have an isomorphism of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$(Y_{L_\infty})_{\mathfrak{p}} \simeq (X_{L_\infty})_{\mathfrak{p}} \oplus \Lambda_{\mathfrak{p}}(\mathcal{G});$$

(ii) *we have* $\mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((X_{L_\infty})_{\mathfrak{p}}) = \mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((Y_{L_\infty})_{\mathfrak{p}}) \leq 1$.

Proof. Sequence (4.1) and Corollary 3.7 imply (i). For (ii) we compute

$$\mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((X_{L_\infty})_{\mathfrak{p}}) = \mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((Y_{L_\infty})_{\mathfrak{p}}) \leq \mathrm{pd}_{\Lambda(\mathcal{G})}(Y_{L_\infty}) \leq 1.$$

□

We denote the group of p-power roots of unity in L_∞ by $\mu_p(L_\infty)$. If M is a $\mathbb{Z}_p$-module, we let $M^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ be its Pontryagin dual. If M is a $\mathcal{G}$-module, we endow $M^\vee$ with the contragredient $\mathcal{G}$-action.

Theorem 4.3. *Put $n := [K : \mathbb{Q}_p]$. Then for every $\mathfrak{p} \in \boldsymbol{P}_0$ the following hold.*

(i) *If $\mu_p(L_\infty)$ is finite, then we have isomorphisms of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$(Y_{L_\infty})_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}(\mathcal{G})^{n+1}, \quad (X_{L_\infty})_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}(\mathcal{G})^{n}.$$

(ii) *If $\mu_p(L_\infty)$ is infinite (and thus L_∞/L is the cyclotomic $\mathbb{Z}_p$-extension), then we have isomorphisms of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$(Y_{L_\infty})_{\mathfrak{p}} \simeq (\mathbb{Z}_p(1))_{\mathfrak{p}} \oplus \Lambda_{\mathfrak{p}}(\mathcal{G})^{n+1}, \quad (X_{L_\infty})_{\mathfrak{p}} \simeq (\mathbb{Z}_p(1))_{\mathfrak{p}} \oplus \Lambda_{\mathfrak{p}}(\mathcal{G})^{n}.$$

Proof. We first note that it suffices to prove the result for $(Y_{L_\infty})_{\mathfrak{p}}$. As on earlier occasions, we may then use [CR81, Proposition 30.17 and Corollary 6.15] to deduce the result for $(X_{L_\infty})_{\mathfrak{p}}$ from Lemma 4.2(i).

As the p-dualizing module of G_K naturally identifies with $\mathbb{Q}_p/\mathbb{Z}_p(1)$ by [NSW08, Theorem 7.2.4], we have a homotopy equivalence of $\Lambda(\mathcal{G})$-modules

$$Y_{L_\infty} \sim D(\mu_p(L_\infty)^\vee) \tag{4.3}$$

by [NSW08, Proposition 5.6.9]. We first assume that $\mu_p(L_\infty)$ is finite. Then (4.3) implies that $(Y_{L_\infty})_{\mathfrak{p}} \sim 0$. This means that $(Y_{L_\infty})_{\mathfrak{p}}$ is a projective $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module. As $\mathcal{Q}(\mathcal{G})$ is semisimple, the $\mathcal{Q}(\mathcal{G})$-module $\mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} Y_{L_\infty}$ is free of rank $n+1$ by [NSW08, Theorem 7.4.2]. Corollary 2.12 then gives the result.

Now assume that $\mu_p(L_\infty)$ is infinite. Then (4.3) gives

$$Y_{L_\infty} \sim D(\mathbb{Z}_p(-1)).$$

As the functor D induces an autoduality, we have $E^1(DY_{L_\infty}) = E^1(\mathbb{Z}_p(-1)) = \mathbb{Z}_p(1)$ and likewise $E^2(DY_{L_\infty}) = E^2(\mathbb{Z}_p(-1)) = 0$ by [NSW08, Proposition 5.5.3 (iv) and Corollary 5.5.7]. Thus sequence (3.1) specializes to

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow Y_{L_\infty} \longrightarrow Y_{L_\infty}^{++} \longrightarrow 0. \tag{4.4}$$

