

Integrality of Stickelberger elements and the equivariant Tamagawa number conjecture

Andreas Nickel*

Abstract

Let L/K be a finite Galois CM-extension of number fields with Galois group G . In an earlier paper, the author has defined a module $SKu(L/K)$ over the center of the group ring $\mathbb{Z}G$ which coincides with the Sinnott-Kurihara ideal if G is abelian and, in particular, contains many Stickelberger elements. It was shown that a certain conjecture on the integrality of $SKu(L/K)$ implies the minus part of the equivariant Tamagawa number conjecture at an odd prime p for an infinite class of (non-abelian) Galois CM-extensions of number fields which are at most tamely ramified above p , provided that Iwasawa's μ -invariant vanishes. Here, we prove a relevant part of this integrality conjecture which enables us to deduce the minus- p -part of the equivariant Tamagawa number conjecture from the vanishing of μ for the same class of extensions. As an application we prove the non-abelian Brumer and Brumer-Stark conjecture outside the 2-primary part for every monomial Galois extension of \mathbb{Q} provided that certain μ -invariants vanish.

Introduction

Let L/K be a finite Galois extension of number fields with Galois group G . Burns [Bu01] used complexes arising from étale cohomology of the constant sheaf \mathbb{Z} to define a canonical element $T\Omega(L/K)$ of the relative K -group $K_0(\mathbb{Z}G, \mathbb{R})$. This element relates the leading terms at zero of Artin L -functions attached to L/K to natural arithmetic invariants. It was shown that the vanishing of $T\Omega(L/K)$ is equivalent to the equivariant Tamagawa number conjecture (ETNC) for the pair $(h^0(\text{Spec}(L)), \mathbb{Z}G)$ (cf. [Bu01, Theorem 2.4.1]).

The vanishing of $T\Omega(L/K)$ is known to be true if L is absolutely abelian as proved by Burns and Greither [BG03] with the exclusion of the 2-primary part; Flach [Fl04, Fl11] extended the argument to cover the 2-primary part as well. Slightly weaker results in this cyclotomic case have been settled independently by Ritter and Weiss [RW02, RW03], Huber and Kings [HK03]. Some relatively abelian results are due to Bley [Bl06]; he showed that if L/K is a finite abelian extension, where K is an imaginary quadratic field which has class number one, then the ETNC holds for all intermediate extensions L/E such that $[L : E]$ is odd and divisible only by primes which split completely in K/\mathbb{Q} . Finally, if L/K is a CM-extension and p is odd, the ETNC at p naturally decomposes into a plus and a

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minus part; it was shown by the author [Ni11a] that the minus part of the ETNC at p holds if L/K is abelian and at most tamely ramified above p , and the Iwasawa μ -invariant vanishes if p divides $|G|$ (and some additional technical condition is fulfilled). Note that the vanishing of μ is a long standing conjecture of Iwasawa theory; the most general result is still due to Ferrero and Washington [FW79] and says that $\mu = 0$ for absolutely abelian extensions.

For non-abelian extensions, the results are rather sparse. Burns and Flach [BF03] have given a proof for an infinite class of quaternion extensions over the rationals and Navilarekallu [Na06] has treated a specific A_4 -extension over \mathbb{Q} . Further progress has recently been made by Johnston and the author [JN].

If L/K is a CM extension, the author [Ni11b] has introduced a module $SKu(L/K)$ over the center of the group ring $\mathbb{Z}G$ which is a noncommutative analogue of the Sinnott-Kurihara ideal (cf. [Si80, p. 193]) and was already implicitly used in [Ni11d] and [BJ11]. An integrality conjecture on $SKu(L/K)$ has been formulated and it was shown that it is implied by the ETNC in many cases and follows from the results in [Ba77], [Ca79], [DR80] if G is abelian. Assuming the validity of this integrality conjecture, the minus part of the ETNC at p was deduced from the conjectural vanishing of μ , provided that the ramification above p is at most tame (and, as in the abelian case, some technical extra assumption holds). Moreover, it follows from the results in [Ni11d] that for the case at hand the non-abelian analogues of Brumer's conjecture, of the Brumer-Stark conjecture and of the strong Brumer-Stark property (as formulated in [Ni11d]) hold, provided that $\mu = 0$ and the integrality conjecture holds.

Most of these results make heavily use of the validity of the equivariant Iwasawa main conjecture (EIMC) attached to the extension L_∞^+/K , where L_∞^+ is the cyclotomic \mathbb{Z}_p -extension of L^+ which is the maximal real subfield of L . Note that the EIMC is known for abelian extensions of totally real number fields with Galois group \mathcal{G} such that \mathcal{G} is a p -adic Lie group of dimension 1 (cf. [Wi90, RW02]). More recently, Ritter and Weiss [RW11] have shown that the EIMC (up to its uniqueness statement) holds for arbitrary p -adic Lie groups of dimension 1 provided that μ vanishes. In fact, this can be generalized to higher dimensional p -adic Lie groups as shown by Kakde [Ka13] and, independently, by Burns [Bu]. Note that Kakde in fact provides an independent proof also in the case of dimension 1.

In this paper, we define a variant $SKu'(L/K)$ of the Sinnott-Kurihara module which is contained in $SKu(L/K)$ and in fact equals $SKu(L/K)$ for abelian G . Let $\mathcal{M}(G)$ be a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}G$; for any ring Λ , we write $\zeta(\Lambda)$ for the subring of all elements which are central in Λ . The first main result is the following theorem which will be proved in §4. Recall that a character of a finite group is called *monomial* if it is induced by a linear character of a subgroup. A finite group is called *monomial* if each of its (complex) irreducible characters is monomial.

Theorem 0.1. *Let L/K be a Galois extension of number fields with Galois group G . If G is monomial, then*

$$SKu'(L/K) \subseteq \zeta(\mathcal{M}(G)).$$

Now let S and T be two finite sets of places of K such that S and T are disjoint and

S contains the set S_∞ of all infinite places of K . One can associate to S and T so-called Stickelberger elements θ_S^T which lie in the center of the group ring algebra $\mathbb{Q}G$. These Stickelberger elements are defined via values of Artin L -functions at zero and are closely related to the Sinnott-Kurihara ideal $SKu(L/K)$; more precisely, they lie in $SKu(L/K)$ under suitable hypotheses on S and T . For instance, it suffices to assume the following hypotheses to which we will refer as $Hyp(S, T)$: S contains the set S_{ram} of all ramified primes and no non-trivial root of unity in L is congruent to 1 modulo all primes $\mathfrak{P} \in T(L)$; here, for any set T of places of K , we write $T(L)$ for the set of places of L which lie above those in T .

Now assume that the Galois group G decomposes as $G = H \times C$, where H is monomial and C is abelian. As before, let $\mathcal{M}(H)$ be a maximal order in $\mathbb{Q}H$ containing $\mathbb{Z}H$. Then we may view $\mathcal{M}(H)[C]$ as an order in $\mathbb{Q}G$ and we have the following integrality statement for Stickelberger elements.

Theorem 0.2. *Let L/K be a Galois extension of number fields with Galois group $G = H \times C$, where H is monomial and C is abelian. Then*

$$\theta_S^T \in \zeta(\mathcal{M}(H)[C]) = \zeta(\mathcal{M}(H))[C]$$

whenever $Hyp(S, T)$ is satisfied.

We will give a more precise statement and its proof in §4. In fact, we will prove a more general result involving also Stickelberger elements which are defined via values of Artin L -functions at negative integers.

Now let p be a prime and let L be either a totally real field or a CM-field. Let \mathcal{L} be the cyclotomic \mathbb{Z}_p -extension of $L(\zeta_p)$, where ζ_p denotes a primitive p -th root of unity. Moreover, let X_{std} be the ‘standard’ Iwasawa module which is the projective limit of the p -parts of the class groups in the cyclotomic tower of $L(\zeta_p)$. We will say that the Iwasawa μ -invariant attached to L and p vanishes if the μ -invariant of X_{std} vanishes. Now let L/K be a Galois CM-extension of number fields with arbitrary Galois group G . Then Theorem 0.2 is the key in proving our main result.