Since $(\mathbb{Z}_p(1))_{(p)}$ vanishes, we have $\mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((\mathbb{Z}_p(1))_{\mathfrak{p}}) \leq 1$ for every $\mathfrak{p} \in \boldsymbol{P}_0$ by Corollary 3.5. The projective dimension of $(Y_{L_\infty})_{\mathfrak{p}}$ is at most 1 by Lemma 4.2(ii) and thus $(Y_{L_\infty}^{++})_{\mathfrak{p}}$ also has finite projective dimension. Lemma 3.2 implies that the $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module $(Y_{L_\infty}^{++})_{\mathfrak{p}}$ is indeed projective. We now may deduce from Corollary 2.12 as above that $(Y_{L_\infty}^{++})_{\mathfrak{p}}$ is free of rank $n+1$. By (4.4) we get an isomorphism

$$(Y_{L_\infty})_{\mathfrak{p}} \simeq (\mathbb{Z}_p(1))_{\mathfrak{p}} \oplus \Lambda_{\mathfrak{p}}(\mathcal{G})^{n+1}$$

as desired. ∎

Corollary 4.4. *For every $\mathfrak{p} \in \boldsymbol{P}_0$ we have an isomorphism of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$U^1(L_\infty)_{\mathfrak{p}} \simeq (X_{L_\infty})_{\mathfrak{p}}.$$

In particular, the following hold.

(i) *If $\mu_p(L_\infty)$ is finite, then we have an isomorphism of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$U^1(L_\infty)_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}(\mathcal{G})^{n}.$$

(ii) *If $\mu_p(L_\infty)$ is infinite (and thus L_∞/L is the cyclotomic $\mathbb{Z}_p$-extension), then we have an isomorphism of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$U^1(L_\infty)_{\mathfrak{p}} \simeq (\mathbb{Z}_p(1))_{\mathfrak{p}} \oplus \Lambda_{\mathfrak{p}}(\mathcal{G})^{n}.$$

Proof. If L_∞ is the unramified $\mathbb{Z}_p$-extension, then $U^1(L_\infty) \simeq X_{L_\infty}$ and the result immediately follows from Theorem 4.3. Now suppose that L_∞/L is ramified. Let us put $Z_{\mathfrak{p}^\sharp} := E^1((X_{L_\infty})_{\mathfrak{p}})$. Then Theorem 4.3 implies that $Z_{\mathfrak{p}^\sharp}$ vanishes unless $\mu_p(L_\infty)$ is infinite, where we have an isomorphism of $\Lambda_{\mathfrak{p}^\sharp}(\mathcal{G})$-modules $Z_{\mathfrak{p}^\sharp} \simeq \mathbb{Z}_p(-1)_{\mathfrak{p}^\sharp}$. In both cases we have that

$$\mathrm{Hom}_{\Lambda_{\mathfrak{p}^\sharp}(\mathcal{G})}((\mathbb{Z}_p)_{\mathfrak{p}^\sharp}, Z_{\mathfrak{p}^\sharp}) = 0. \tag{4.5}$$

The exact sequence (4.2) localized at $\mathfrak{p}$ induces a long exact sequence of $\Lambda_{\mathfrak{p}^\sharp}(\mathcal{G})$-modules

$$\cdots \longrightarrow E^1((\mathbb{Z}_p)_{\mathfrak{p}}) \longrightarrow E^1((X_{L_\infty})_{\mathfrak{p}}) \longrightarrow E^1(U^1(L_\infty)_{\mathfrak{p}}) \longrightarrow E^2((\mathbb{Z}_p)_{\mathfrak{p}}) \longrightarrow \cdots$$

As we have an isomorphism $E^1((\mathbb{Z}_p)_{\mathfrak{p}}) \simeq (\mathbb{Z}_p)_{\mathfrak{p}^\sharp}$, the second arrow is the zero map by (4.5). Since $E^i((\mathbb{Z}_p)_{\mathfrak{p}})$ vanishes for $i \neq 1$, we obtain an isomorphism of $\Lambda_{\mathfrak{p}^\sharp}(\mathcal{G})$-modules

$$E^1(U^1(L_\infty)_{\mathfrak{p}}) \simeq E^1((X_{L_\infty})_{\mathfrak{p}}) = Z_{\mathfrak{p}^\sharp}.$$