Theorem 0.3. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ -invariant attached to L and p vanishes, then the p -minus part of the ETNC for the pair $(h^0(\text{Spec}(L)), \mathbb{Z}G)$ is true.*

For a fixed extension L/K there is only a finite number of exceptional primes; for a precise definition see §5 (Definition 5.5), where we will prove Theorem 0.3. Finally, we obtain the following corollaries.

Corollary 0.4. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ -invariant attached to L and p vanishes, then the p -part of the integrality conjecture (Conjecture 2.2 below) holds.*

Corollary 0.5. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ -invariant attached to L and p vanishes, then the p -parts of the following conjectures hold:*

- (i) the non-abelian Brumer conjecture [Ni11d, Conjecture 2.1]
- (ii) the non-abelian Brumer-Stark conjecture [Ni11d, Conjecture 2.6]
- (iii) the minus part of the central conjecture (Conjecture 2.4.1) of Burns [Bu11]
- (iv) the minus-part of the Lifted Root Number Conjecture of Gruenberg, Ritter and Weiss [GRW99].

Moreover, L/K fulfills the non-abelian strong Brumer-Stark property at p (cf. [Ni11d, Definition 3.5]).

We will recall the precise statement of the non-abelian Brumer and Brumer-Stark conjecture in §6.

Corollary 0.6. *Let L/K be a tamely ramified Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ -invariant attached to L and p vanishes, then the minus- p -parts of the central conjecture (Conjecture 3.3) of Breuning and Burns [BB07] and of the ETNC for the pair $(h^0(\text{Spec}(L))(1), \mathbb{Z}G)$ are valid.*

Finally, a further nice consequence of our results is the following theorem.

Theorem 0.7. *Let L be any monomial Galois CM-extension of \mathbb{Q} . Assume that the Iwasawa μ -invariant attached to L and p vanishes for every odd prime p which ramifies in L or divides $[L : \mathbb{Q}]$. Then the non-abelian Brumer conjecture and the non-abelian Brumer-Stark conjecture are true outside the 2-primary parts.*

If L is abelian over \mathbb{Q} , we know the vanishing of the μ -invariants by the aforementioned result of Ferrero and Washington [FW79] and the above theorem recovers Stickelberger's classical theorem (cf. [Wa82, Theorem 6.10]). So the above result is a 'non-abelian Stickelberger theorem' without the extra factors occurring in [BJ11].

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1 Preliminaries

1.1 K -theory

1.1.1 Localization Sequences

Let Λ be a left noetherian ring with 1 and $\text{PMod}(\Lambda)$ the category of all finitely generated projective Λ -modules. We write $K_0(\Lambda)$ for the Grothendieck group of $\text{PMod}(\Lambda)$, and $K_1(\Lambda)$ for the Whitehead group of Λ which is the abelianized infinite general linear group. If S is

a multiplicatively closed subset of the center of Λ which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all finitely generated S -torsion Λ -modules of finite projective dimension by $K_0S(\Lambda)$. Writing Λ_S for the ring of quotients of Λ with denominators in S , we have the following Localization Sequence (cf. [CR87, p. 65])

$$K_1(\Lambda) \rightarrow K_1(\Lambda_S) \xrightarrow{\partial_S} K_0S(\Lambda) \rightarrow K_0(\Lambda) \rightarrow K_0(\Lambda_S). \quad (1)$$

In the special case where Λ is an \mathfrak{o} -order over a commutative ring \mathfrak{o} and S is the set of all nonzerodivisors of \mathfrak{o} , we also write $K_0T(\Lambda)$ instead of $K_0S(\Lambda)$. Moreover, we denote the relative K -group corresponding to a ring homomorphism $\Lambda \rightarrow \Lambda'$ by $K_0(\Lambda, \Lambda')$ (cf. [Sw68]). Then we have a Localization Sequence (cf. [CR87, p. 72])

$$K_1(\Lambda) \rightarrow K_1(\Lambda') \xrightarrow{\partial_{\Lambda, \Lambda'}} K_0(\Lambda, \Lambda') \rightarrow K_0(\Lambda) \rightarrow K_0(\Lambda'). \quad (2)$$

The maps ∂_S and $\partial_{\Lambda, \Lambda'}$ in (1) and (2) are called *boundary homomorphisms*. It is also shown in [Sw68] that there is an isomorphism $K_0(\Lambda, \Lambda_S) \simeq K_0S(\Lambda)$.

Let G be a finite group; in the case where Λ' is the group ring $\mathbb{R}G$, the reduced norm map $\text{nr}_{\mathbb{R}G} : K_1(\mathbb{R}G) \rightarrow \zeta(\mathbb{R}G)^\times$ is injective, and there exists a canonical map $\hat{\partial}_G : \zeta(\mathbb{R}G)^\times \rightarrow K_0(\mathbb{Z}G, \mathbb{R}G)$ such that the restriction of $\hat{\partial}_G$ to the image of the reduced norm equals $\partial_{\mathbb{Z}G, \mathbb{R}G} \circ \text{nr}_{\mathbb{R}G}^{-1}$. This map is called the *extended boundary homomorphism* and was introduced by Burns and Flach [BF01].

1.1.2 Refined Euler characteristics

For any ring Λ we write $\mathcal{D}(\Lambda)$ for the derived category of Λ -modules. Let $\mathcal{C}^b(\text{PMod}(\Lambda))$ be the category of bounded complexes of finitely generated projective Λ -modules. A complex of Λ -modules is called perfect if it is isomorphic in $\mathcal{D}(\Lambda)$ to an element of $\mathcal{C}^b(\text{PMod}(\Lambda))$. We denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$ comprising perfect complexes by $\mathcal{D}^{\text{perf}}(\Lambda)$. For any $C^\bullet \in \mathcal{C}^b(\text{PMod}(\Lambda))$ we define Λ -modules

$$C^{\text{ev}} := \bigoplus_{i \in \mathbb{Z}} C^{2i}, \quad C^{\text{odd}} := \bigoplus_{i \in \mathbb{Z}} C^{2i+1}.$$

Similarly, we define $H^{\text{ev}}(C^\bullet)$ and $H^{\text{odd}}(C^\bullet)$ to be the direct sum over all even and odd degree cohomology groups of C^\bullet , respectively.

For the following let R be a Dedekind domain of characteristic 0, F its field of fractions, A a finite dimensional F -algebra and Λ an R -order in A . Let K be a field containing F , and write $K_0(\Lambda, K)$ for $K_0(\Lambda, K \otimes_F A)$. A pair (C^\bullet, t) consisting of a complex $C^\bullet \in \mathcal{D}^{\text{perf}}(\Lambda)$ and an isomorphism $t : H^{\text{odd}}(C_K^\bullet) \rightarrow H^{\text{ev}}(C_K^\bullet)$ is called a trivialized complex, where $C_K^\bullet := K \otimes_R^{\mathbb{L}} C^\bullet$ is the left derived tensor product of C^\bullet with K . We refer to t as a trivialization of C^\bullet . One defines the refined Euler characteristic $\chi_{\Lambda, K}(C^\bullet, t) \in K_0(\Lambda, K)$ of a trivialized complex as follows: Choose a complex $P^\bullet \in \mathcal{C}^b(\text{PMod}(\Lambda))$ which is quasi-isomorphic to C^\bullet . Let $B^i(P_K^\bullet)$ and $Z^i(P_K^\bullet)$ denote the i^{th} coboundaries and i^{th} cocycles of P_K^\bullet , respectively. We have the obvious exact sequences

$$B^i(P_K^\bullet) \rightarrow Z^i(P_K^\bullet) \rightarrow H^i(P_K^\bullet), \quad Z^i(P_K^\bullet) \rightarrow P_K^i \rightarrow B^{i+1}(P_K^\bullet).$$

If we choose splittings of the above sequences, we get an isomorphism

$$\phi_t : P_K^{odd} \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P_K^\bullet) \oplus H^{odd}(P_K^\bullet) \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P_K^\bullet) \oplus H^{ev}(P_K^\bullet) \simeq P_K^{ev},$$

where the second map is induced by t . Then the refined Euler characteristic is defined to be

$$\chi_{\Lambda, K}(C^\bullet, t) := (P^{odd}, \phi_t, P^{ev}) \in K_0(\Lambda, K)$$

which indeed is independent of all choices made in the construction. For further information concerning refined Euler characteristics we refer the reader to [Bu03].