In particular $E^1(E^1(U^1(L_\infty)_{\mathfrak{p}})) \simeq E^1(Z_{\mathfrak{p}^\sharp})$ vanishes unless $\mu_p(L_\infty)$ is infinite, where we have $E^1(Z_{\mathfrak{p}^\sharp}) \simeq \mathbb{Z}_p(1)_{\mathfrak{p}}$. Now [NSW08, Proposition 5.5.8] and (3.1) imply that

$$E^1(DU^1(L_\infty)_{\mathfrak{p}}) \simeq E^1(Z_{\mathfrak{p}^\sharp})$$

which in particular has projective dimension at most 1. It follows that $U^1(L_\infty)_{\mathfrak{p}}$ and $U^1(L_\infty)_{\mathfrak{p}}^{++}$ have finite projective dimension by (4.2), Lemma 4.2 and the exact sequence

$$0 \longrightarrow E^1(Z_{\mathfrak{p}^\sharp}) \longrightarrow U^1(L_\infty)_{\mathfrak{p}} \longrightarrow U^1(L_\infty)_{\mathfrak{p}}^{++} \longrightarrow 0. \tag{4.6}$$

Thus $U^1(L_\infty)_{\mathfrak{p}}^{++}$ is indeed a projective $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module by Lemma 3.2. We have

$$\mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} U^1(L_\infty)^{++} \simeq \mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} U^1(L_\infty) \simeq \mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} X_{L_\infty} \simeq \mathcal{Q}(\mathcal{G})^n$$

by Theorem 4.3. It now follows from Corollary 2.12 that $U^1(L_\infty)_{\mathfrak{p}}^{++}$ is a free $\Lambda_{\mathfrak{p}}(\mathcal{G})$-module of rank n. Sequence (4.6) splits, giving the claim. □

4.3. **Iwasawa theory of ℓ-adic fields.** We briefly discuss the case where L/K is a finite Galois extension of ℓ-adic fields with $p \neq \ell$. Then L has a unique $\mathbb{Z}_p$-extension L_∞, namely the unramified $\mathbb{Z}_p$-extension. We again define $\mathcal{G} := \mathrm{Gal}(L_\infty/K)$. For each $n \geq 0$, the valuation map induces an exact sequence

$$0 \longrightarrow \mu_p(L_n) \longrightarrow \widehat{L_n^\times} \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

Taking inverse limits over all n yields an isomorphism of $\Lambda(\mathcal{G})$-modules

$$\varprojlim_n \mu_p(L_n) \simeq \varprojlim_n \widehat{L_n^\times} =: X_{L_\infty}.$$

The following result is therefore clear (see also [NSW08, Theorem 11.2.3(ii)]). We let ζ_p be a primitive p-th roof of unity.

Lemma 4.5. *For $\ell \neq p$ we have $X_{L_\infty} \simeq \mathbb{Z}_p(1)$ if $\zeta_p \in L$ and $X_{L_\infty} = 0$ otherwise.*

5. Iwasawa theory of number fields

5.1. **The relevant Galois groups.** In this section we consider a finite Galois extension L/K of number fields with Galois group G. Let p be a prime and let L_∞ be the cyclotomic $\mathbb{Z}_p$-extension of L with n-th layer L_n. We will assume throughout that

$$K \textit{ is totally imaginary if } p = 2.$$

We put $\mathcal{G} := \mathrm{Gal}(L_\infty/K)$ which is a one-dimensional p-adic Lie group. We may write $\mathcal{G} \simeq H \rtimes \Gamma$, where H naturally identifies with a subgroup of G and $\Gamma \simeq \mathbb{Z}_p$. For every place v of K we choose a place w_∞ of L_∞ above v and let $\mathcal{G}_v$ be the decomposition group at w_∞. We denote the place of L below w_∞ by w and the completion of L at w by L_w.

We choose a finite set S of places of K containing all archimedean places and all places that ramify in L_∞/K. In particular, all p-adic places lie in S. We denote the ring of integers in L by $\mathcal{O}_L$ and the ring of $S(L)$-integers by $\mathcal{O}_{L,S}$, where $S(L)$ denotes the set of places of L above those in S.

We let M_S be the maximal pro-p-extension of L which is unramified outside S. We put $G_S := \mathrm{Gal}(M_S/K)$ and $H_S := \mathrm{Gal}(M_S/L_\infty)$. Since K is totally imaginary if $p = 2$, the cohomological p-dimension of G_S equals 2 by [NSW08, Proposition 10.11.3] (note that our definition of G_S follows [NSW08, Chapter XI, §3, p.739], but slightly differs from the profinite group G_S considered in [NSW08, Chapter X, §11]; however, the proof of [Jan89, Lemma 5.3] shows that both groups have the same cohomological p-dimension). Choose a presentation $F_d \twoheadrightarrow G_S$ of G_S by a free profinite group F_d of finite rank d. Then we obtain a commutative diagram (compare [NSW08, p. 740])

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
1 \arrow[r] & R \arrow[r] \arrow[d] & F_d \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H_S \arrow[r] \arrow[d] & G_S \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

with exact rows and columns, where R and N are the kernels of $F_d \twoheadrightarrow \mathcal{G}$ and $F_d \twoheadrightarrow G_S$, respectively. Then G_S acts on $N^{\mathrm{ab}}(p)$, the maximal abelian pro-p-quotient of N. The module $N^{\mathrm{ab}}_{H_S}(p)$ of H_S-coinvariants of $N^{\mathrm{ab}}(p)$ is a projective $\Lambda(\mathcal{G})$-module by [NSW08, Proposition 5.6.7]. We let r_1 and r_2 be the number of real and complex places of K, respectively. We let S'_∞ be the set of real places of K becoming complex in L_∞ and put $r'_1 := |S'_\infty|$. If we choose d greater than or equal to $r_2 + r'_1 + 1$, then we have an isomorphism of $\Lambda(\mathcal{G})$-modules

$$N^{\mathrm{ab}}_{H_S}(p) \simeq \Lambda(\mathcal{G})^{d-r_2-r'_1-1} \oplus \bigoplus_{v \in S'_\infty} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{G}_v} \mathbb{Z}_p \tag{5.1}$$

by [Jan89, Theorem 5.4] (see also [NSW08, Theorem 11.3.10(iii)]; the assumption that p does not divide $[L : K]$ is not needed for this part of the theorem). Here, for a closed subgroup $\mathcal{H}$ of $\mathcal{G}$ and a compact $\Lambda(\mathcal{H})$-module M we let

$$\mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}} M := \Lambda(\mathcal{G}) \widehat{\otimes}_{\Lambda(\mathcal{H})} M$$

denote compact induction of M from $\mathcal{H}$ to $\mathcal{G}$.

5.2. **Global and semi-local Iwasawa modules.** Let $X_S := H_S^{\mathrm{ab}}$ be the abelianization of H_S. Then X_S is a finitely generated $\Lambda(\mathcal{G})$-module by [NSW08, Proposition 11.3.1]. We also consider the 'standard' Iwasawa module X_{nr}, which is the Galois group over L_∞ of the maximal unramified abelian pro-p-extension of L_∞, and the quotient X_{cs}^S of X_{nr} that corresponds to the maximal subextension which is completely decomposed at all primes above S. For a finite place v of K we define

$$A_v := \varprojlim_n \prod_{w_n | v} \widehat{L_{n,w_n}^\times} \simeq \mathrm{Ind}_{\mathcal{G}_v}^{\mathcal{G}} X_{L_{w,\infty}},$$

$$U_v := \varprojlim_n \prod_{w_n | v} \widehat{\mathcal{O}_{L_{n,w_n}}^\times} \simeq \begin{cases} \mathrm{Ind}_{\mathcal{G}_v}^{\mathcal{G}} X_{L_{w,\infty}} & \text{if} \quad v \nmid p \\ \mathrm{Ind}_{\mathcal{G}_v}^{\mathcal{G}} U^1(L_{w,\infty}) & \text{if} \quad v \mid p. \end{cases}$$

Here, $L_{w,\infty}$ always denotes the cyclotomic $\mathbb{Z}_p$-extension of L_w. We let S_f be the subset of S comprising all finite places in S. We then define $\Lambda(\mathcal{G})$-modules

$$A_S := \prod_{v \in S_f} A_v, \quad U_S := \prod_{v \in S_f} U_v.$$

Finally, we let

$$E_S := \varprojlim_n (\mathcal{O}_{L_n,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p), \quad E := \varprojlim_n (\mathcal{O}_{L_n}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

Since the weak Leopoldt conjecture holds for the cyclotomic $\mathbb{Z}_p$-extension by [NSW08, Theorem 10.3.25], we obtain from [Jan89, Theorem 5.4] the following commutative diagram of $\Lambda(\mathcal{G})$-modules with exact rows (see also [NSW08, Theorem 11.3.10(i)]; the assumption $p \nmid [L : K]$ is irrelevant, since all maps are certainly $\mathcal{G}$-equivariant)