We define $DT(\Lambda)$ to be the torsion subgroup of $K_0(\Lambda, F)$.

1.1.3 Hom-description

In this paper we use a formulation of the ETNC in terms of relative K -theory and reduced norms. However, we will frequently refer to [Ni11a], where the Hom-description is used. Here we summarize some basic facts of this equivalent theory for convenience of the reader.

Let G be a finite group and let p be a prime. We denote the ring of virtual characters of G with values in \mathbb{C}_p by $R_p(G)$. We choose a finite Galois extension E of \mathbb{Q}_p such that all representations of G can be realized over E , and put $\Gamma := \text{Gal}(E/\mathbb{Q}_p)$. If χ is a character of G , we let V_χ be an EG -module with character χ . Then by [GRW99, Appendix A] there is an isomorphism

$$\begin{aligned} \text{Det} : K_1(\mathbb{Q}_p G) &\xrightarrow{\simeq} \text{Hom}_\Gamma(R_p(G), E^\times) \\ x &\mapsto [\chi \mapsto \det(X \mid \text{Hom}_{EG}(V_\chi, (EG)^n))], \end{aligned}$$

where $X \in \text{GL}_n(\mathbb{Q}_p G)$ maps to x under the natural map $\text{GL}_n(\mathbb{Q}_p G) \rightarrow K_1(\mathbb{Q}_p G)$. We have an exact sequence

$$K_1(\mathbb{Z}_p G) \rightarrow K_1(\mathbb{Q}_p G) \rightarrow K_0 T(\mathbb{Z}_p G) \rightarrow 0,$$

as the boundary homomorphism in the localization sequence (1) is surjective in this case by a theorem of Swan [CR81, Theorem 32.1]. As $\mathbb{Z}_p G^\times$ surjects onto $K_1(\mathbb{Z}_p G)$ by [CR81, Theorem 40.31], we obtain the local Hom-description

$$K_0 T(\mathbb{Z}_p G) \simeq \text{Hom}_\Gamma(R_p(G), E^\times) / \text{Det}(\mathbb{Z}_p G^\times).$$

When $f \in \text{Hom}_\Gamma(R_p(G), E^\times)$ corresponds to $T \in K_0 T(\mathbb{Z}_p G)$ under this isomorphism, then we say that f is a *representing homomorphism* for T . Similarly, by [CR87, Theorem 45.3] the reduced norm induces isomorphisms

$$\begin{aligned} \text{nr} : K_1(\mathbb{Q}_p G) &\xrightarrow{\simeq} \zeta(\mathbb{Q}_p G)^\times \\ K_0 T(\mathbb{Z}_p G) &\simeq \zeta(\mathbb{Q}_p G)^\times / \text{nr}(\mathbb{Z}_p G^\times). \end{aligned}$$

We therefore have a commutative triangle

$$\begin{array}{ccc} & K_1(\mathbb{Q}_p G) & \\ \text{nr} \swarrow & & \searrow \text{Det} \\ \zeta(\mathbb{Q}_p G)^\times & \xrightarrow{f} & \text{Hom}_\Gamma(R_p(G), E^\times) \end{array} \quad (3)$$

in which each map is an isomorphism. We now describe the isomorphism f in more detail. For this let $z \in \zeta(\mathbb{Q}_p G)^\times$. Let $\text{Irr}_p(G)$ be the set of absolutely irreducible \mathbb{C}_p -valued characters of G . For $\chi \in \text{Irr}_p(G)$ we write $e_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$ for the associated central idempotent in $\mathbb{C}_p G$. Note that e_χ actually belongs to EG and each generates one of the minimal ideals of $\zeta(EG)$; hence

$$\zeta(EG) = \bigoplus_{\chi \in \text{Irr}_p(G)} Ee_\chi.$$

Consider z as an element in $\zeta(EG)^\times$ and write $z = \sum_\chi z_\chi e_\chi$ with each $z_\chi \in E^\times$. Then f maps z to the unique homomorphism $f_z \in \text{Hom}_\Gamma(R_p(G), E^\times)$ which on irreducible characters is given by

$$f_z : \chi \mapsto z_\chi, \quad \chi \in \text{Irr}_p(G). \quad (4)$$

1.2 Equivariant L -values

Let us fix a finite Galois extension L/K of number fields with Galois group G . For every prime \mathfrak{p} of K we fix a prime \mathfrak{P} of L above \mathfrak{p} and write $G_{\mathfrak{P}}$ and $I_{\mathfrak{P}}$ for the decomposition group and inertia subgroup of L/K at \mathfrak{P} , respectively. Moreover, we denote the residual group at \mathfrak{P} by $\overline{G}_{\mathfrak{P}} = G_{\mathfrak{P}}/I_{\mathfrak{P}}$ and choose a lift $\phi_{\mathfrak{P}} \in G_{\mathfrak{P}}$ of the Frobenius automorphism at \mathfrak{P} .

1.2.1 Complex L -series

If S is a finite set of places of K containing the set S_∞ of all infinite places of K , and χ is a (complex) character of G , we denote the S -truncated Artin L -function attached to χ and S by $L_S(s, \chi)$ and define $L_S^*(r, \chi)$ to be the leading coefficient of the Taylor expansion of $L_S(s, \chi)$ at $s = r$, $r \in \mathbb{Z}_{\leq 0}$. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}G) = \prod_{\chi \in \text{Irr}(G)} \mathbb{C}$, where $\text{Irr}(G)$ denotes the set of complex valued irreducible characters of G . We define the equivariant Artin L -function to be the meromorphic $\zeta(\mathbb{C}G)$ -valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \text{Irr}(G)}.$$

We put $L_S^*(r) = (L_S^*(r, \chi))_{\chi \in \text{Irr}(G)}$ and abbreviate $L_{S_\infty}(s)$ by $L(s)$. If T is a second finite set of places of K such that $S \cap T = \emptyset$, we define $\delta_T(s) := (\delta_T(s, \chi))_{\chi \in \text{Irr}(G)}$, where $\delta_T(s, \chi) = \prod_{\mathfrak{p} \in T} \det(1 - N(\mathfrak{p})^{1-s} \phi_{\mathfrak{P}}^{-1} |V_\chi^{I_{\mathfrak{P}}})$ and V_χ is a G -module with character χ . We put

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\sharp,$$

where we denote by $^\sharp : \mathbb{C}G \rightarrow \mathbb{C}G$ the anti-involution induced by $g \mapsto g^{-1}$. These functions are the so-called (S, T) -modified G -equivariant L -functions, and for every integer $r \leq 0$ we define Stickelberger elements

$$\theta_S^T(r) := \Theta_{S,T}(r) \in \zeta(\mathbb{C}G).$$

For convenience, we also put $L_S^T(s, \chi) := \delta_T(s, \check{\chi}) \cdot L_S(s, \chi)$, where we write $\check{\chi}$ for the character contragredient to χ . Thus

$$\theta_S^T(r)^\sharp = (L_S^T(r, \chi))_{\chi \in \text{Irr}(G)}.$$

We will also write $L_S^T(L/K, s, \chi)$ for $L_S^T(s, \chi)$ if the extension L/K is not clear from the context, and similarly for $\theta_S^T(r)$. If T is empty, we abbreviate $\theta_S^T(r)$ by $\theta_S(r)$. Now a result of Siegel [Si70] implies that

$$\theta_S^T(r) \in \zeta(\mathbb{Q}G) \quad (5)$$

for all $r \leq 0$. Let us fix an embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$; then the image of θ_S^T in $\zeta(\mathbb{Q}_p G)$ via the canonical embedding

$$\zeta(\mathbb{Q}G) \hookrightarrow \zeta(\mathbb{Q}_p G) = \bigoplus_{\chi \in \text{Irr}_p(G)/\sim} \mathbb{Q}_p(\chi),$$

is given by $\sum_{\chi} L_S^T(r, \check{\chi}^{\iota^{-1}})^{\iota}$. Here, the sum runs over all \mathbb{C}_p -valued irreducible characters of G modulo Galois action. Note that we will frequently drop ι and ι^{-1} from the notation. Finally, for an irreducible character χ with values in either \mathbb{C} or \mathbb{C}_p we put $e_{\chi} = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$ which is a central idempotent in either $\mathbb{C}G$ or $\mathbb{C}_p G$.