(5.2)
$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, hook] & U_S \arrow[r] \arrow[d, hook] & X_S \arrow[r] \arrow[d, equal] & X_{nr} \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & E_S \arrow[r] & A_S \arrow[r] & X_S \arrow[r] & X_{cs}^S \arrow[r] & 0. \end{tikzcd}$$

As in the local case, there is an exact sequence of $\Lambda(\mathcal{G})$-modules (see [NSW08, Proposition 5.6.7])

$$0 \longrightarrow X_S \longrightarrow Y_S \longrightarrow \Delta(\mathcal{G}) \longrightarrow 0, \tag{5.3}$$

where $Y_S := \Delta(G_S)_{H_S}$ is a finitely generated $\Lambda(\mathcal{G})$-module of projective dimension at most 1.

5.3. **Structure of global Iwasawa modules.** We now determine the structure of the above Iwasawa modules after localization at a prime ideal $\mathfrak{p} \in \boldsymbol{P}_0$. We begin with the semi-local Iwasawa modules.

Proposition 5.1. *Let $S_f(\zeta_p)$ be the set of all finite places v in S such that $\zeta_p \in L_w$ and put $n := [K : \mathbb{Q}]$. Then for every $\mathfrak{p} \in \boldsymbol{P}_0$ we have isomorphisms of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$(A_S)_{\mathfrak{p}} \simeq (U_S)_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}(\mathcal{G})^n \oplus \bigoplus_{v \in S_f(\zeta_p)} \left(\mathrm{Ind}_{\mathcal{G}_v}^{\mathcal{G}} \mathbb{Z}_p(1)\right)_{\mathfrak{p}}.$$

In particular, we have $\mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((A_S)_{\mathfrak{p}}) = \mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((U_S)_{\mathfrak{p}}) \leq 1$.

Proof. This follows from Theorem 4.3, Corollary 4.4, Lemma 4.5 and the well known formula $[K : \mathbb{Q}] = \sum_{v|p} [K_v : \mathbb{Q}_p]$. □

We let $D_2^{(p)}(G_S)$ be the p-dualizing module of G_S and put $Z_S := (D_2^{(p)}(G_S)^{H_S})^\vee$.

Lemma 5.2. *For every $\mathfrak{p} \in \boldsymbol{P}_0$ the following hold.*

(i) *We have an isomorphism of* $\Lambda_{\mathfrak{p}}(\mathcal{G})$*-modules*

$$(Y_S)_{\mathfrak{p}} \simeq (X_S)_{\mathfrak{p}} \oplus \Lambda_{\mathfrak{p}}(\mathcal{G});$$

(ii) *we have* $\mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((X_S)_{\mathfrak{p}}) = \mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((Y_S)_{\mathfrak{p}}) \leq 1$*;*

(iii) *we have a homotopy equivalence* $(X_S)_{\mathfrak{p}} \sim (DZ_S)_{\mathfrak{p}}$ *and an isomorphism of* $\Lambda_{\mathfrak{p}^\sharp}(\mathcal{G})$*-modules*

$$E^1((X_S)_{\mathfrak{p}}) \simeq (Z_S)_{\mathfrak{p}^\sharp}.$$

Proof. Sequence (5.3) and Corollary 3.7 imply (i). As Y_S is a $\Lambda(\mathcal{G})$-module of projective dimension at most 1, (i) implies (ii). By (i) we have $(X_S)_{\mathfrak{p}} \sim (Y_S)_{\mathfrak{p}}$ and in particular $E^1((X_S)_{\mathfrak{p}}) \simeq E^1((Y_S)_{\mathfrak{p}})$. Hence (iii) is a consequence of [NSW08, Proposition 5.6.9]. □

We let μ_L be the Iwasawa μ-invariant of the standard Iwasawa module X_{nr}. We recall the following conjecture of Iwasawa.

Conjecture 5.3 (Iwasawa)**.** *For every number field* L *the* μ*-invariant* μ_L *vanishes.*

The following two results are analogues of [NSW08, Theorem 11.3.11] for arbitrary one-dimensional p-adic Lie extensions (containing the cyclotomic $\mathbb{Z}_p$-extension).

Theorem 5.4. *Let* $\mathfrak{p} \in \boldsymbol{P}_0$ *and assume that* $\mu_{L(\zeta_p)} = 0$ *if* $\mathfrak{p} = (p)$*. Then the following hold.*

(i) *We have an isomorphism of* $\Lambda_{\mathfrak{p}}(\mathcal{G})$*-modules*

$$(X_S)_{\mathfrak{p}} \simeq E^1(Z_S)_{\mathfrak{p}} \oplus (X_S)_{\mathfrak{p}}^{++};$$

moreover, we have $(Z_S)_{(p)} = 0$ *so that in particular* $(X_S)_{(p)} \simeq (X_S)_{(p)}^{++}$*;*

(ii) *we have isomorphisms of* $\Lambda_{\mathfrak{p}}(\mathcal{G})$*-modules*

$$((X_S)_{\mathfrak{p}^\sharp})^+ \simeq (X_S)_{\mathfrak{p}}^{++} \simeq \Lambda_{\mathfrak{p}}(\mathcal{G})^{r_2} \oplus \bigoplus_{v \in S'_\infty} \left(\mathrm{Ind}_{\mathcal{G}_v}^{\mathcal{G}} \mathbb{Z}_p^-\right)_{\mathfrak{p}},$$

where $\mathbb{Z}_p^-$ *is the* $\mathcal{G}_v$*-module* $\mathbb{Z}_p$ *upon which the generator of* $\mathcal{G}_v \simeq \mathbb{Z}/2\mathbb{Z}$ *acts by multiplication by* -1*.*

Proof. By Lemma 5.2(iii) we have $E^1(DX_S)_{\mathfrak{p}} \simeq E^1(Z_S)_{\mathfrak{p}}$ so that (i) follows from Corollary 3.6 if $\mathfrak{p} \neq (p)$. We claim that $(Z_S)_{(p)}$ vanishes. Then Lemma 3.1 implies (i) in the case $\mathfrak{p} = (p)$. We first assume that $\zeta_p \in L$. Then by [Jan89, Theorem 5.4 (d)] there is an exact sequence of $\Lambda(\Gamma_0)$-modules

$$0 \longrightarrow X_{cs}^S(-1) \longrightarrow Z_S \longrightarrow T \longrightarrow 0,$$

where T is finitely generated and free as $\mathbb{Z}_p$-module. As μ_L vanishes by assumption and X_{nr} surjects onto X_{cs}^S, the latter module is also finitely generated over $\mathbb{Z}_p$. Hence the same is true for Z_S and so $(Z_S)_{(p)} = 0$ as desired. If ζ_p is not in L, we put $L' := L(\zeta_p)$ and $\Delta := \mathrm{Gal}(L'/L) \simeq \mathrm{Gal}(L'_\infty/L_\infty)$. Let Z'_S be the Iwasawa module Z_S that corresponds to L'. We have shown that Z'_S is a finitely generated $\mathbb{Z}_p$-module. However, there is a natural isomorphism $(Z'_S)_\Delta \simeq Z_S$ so that the μ-invariant of Z_S also vanishes. This proves the claim and thus (i). Lemmas 3.2, 3.3 and 5.2(ii) imply that both $((X_S)_{\mathfrak{p}^\sharp})^+$ and $(X_S)_{\mathfrak{p}}^{++}$ are projective $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules. By Corollary 2.12 it now suffices to compute $\mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} X_S^{++}$. By [NSW08, Proposition 5.6.7] we have

$$\mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} (X_S^{++} \oplus N_{H_S}^{\mathrm{ab}}(p)) = \mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} (X_S \oplus N_{H_S}^{\mathrm{ab}}(p)) \simeq \mathcal{Q}(\mathcal{G})^{d-1}.$$

Since $\mathcal{Q}(\mathcal{G})$ is semisimple, (ii) now follows from (5.1). □

Theorem 5.5. *Let $\mathfrak{p} \in \boldsymbol{P}_0$ and assume that $\mu_{L(\zeta_p)} = 0$ if $\mathfrak{p} = (p)$. Then the following hold.*

(i) *We have* $\mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}(E_{\mathfrak{p}}) = \mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}((E_S)_{\mathfrak{p}}) \leq 1$*;*

(ii) *if $\zeta_p \in L$ we have isomorphisms of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$E_{\mathfrak{p}} \simeq (E_S)_{\mathfrak{p}} \simeq \mathbb{Z}_p(1)_{\mathfrak{p}} \oplus \Lambda_{\mathfrak{p}}(\mathcal{G})^{r_2+r_1-r_1'} \oplus \bigoplus_{v \in S_\infty'} \left(\mathrm{Ind}_{\mathcal{G}_v}^{\mathcal{G}} \mathbb{Z}_p\right)_{\mathfrak{p}};$$