1.2.2 p -adic L -series

Now let L/K be a Galois CM-extension, i.e. L is a CM-field, K is totally real and complex conjugation induces a unique automorphism j of L which lies in the center of G . Recall that a character χ of G is called *even* when $\chi(j) = \chi(1)$, and it is called *odd* when $\chi(j) = -\chi(1)$. Fix an odd prime p and suppose that S also contains all p -adic places of K . Let χ be an even character of G and denote the S -truncated p -adic Artin L -series of χ by $L_{p,S}(s, \chi)$. Then for every integer $r \geq 2$ one has

$$L_{p,S}(1-r, \chi) = L_S(1-r, \chi\omega^{-r}), \quad (6)$$

where ω denotes the Teichmüller character. When χ is a linear character, then the interpolation property (6) for every $r \geq 1$ follows from the work of Deligne and Ribet [DR80]. The general case for $r \geq 2$ is then established by using Serre's variant of Brauer induction (see [Ta84, Chapitre III, §1]). In the case $r = 1$, however, this argument fails due to the potential presence of trivial zeros of the p -adic L -series of χ at zero. One nevertheless expects that the identity

$$L_{p,S}(0, \chi) = L_S(0, \chi\omega^{-1}) \quad (7)$$

holds in general. As both sides behave well under direct sum, inflation and induction of characters, we see that (7) at least holds when χ is a monomial character (see the discussion in [Gro81, §2]).

Note that the identity (7) is implicitly assumed to hold in [Ni11b] and [Ni13, §4]. This will not affect our main Theorem 0.3, as we will first reduce to monomial Galois groups. However, such a reduction step is not possible for the Brumer-Stark conjecture. As a consequence we have to restrict to monomial extensions in Theorem 0.7.

1.3 Ray class groups

Let T and S be finite sets of places as in §1.2.1; so S contains all infinite places and $S \cap T = \emptyset$. We write cl_L^T for the ray class group of L to the ray $\mathfrak{M}_T := \prod_{\mathfrak{p} \in T(L)} \mathfrak{P}$ and \mathcal{O}_S for the ring of $S(L)$ -integers of L . We denote the $S(L)$ -units of L by E_S and define $E_S^T := \{x \in E_S : x \equiv 1 \pmod{\mathfrak{M}_T}\}$. If $S = S_{\infty}$, we also write E_L^T for $E_{S_{\infty}}^T$. All these modules

are equipped with a natural G -action. Now suppose that L/K is a Galois CM-extension and let $j \in G$ denote complex conjugation. If R is a subring of either \mathbb{C} or \mathbb{C}_p for a prime p such that 2 is invertible in R , we put $RG_+ := RG/(1-j)$ and $RG_- := RG/(1+j)$ which are rings, since the idempotents $(1 \pm j)/2$ lie in RG . For any RG -module M we define $M^+ = RG_+ \otimes_{RG} M$ and $M^- = RG_- \otimes_{RG} M$ which are exact functors since $2 \in R^\times$. We define

$$A_L^T := (\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \text{cl}_L^T)^-.$$

If M is a finitely generated \mathbb{Z} -module and p is a prime, we put $M(p) := \mathbb{Z}_p \otimes_{\mathbb{Z}} M$. For odd primes p , we will in particular consider $A_L^T(p)$, the minus p -part of the ray class group cl_L^T .

1.4 Noncommutative Fitting invariants

For the following we refer the reader to [Ni10] and [JN13]. We denote the set of all $m \times n$ matrices with entries in a ring R by $M_{m \times n}(R)$ and in the case $m = n$ the group of all invertible elements of $M_{n \times n}(R)$ by $\text{Gl}_n(R)$.

1.4.1 $\text{nr}(\Lambda)$ -equivalence

Let A be a separable K -algebra and Λ be an \mathfrak{o} -order in A , finitely generated as \mathfrak{o} -module, where \mathfrak{o} is an integrally closed complete commutative noetherian local domain with field of quotients K . The group ring $\mathbb{Z}_p G$ of a finite group G will serve as a standard example. Let N and M be two $\zeta(\Lambda)$ -submodules of an \mathfrak{o} -torsionfree $\zeta(\Lambda)$ -module. Then N and M are called $\text{nr}(\Lambda)$ -equivalent if there exists an integer n and a matrix $U \in \text{Gl}_n(\Lambda)$ such that $N = \text{nr}(U) \cdot M$, where $\text{nr} : A \rightarrow \zeta(A)$ denotes the reduced norm map which extends to matrix rings over A in the obvious way. We denote the corresponding equivalence class by $[N]_{\text{nr}(\Lambda)}$. We say that N is $\text{nr}(\Lambda)$ -contained in M (and write $[N]_{\text{nr}(\Lambda)} \subseteq [M]_{\text{nr}(\Lambda)}$) if for all $N' \in [N]_{\text{nr}(\Lambda)}$ there exists $M' \in [M]_{\text{nr}(\Lambda)}$ such that $N' \subseteq M'$. Note that it suffices to check this property for one $N_0 \in [N]_{\text{nr}(\Lambda)}$. We will say that x is contained in $[N]_{\text{nr}(\Lambda)}$ (and write $x \in [N]_{\text{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\text{nr}(\Lambda)}$ such that $x \in N_0$.

1.4.2 Noncommutative Fitting invariants

Now let M be a finitely presented (left) Λ -module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \rightarrow M \tag{8}$$

be a finite presentation of M . We identify the homomorphism h with the corresponding matrix in $M_{a \times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geq b$. In the case $a = b$ we call (8) a *quadratic presentation*. The Fitting invariant of h over Λ is defined to be

$$\text{Fitt}_\Lambda(h) = \begin{cases} [0]_{\text{nr}(\Lambda)} & \text{if } a < b \\ [\langle \text{nr}(H) | H \in S(h) \rangle_{\zeta(\Lambda)}]_{\text{nr}(\Lambda)} & \text{if } a \geq b. \end{cases}$$

We call $\text{Fitt}_\Lambda(h)$ a *Fitting invariant* of M over Λ . One defines $\text{Fitt}_\Lambda^{\max}(M)$ to be the unique Fitting invariant of M over Λ which is maximal among all Fitting invariants of M with

respect to the partial order “ \subseteq ”. If M admits a quadratic presentation h , one also puts $\text{Fitt}_\Lambda(M) := \text{Fitt}_\Lambda(h)$ which is independent of the chosen quadratic presentation.

Remark 1.1. *Suppose that $\Lambda = \mathbb{Z}_p G$ for a finite group G and that M is a finite $\mathbb{Z}_p G$ -module with projective dimension at most 1. Then M admits a quadratic presentation h which must be injective. The corresponding matrix H then belongs to $\text{Gl}_a(\mathbb{Q}_p G)$ and $z := \text{nr}(H)$ is a generator of the Fitting invariant $\text{Fitt}_{\mathbb{Z}_p G}(M)$. Using the Hom-description of §1.1.3, the commutative triangle (3) shows that the homomorphism f_z defined in (4) is a representing homomorphism of the class of M in $K_0 T(\mathbb{Z}_p G)$. Conversely, if f is a representing homomorphism for the class of M , then there is $z \in \zeta(\mathbb{Q}_p G)^\times$ such that $f = f_z$, and z generates the Fitting invariant of M .*

1.4.3 Denominator ideals and the integrality ring

Assume now that \mathfrak{o} is an integrally closed commutative noetherian domain, but not necessarily complete or local. We denote by $\mathcal{I}(\Lambda)$ the $\zeta(\Lambda)$ -submodule of $\zeta(A)$ generated by the elements $\text{nr}(H)$, $H \in M_{b \times b}(\Lambda)$, $b \in \mathbb{N}$. As the reduced norm is multiplicative, we see that $\mathcal{I}(\Lambda)$ is in fact a commutative ring which we call the *integrality ring* of Λ .

We may decompose the separable K -algebra A into its simple components

$$A = A_1 \oplus \cdots \oplus A_t,$$

i.e. each A_i is a simple K -algebra and $A_i = Ae_i = e_i A$ with central primitive idempotents e_i , $1 \leq i \leq t$. Each A_i is isomorphic to an algebra of $n_i \times n_i$ matrices over a skewfield D_i and $K_i = \zeta(D_i)$ is a finite field extension of K . We denote the Schur index of D_i by s_i such that $[D_i : K_i] = s_i^2$. We choose a maximal order Λ' containing Λ . Then also Λ' decomposes into $\Lambda' = \bigoplus_{i=1}^t \Lambda'_i$, where $\Lambda'_i = \Lambda' e_i$. Now let $H \in M_{b \times b}(\Lambda)$ and write $H = \sum_{i=1}^t H_i$, where each H_i is a $b \times b$ matrix with entries in Λ'_i . Let $m_i = n_i \cdot s_i \cdot b$ and let $f_i(X) = \sum_{j=0}^{m_i} \alpha_{ij} X^j$ be the reduced characteristic polynomial of H_i . We put

$$H_i^* := (-1)^{m_i+1} \cdot \sum_{j=1}^{m_i} \alpha_{ij} H_i^{j-1}, \quad H^* := \sum_{i=1}^t H_i^*.$$

Then by [JN13, Lemma 3.4], we have $H^* \in M_{b \times b}(\Lambda')$ and $H^* H = H H^* = \text{nr}(H) \cdot \mathbf{1}_{b \times b}$; note that the condition on \mathfrak{o} to be a complete local ring is not necessary for this result. Moreover, we point out that this definition follows [JN13], but slightly differs from the corresponding notion in [Ni10]. If $\tilde{H} \in M_{b \times b}(\Lambda)$ is a second matrix, then $(H \tilde{H})^* = \tilde{H}^* H^*$. We define

$$\mathcal{H}(\Lambda) := \{x \in \zeta(\Lambda) \mid x H^* \in M_{b \times b}(\Lambda) \forall b \in \mathbb{N} \forall H \in M_{b \times b}(\Lambda)\}.$$

Since $x \cdot \text{nr}(H) = x H^* H$, we have in particular

$$\mathcal{H}(\Lambda) \cdot \mathcal{I}(\Lambda) = \mathcal{H}(\Lambda) \subseteq \zeta(\Lambda) \tag{9}$$

and we call $\mathcal{H}(\Lambda)$ the *denominator ideal* of Λ . For convenience, we put $\mathcal{H}_p(G) := \mathcal{H}(\mathbb{Z}_p G)$ and $\mathcal{H}(G) := \mathcal{H}(\mathbb{Z} G)$, and similarly $\mathcal{I}_p(G) := \mathcal{I}(\mathbb{Z}_p G)$ and $\mathcal{I}(G) := \mathcal{I}(\mathbb{Z} G)$. The importance of the denominator ideal $\mathcal{H}(\Lambda)$ will become clear by means of the following result which is [JN13, Theorem 3.6] (see also [Ni10, Theorem 4.2]).

Theorem 1.2. *If \mathfrak{o} is an integrally closed complete commutative noetherian local domain and M is a finitely presented Λ -module, then*

$$\mathcal{H}(\Lambda) \cdot \text{Fitt}_\Lambda^{\max}(M) \subseteq \text{Ann}_\Lambda(M).$$

We will need the following lemma whose last claim is [Ni11b, Lemma 6.6].

Lemma 1.3. *Let G and C be finite groups with C abelian. Let p be a prime and choose a maximal order $\mathcal{M}_p(G)$ in $\mathbb{Q}_p G$ which contains $\mathbb{Z}_p G$. Then we have inclusions*

$$\mathcal{I}_p(G)[C] \subseteq \mathcal{I}_p(G \times C) \subseteq \zeta(\mathcal{M}_p(G))[C].$$

In particular we have $|G| \cdot \mathcal{I}_p(G \times C) \subseteq \zeta(\mathbb{Z}_p[G \times C])$ for all primes p .

Proof. The ring $\mathcal{I}_p(G)[C]$ (resp. $\mathcal{I}_p(G \times C)$) is generated over $\zeta(\mathbb{Z}_p[G])[C] = \zeta(\mathbb{Z}_p[G \times C])$ by the elements $\text{nr}(H)$, where H runs through $M_{n \times n}(\mathbb{Z}_p G)$ (resp. $M_{n \times n}(\mathbb{Z}_p[G \times C])$), $n \in \mathbb{N}$. So we have $\mathcal{I}_p(G)[C] \subseteq \mathcal{I}_p(G \times C)$. The proof of the second inclusion is essentially the same as that of [Ni11b, Lemma 6.6]; we include it here for convenience. Up to conjugation the maximal order $\mathcal{M}_p(G)$ is a direct sum of matrix rings of type $M_{n \times n}(\mathcal{O}_D)$, where \mathcal{O}_D denotes the valuation ring of a skew field D . Note that conjugation does neither change the center of the order nor the image of the reduced norm. We have

$$\zeta(M_{n \times n}(\mathcal{O}_D)) = \zeta(\mathcal{O}_D) = \mathcal{O}_F,$$

where \mathcal{O}_F is the ring of integers of the field $F = \zeta(D)$ which is finite over \mathbb{Q}_p . Since the reduced norm maps $\mathcal{M}_p(G)$ into its center and $|G| \cdot \zeta(\mathcal{M}_p(G)) \subseteq \zeta(\mathbb{Z}_p G)$, it suffices to show that the reduced norm maps $M_{m \times m}(M_{n \times n}(\mathcal{O}_D)[C])$ into $\mathcal{O}_F[C]$. Let us first assume that $D = F$. Then the map

$$\begin{aligned} \sigma : M_{n \times n}(F)[C] &\longrightarrow M_{n \times n}(F[C]) \\ \sum_{c \in C} M_c c &\longmapsto \left(\sum_{c \in C} \alpha_{ij}(c) c \right)_{i,j} \end{aligned}$$

is an isomorphism of rings, where $M_c = (\alpha_{ij}(c))_{i,j}$ lies in $M_{n \times n}(F)$. Likewise, σ induces an isomorphism

$$\sigma : M_{n \times n}(\mathcal{O}_F)[C] \simeq M_{n \times n}(\mathcal{O}_F[C]).$$

Therefore, we have

$$\text{nr}(M_{m \times m}(M_{n \times n}(\mathcal{O}_F)[C])) = \text{nr}(M_{nm \times nm}(\mathcal{O}_F[C])) = \text{nr}(\mathcal{O}_F[C]) = \mathcal{O}_F[C].$$

For arbitrary D , there is a field E , Galois over F such that $E \otimes_F D \simeq M_{s \times s}(E)$ for some integer s . We have just proven that the reduced norm maps $M_{m \times m}(M_{n \times n}(\mathcal{O}_D)[C])$ into $\mathcal{O}_E[C]$. However, the image is invariant under the action of $\text{Gal}(E/F)$ and is therefore contained in $\mathcal{O}_F[C]$. \square

Corollary 1.4. *Let p be a prime and let G and C be finite groups with C abelian. If $\mathcal{I}_p(G) = \zeta(\mathcal{M}_p(G))$, then $\mathcal{I}_p(G \times C) = \mathcal{I}_p(G)[C] = \zeta(\mathcal{M}_p(G))[C]$.*

The following result determines all primes p for which the denominator ideal $\mathcal{H}_p(G)$ is best possible.