(iii) *if $\zeta_p \notin L$ we have isomorphisms of $\Lambda_{\mathfrak{p}}(\mathcal{G})$-modules*

$$E_{\mathfrak{p}} \simeq (E_S)_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}(\mathcal{G})^{r_2+r_1-r_1'} \oplus \bigoplus_{v \in S_\infty'} \left(\mathrm{Ind}_{\mathcal{G}_v}^{\mathcal{G}} \mathbb{Z}_p\right)_{\mathfrak{p}}.$$

Proof. We first show that the projective dimension of $E_{\mathfrak{p}}$ and $(E_S)_{\mathfrak{p}}$ is at most 1. For this we only have to treat the case $\mathfrak{p} = (p)$. Otherwise we apply Corollary 3.5. As the μ-invariant of X_{nr} vanishes by assumption, we obtain from diagram (5.2) two exact sequences of $\Lambda_{(p)}(\mathcal{G})$-modules

$$0 \longrightarrow E_{(p)} \longrightarrow (U_S)_{(p)} \longrightarrow (X_S)_{(p)} \longrightarrow 0,$$

$$0 \longrightarrow (E_S)_{(p)} \longrightarrow (A_S)_{(p)} \longrightarrow (X_S)_{(p)} \longrightarrow 0.$$

Since the projective dimension of $(U_S)_{(p)}$, $(A_S)_{(p)}$ and $(X_S)_{(p)}$ is at most 1 by Proposition 5.1 and Lemma 5.2(ii), the same is true for $E_{(p)}$ and $(E_S)_{(p)}$.

Now let $\mathfrak{p} \in \boldsymbol{P}_0$ be arbitrary. It follows as in the proof of [NSW08, Theorem 11.3.11(ii)] that $E^1(D(E_S)_{\mathfrak{p}}) \simeq \mathbb{Z}_p(1)_{\mathfrak{p}}$ if $\zeta_p \in L$ and that $E^1(D(E_S)_{\mathfrak{p}})$ vanishes otherwise. In both cases we have $\mathrm{pd}_{\Lambda_{\mathfrak{p}}(\mathcal{G})}(E^1(D(E_S)_{\mathfrak{p}})) \leq 1$ and thus $(E_S)_{\mathfrak{p}}^{++}$ is projective by Lemma 3.2. It follows that $(E_S)_{\mathfrak{p}}$ decomposes into a direct sum

$$(E_S)_{\mathfrak{p}} \simeq E^1(D(E_S)_{\mathfrak{p}}) \oplus (E_S)_{\mathfrak{p}}^{++}.$$

The inclusions $E^1(DE_{\mathfrak{p}}) \subseteq E^1(D(E_S)_{\mathfrak{p}}) \subseteq E_{\mathfrak{p}}$ imply that in fact $E^1(DE_{\mathfrak{p}}) = E^1(D(E_S)_{\mathfrak{p}})$. It follows as above that the module $E_{\mathfrak{p}}^{++}$ is projective and that we have an isomorphism

$$E_{\mathfrak{p}} \simeq E^1(DE_{\mathfrak{p}}) \oplus E_{\mathfrak{p}}^{++}.$$

In particular, we obtain (i). By Corollary 2.12 it now suffices to compute

$$\mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} E_S^{++} = \mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} E_S = \mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} E = \mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} E^{++}.$$

We deduce from diagram (5.2) and Proposition 5.1 that we have isomorphisms of $\mathcal{Q}(\mathcal{G})$-modules

$$\mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} (E_S \oplus X_S) \simeq \mathcal{Q}(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} A_S \simeq \mathcal{Q}(\mathcal{G})^n.$$

As $\mathcal{Q}(\mathcal{G})$ is semisimple, the result follows from Theorem 5.4. □

Remark 5.6. Let L_∞ be an arbitrary $\mathbb{Z}_p$-extension of L such that L_∞/K is again a Galois extension. Assuming the validity of the weak Leopoldt conjecture, it seems to be likely that one can prove analogues of Theorems 5.4 and 5.5. The main obstacle occurs in the case $\mathfrak{p} = (p)$ because the relevant μ-invariant does not vanish in general.

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