Proposition 1.5. *We have $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_p G)$ if and only if p does not divide the order of the commutator subgroup G' of G . Furthermore, when this is the case we have $\mathcal{I}_p(G) = \zeta(\mathbb{Z}_p G)$.*

Proof. The first assertion is a special case of [JN13, Proposition 4.8]. The second assertion then follows from (9). \square

2 The integrality conjectures

Let L/K be a Galois extension with Galois group G . Let S and T be two finite sets of places of K such that

- (i) S contains all the infinite places of K and all the places which ramify in L/K , i.e. $S \supseteq S_{\text{ram}} \cup S_{\infty}$.
- (ii) $S \cap T = \emptyset$.
- (iii) E_S^T is torsionfree.

We refer to the above hypotheses as $\text{Hyp}(S, T)$. For a fixed set S we define \mathfrak{A}_S to be the $\zeta(\mathbb{Z}G)$ -submodule of $\zeta(\mathbb{Q}G)$ generated by the elements $\delta_T(0)$, where T runs through the finite sets of places of K such that $\text{Hyp}(S, T)$ is satisfied. Note that \mathfrak{A}_S equals the $\mathbb{Z}G$ -annihilator of the roots of unity of L if G is abelian by [Ta84, Lemma 1.1, p. 82].

2.1 The Sinnott-Kurihara ideal

For any finite group H we put $N_H := \sum_{h \in H} h$. For a finite prime \mathfrak{p} of K , we define a $\mathbb{Z}G_{\mathfrak{p}}$ -module $U_{\mathfrak{p}}$ by

$$U_{\mathfrak{p}} := \langle N_{I_{\mathfrak{p}}}, 1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1} \rangle_{\mathbb{Z}G_{\mathfrak{p}}} \subset \mathbb{Q}G_{\mathfrak{p}},$$

where $\varepsilon_{\mathfrak{p}} = |I_{\mathfrak{p}}|^{-1} N_{I_{\mathfrak{p}}}$. Note that $U_{\mathfrak{p}} = \mathbb{Z}G_{\mathfrak{p}}$ if \mathfrak{p} is unramified in L/K such that the definition of the following $\mathcal{I}(G)$ -module is indeed independent of the set S as long as S contains the ramified primes:

$$U := \left\langle \prod_{\mathfrak{p} \in S \setminus S_{\infty}} \text{nr}(u_{\mathfrak{p}}) | u_{\mathfrak{p}} \in U_{\mathfrak{p}} \right\rangle_{\mathcal{I}(G)} \subset \zeta(\mathbb{Q}G).$$

Definition 2.1. *Let S be a finite set of primes which contains $S_{\text{ram}} \cup S_{\infty}$. We define an $\mathcal{I}(G)$ -module by*

$$SKu(L/K, S) := \mathfrak{A}_S \cdot U \cdot L(0)^{\sharp} \subset \zeta(\mathbb{Q}G).$$

We call $SKu(L/K) := SKu(L/K, S_{\text{ram}} \cup S_{\infty})$ the (fractional) Sinnott-Kurihara ideal.

For abelian G , this definition coincides with the Sinnott-Kurihara ideal $SKu(L/K)$ in [Gr07] (see also [Si80, p. 193]) and is closely related to the Fitting ideal of the Pontryagin dual of the class group (see [Gr07, Theorem 8.8]).

Note that our definition slightly differs from the original definition in [Ni11b], where in the definition of U the integrality ring $\mathcal{I}(G)$ is replaced with $\zeta(\mathbb{Z}G)$. However, as observed by the reviewer, it is then not clear whether the definition of U does not depend on S . We assure the reader that this redefinition does not affect any of the results in [Ni11b].

The integrality conjecture as formulated in [Ni11b] (where L/K is assumed to be a CM-extension; but we will not assume this here) now asserts the following:

Conjecture 2.2. *The Sinnott-Kurihara ideal $SKu(L/K)$ is contained in $\mathcal{I}(G)$.*

Remark 2.3. (i) *Since clearly $SKu(L/K, S) \subseteq SKu(L/K, S')$ whenever $S' \subseteq S$, Conjecture 2.2 implies $SKu(L/K, S) \subseteq \mathcal{I}(G)$ for all admissible sets S .*

(ii) *If the sets S and T satisfy $Hyp(S, T)$, the Stickelberger element $\theta_S^T(0)$ is contained in $SKu(L/K, S)$. Hence Conjecture 2.2 predicts that $\theta_S^T(0) \in \mathcal{I}(G)$ which is part of [Ni11d, Conjecture 2.1].*

(iii) *In the above definitions, we may replace \mathbb{Z} and \mathbb{Q} by \mathbb{Z}_p and \mathbb{Q}_p , respectively. We obtain a local Sinnott-Kurihara ideal $SKu_p(L/K)$ contained in $\zeta(\mathbb{Q}_p G)$. Since we have an equality*

$$\mathcal{I}(G) = \bigcap_p (\mathcal{I}_p(G) \cap \zeta(\mathbb{Q}G)),$$

we have an equivalence

$$SKu(L/K) \subseteq \mathcal{I}(G) \iff SKu_p(L/K) \subseteq \mathcal{I}_p(G) \forall p.$$

If G is abelian, we obviously have $\mathcal{I}(G) = \zeta(\mathbb{Z}G) = \mathbb{Z}G$ and the results in [Ba77], [Ca79], [DR80] each imply the following theorem (cf. [Gr07, §2]).

Theorem 2.4. *Conjecture 2.2 holds if L/K is an abelian extension.*

2.2 The modified Sinnott-Kurihara ideal

We also define a modified version of the Sinnott-Kurihara ideal as follows. For a finite prime \mathfrak{p} of K , define an $\mathcal{I}(G)$ -module $U'_\mathfrak{p}$ by

$$U'_\mathfrak{p} := \langle \text{nr}(N_{I_\mathfrak{p}}), \text{nr}(1 - \varepsilon_\mathfrak{p} \phi_\mathfrak{p}^{-1}) \rangle_{\mathcal{I}(G)} \subseteq \zeta(\mathbb{Q}G).$$

If S contains $S_{\text{ram}} \cup S_\infty$, we define

$$U' := \prod_{\mathfrak{p} \in S \setminus S_\infty} U'_\mathfrak{p},$$

$$SKu'(L/K, S) := \mathfrak{A}_S \cdot U' \cdot L(0)^\# \subseteq SKu(L/K, S).$$

As before, the definition of U' does not depend on S and all the above remarks remain true if we replace $SKu(L/K, S)$ by $SKu'(L/K, S)$ throughout. We put $SKu'(L/K) := SKu'(L/K, S_{\text{ram}} \cup S_\infty)$. If G is abelian, the reduced norm is just the identity on $\mathbb{Q}G$. Moreover, \mathfrak{A}_S is the whole $\mathbb{Z}G$ -annihilator of μ_L , the roots of unity in L , and hence independent of S . This implies the following proposition.

Proposition 2.5. *If L/K is an abelian extension, then*

$$SKu(L/K) = SKu(L/K, S) = SKu'(L/K, S) \subseteq \mathbb{Z}G$$

for all admissible sets S .

2.3 Negative integers

We now discuss a (partial) analogue of Conjecture 2.2 in the case, where $r < 0$ is a negative integer. We denote the absolute Galois group of L by G_L and put $\mu_{1-r}(L) := (\mathbb{Q}/\mathbb{Z})(1-r)^{G_L}$, where $(\mathbb{Q}/\mathbb{Z})(1-r)$ denotes the usual $(1-r)$ -fold Tate twist of \mathbb{Q}/\mathbb{Z} .

Conjecture 2.6. *Let L/K be a Galois extension of number fields with Galois group G and let $r < 0$ be an integer. Then for every $x \in \text{Ann}_{\mathbb{Z}G}(\mu_{1-r}(L))$ one has*

$$\text{nr}(x) \cdot \theta_S(r) \in \mathcal{I}(G)$$

for all finite sets S of primes of K containing $S_{\text{ram}} \cup S_{\infty}$.

Remark 2.7. (i) *If $S \subseteq S'$, then we have an equality*

$$\theta_{S'}(r) = \left(\prod_{\mathfrak{p} \in S' \setminus S} \text{nr}(1 - N(\mathfrak{p})^{-r} \phi_{\mathfrak{p}}^{-1}) \right) \theta_S(r).$$

As $\text{nr}(1 - N(\mathfrak{p})^{-r} \phi_{\mathfrak{p}}^{-1})$ lies in $\mathcal{I}(G)$, it suffices to consider the set $S = S_{\text{ram}} \cup S_{\infty}$ in Conjecture 2.6.

- (ii) *Since the $\mathbb{Z}G$ -annihilator of $\mu_{1-r}(L)$ is generated by the elements $\prod_{\mathfrak{p} \in T} (1 - \phi_{\mathfrak{p}} N(\mathfrak{p})^{1-r})$, where T runs through all finite sets of primes in K such that $\text{Hyp}(S, T)$ is satisfied (cf. [Co77]), Conjecture 2.6 in particular implies that $\theta_S^T(r) \in \mathcal{I}(G)$ for all finite sets of primes S and T such that $\text{Hyp}(S, T)$ holds.*
- (iii) *Note that Conjecture 2.6 outside its 2-primary part implicitly is a part of [Ni11c, Conjecture 2.11] if either L/K is a CM-extension and r is even or L/K is an extension of totally real fields and r is odd.*

Again, the results in [Ba77], [Ca79], [DR80] each imply the following theorem.

Theorem 2.8. *Conjecture 2.6 holds if L/K is an abelian extension.*

3 A reduction step

In order to prove one of the conjectures of the preceding paragraph, we may henceforth assume that the field K is totally real, as otherwise $\theta_{S_{\infty}}(r) = L(r)^{\sharp} = 0$; hence also $SKu(L/K) = 0$ and $\theta_S(r) = 0$ for all finite sets S containing S_{∞} . By the same reason, we may assume that L is totally complex if r is even. Note that we actually have to exclude the special case, where $r = 0$ and L/\mathbb{Q} is a CM-extension of degree 2. In this case, however, the occurring Galois group is abelian and everything is known by Theorem 2.4.

Let us denote the set of complex places of L by $S_{\mathbb{C}}(L)$. For every $w \in S_{\mathbb{C}}(L)$, the decomposition group G_w is cyclic of order two and we denote its generator by j_w . If r is even we define

$$H = H(r) := \langle j_w \cdot j_{w'} \mid w, w' \in S_{\mathbb{C}}(L) \rangle.$$

If r is odd, we define

$$H = H(r) := \langle j_w \mid w \in S_{\mathbb{C}}(L) \rangle.$$

In both cases, H is normal in G such that the fixed field L^H is a Galois extension of K with Galois group $\overline{G} := G/H$. Note that L^H/K is either a Galois CM-extension or a Galois extension of totally real fields.

Proposition 3.1. *Let L/K be a Galois extension of number fields with Galois group G and let p be an odd prime. Assume that L is totally imaginary if we consider Conjecture 2.2 or Conjecture 2.6 for even r . Assume further that G has a unique 2-Sylow subgroup. Then the p -part of Conjecture 2.2 (resp. Conjecture 2.6) is true for L/K if and only if the p -part of Conjecture 2.2 (resp. Conjecture 2.6) is true for L^H/K .*

Proof. Since H is normal in G , the group ring element $\varepsilon_H := |H|^{-1}N_H$ is a central idempotent in $\mathbb{Q}_p G$. Let G_2 be the unique 2-Sylow subgroup of G . Then j_w lies in G_2 for every $w \in S_{\mathbb{C}}(L)$ such that H is a finite 2-group. Since p is odd, this implies that ε_H actually lies in $\mathbb{Z}_p G$. Now $\mathbb{Z}_p G$ decomposes into $\mathbb{Z}_p G = \varepsilon_H \mathbb{Z}_p G \oplus (1 - \varepsilon_H) \mathbb{Z}_p G$ and the canonical epimorphism $\pi : \mathbb{Z}_p G \rightarrow \mathbb{Z}_p \overline{G}$ induces an isomorphism $\pi : \varepsilon_H \mathbb{Z}_p G \simeq \mathbb{Z}_p \overline{G}$. However, by the definition of H , we have $L_{S_{\infty}}(r, \chi) = 0$ if H is not contained in the kernel of the irreducible character χ . When $r = 0$ this means that $SKu(L/K) = \varepsilon_H \cdot SKu(L/K)$ which identifies with $SKu(L^H/K)$ via π . Using $\mu_{1-r}(L)^H = \mu_{1-r}(L^H)$, a similar observation holds in the case $r < 0$. \square

Remark 3.2. *Note that Proposition 3.1 in particular applies when G is nilpotent.*

4 Integrality of Stickelberger elements

The aim of this section is to prove Theorem 0.1 and Theorem 0.2.

4.1 Admissible sets of places

If p is a prime, we denote by S_p the set of p -adic places of K . We now introduce the following terminology.

Definition 4.1. *If $r = 0$, we will say that S and T are $(p, 0)$ -admissible if the following conditions are satisfied:*

- (i) *The union of S and T contains all non- p -adic ramified primes, i.e. $S_{\text{ram}} \setminus (S_{\text{ram}} \cap S_p) \subseteq S \cup T$,*
- (ii) *S contains all wildly ramified primes in S_p ,*
- (iii) *S contains the set S_{∞} of all archimedean primes,*
- (iv) *$S \cap T = \emptyset$,*
- (v) *$E_S^{T_{\text{nr}}}$ is torsionfree, where T_{nr} denotes the set of all unramified primes in T .*

If $r < 0$, we will say that S and T are (p, r) -admissible if $\text{Hyp}(S, T)$ is satisfied.

Note that S and T are in fact (p, r) -admissible for all primes p and all $r \leq 0$ if $\text{Hyp}(S, T)$ is satisfied.

4.2 Integrality of Stickelberger elements

Recall that a finite group G is called *monomial* if every irreducible character of G is induced by a linear character. Examples of monomial groups are nilpotent groups [CR81, Theorem 11.3] and, more generally, supersolvable groups [We82, Chapter 2, Corollary 3.5]. For more information concerning monomial groups we refer the reader to [We82, Chapter 2].

Now assume that L/K is a finite Galois extension of number fields with Galois group G , where G decomposes as $G = H \times C$ with H monomial and C abelian. As in the introduction let $\mathcal{M}(H)$ (resp. $\mathcal{M}_p(H)$) be a maximal order in $\mathbb{Q}H$ (resp. \mathbb{Q}_pH) containing $\mathbb{Z}H$ (resp. \mathbb{Z}_pH). Then we may view $\mathcal{M}(H)[C]$ (resp. $\mathcal{M}_p(H)[C]$) as an order in $\mathbb{Q}G$ (resp. \mathbb{Q}_pG) and we have the following more general version of Theorem 0.2.

Theorem 4.2. *Let L/K be a finite Galois extension of number fields with Galois group $G = H \times C$, where H is monomial and C is abelian. Let p be a prime and $r \in \mathbb{Z}_{<0}$. If S and T are two finite sets of primes of K which are (p, r) -admissible, then*

$$\theta_S^T(r) \in \zeta(\mathcal{M}_p(H)[C]) = \zeta(\mathcal{M}_p(H))[C].$$

In particular, if $\text{Hyp}(S, T)$ is satisfied, we have

$$\theta_S^T(r) \in \zeta(\mathcal{M}(H)[C]) = \zeta(\mathcal{M}(H))[C].$$

Proof. We first assume that $G = C$ is abelian. Then $\zeta(\mathcal{M}(H))[C] = \mathbb{Z}G$ and the assertion follows easily from Theorem 2.4 if $r = 0$ and from Theorem 2.8 if $r < 0$ as long as $\text{Hyp}(S, T)$ is satisfied. We are left with the case, where $r = 0$ and S and T are $(p, 0)$ -admissible. We claim that $\theta_S^T(0)$ lies in $SKu_p(L/K)$ and hence Theorem 2.4 again implies the desired result. To see this, we write $\theta_S^T(0)$ as

$$\theta_S^T(0) = \delta_{T_{\text{nr}}}(0) \cdot \prod_{\mathfrak{p} \in T \setminus T_{\text{nr}}} \delta_{\mathfrak{p}}(0) \prod_{\mathfrak{p} \in S \setminus S_{\infty}} (1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1}) \cdot L(0)^{\sharp}.$$

The set T_{nr} satisfies $\text{Hyp}(T_{\text{nr}}, S_{\text{ram}} \cup S_{\infty})$ by condition (v) so that $\delta_{T_{\text{nr}}}(0)$ lies in $\mathfrak{A}_{S_{\text{ram}} \cup S_{\infty}}$. Let $\mathfrak{p} \in T \setminus T_{\text{nr}}$ and let $q \in \mathbb{Z}$ be the rational prime below \mathfrak{p} . By local class field theory [Se79, Chapter XV, §2], the local units at \mathfrak{p} surject under the reciprocity map onto $I_{\mathfrak{p}}$. The subgroup of principal units is mapped onto the q -Sylow subgroup of $I_{\mathfrak{p}}$. As the factor group of the local units modulo the principal units has order $N(\mathfrak{p}) - 1$, the ramification index $e_{\mathfrak{p}} := |I_{\mathfrak{p}}|$ divides $N(\mathfrak{p}) - 1$ if $q = p$, and still up to a power of q if $q \neq p$. Hence

$$\delta_{\mathfrak{p}}(0) = 1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1} N(\mathfrak{p}) = 1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1} - \phi_{\mathfrak{p}}^{-1} \frac{N(\mathfrak{p}) - 1}{e_{\mathfrak{p}}} N_{I_{\mathfrak{p}}} \in \mathbb{Z}_p \otimes U_{\mathfrak{p}}.$$

For the tamely ramified primes above p the element

$$e_{\mathfrak{p}} = (e_{\mathfrak{p}} - N_{I_{\mathfrak{p}}})(1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1}) + N_{I_{\mathfrak{p}}} \in U_{\mathfrak{p}}$$

lies in \mathbb{Z}_p^{\times} , since $p \nmid e_{\mathfrak{p}}$. Therefore, we get $\mathbb{Z}_p \otimes U_{\mathfrak{p}} = \mathbb{Z}_p G_{\mathfrak{p}}$ in this case. Finally, we obviously have $(1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1}) \in U_{\mathfrak{p}}$ for the primes $\mathfrak{p} \in S \setminus S_{\infty}$. Putting all this together we find that $\theta_S^T(0)$ belongs to $SKu_p(L/K)$ as desired.

We now treat the general case, where $G = H \times C$. Since we have to deal with Stickelberger elements corresponding to various subextensions of L/K , we will write $\theta_S^T(L/K, r)$ for $\theta_S^T(r)$ for clarity. Each irreducible character of G may be written as $\chi \cdot \lambda$, where $\chi \in \text{Irr}(H)$ and $\lambda \in \text{Irr}(C)$. We have the following decomposition

$$\zeta(\mathbb{Q}_p[H \times C]) = \bigoplus_{\chi \in \text{Irr}_p(H)/\sim} \mathbb{Q}_p(\chi)[C],$$

where the sum runs over all (\mathbb{C}_p -valued) irreducible characters of H modulo Galois action and $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(h) | h \in H)$. We fix an irreducible character χ of H . Then the image of $\theta_S^T(L/K, r)^\#$ in the χ -component of the above decomposition is given by

$$\sum_{\lambda \in \text{Irr}_p(C)} L_S^T(L/K, \chi \cdot \lambda, r) e_\lambda \in \mathbb{Q}_p(\chi)[C] \quad (10)$$

and we wish to show that it actually lies in $\mathbb{Z}_p(\chi)[C]$, where $\mathbb{Z}_p(\chi)$ denotes the ring of integers in $\mathbb{Q}_p(\chi)$. Since H is monomial, there is a subgroup U of H and a linear character ψ of U such that χ is induced by ψ , i.e. $\chi = \text{ind}_U^H \psi$. Let us denote the abelianization of U by U^{ab} ; as ψ is linear, it is inflated by a character ψ^{ab} of U^{ab} and hence $\chi = \text{ind}_U^H \text{infl}_{U^{\text{ab}}}^U \psi^{\text{ab}}$. Note that ψ^{ab} is a linear character and thus irreducible. Moreover, if λ is an irreducible character of C , we have

$$\chi \cdot \lambda = (\text{ind}_U^H \psi) \cdot \lambda = \text{ind}_{U \times C}^G (\psi \cdot \lambda) = \text{ind}_{U \times C}^G \text{infl}_{U^{\text{ab}} \times C}^{U \times C} (\psi^{\text{ab}} \cdot \lambda). \quad (11)$$

We assure the reader that the usual behavior of S -truncated Artin L -series under direct sum, induction and inflation of characters holds for our T -modified version just as well. Thus (11) implies that

$$\sum_{\lambda \in \text{Irr}_p(C)} L_S^T(L/K, \chi \cdot \lambda, r) e_\lambda = \sum_{\lambda \in \text{Irr}_p(C)} L_{S'}^{T'}(L^{[U,U]}/L^{U \times C}, \psi^{\text{ab}} \cdot \lambda, r) e_\lambda,$$

where $[U, U]$ denotes the commutator subgroup of U and $S' = S(L^{U \times C})$ and similarly for T' . However, the righthand side lies in $\mathbb{Z}_p(\psi)[C]$, since it is the ψ^{ab} -component of the Stickelberger element $\theta_{S'}^{T'}(L^{[U,U]}/L^{U \times C}, r)^\#$ attached to the *abelian* subextension $L^{[U,U]}/L^{U \times C}$. This and (10) imply that

$$\sum_{\lambda \in \text{Irr}_p(C)} L_S^T(L/K, \chi \cdot \lambda, r) e_\lambda \in \mathbb{Q}_p(\chi)[C] \cap \mathbb{Z}_p(\psi)[C] = \mathbb{Z}_p(\chi)[C]$$

as desired. In particular, if $\text{Hyp}(S, T)$ is satisfied, then the sets S and T are (p, r) -admissible for all primes p , and hence

$$\theta_S^T(r) \in \bigcap_p (\zeta(\mathcal{M}_p(H))[C] \cap \zeta(\mathbb{Q}H)[C]) = \zeta(\mathcal{M}(H))[C].$$

□

Remark 4.3. *In the case, where the abelian group C is trivial, Nomura [No] has shown that the conditions on the finite sets S and T can be further relaxed.*

Now let J be a subset of S_{ram} and put $S_J := (S_{\infty} \cup S_{\text{ram}}) \setminus J$. Let $K \subseteq L_J \subseteq L$ be the maximal subfield of L such that L_J/K is unramified outside S_J . Then L_J/K is a Galois extension with Galois group $G_J = G/H_J$, where $H_J = \text{Gal}(L/L_J)$. We have the following stronger version of Theorem 0.1.

Theorem 4.4. *Let L/K be a Galois extension of number fields with Galois group G and let J be a subset of S_{ram} . If the Galois group G_J of the subextension L_J/K is monomial, then*

$$\prod_{\mathfrak{p} \in J} \text{nr}(N_{I_{\mathfrak{p}}}) \cdot \theta_{S_J}^T(r) \in \zeta(\mathcal{M}(G)),$$

whenever $r \leq 0$ and $\text{Hyp}(S_{\text{ram}} \cup S_{\infty}, T)$ is satisfied. In particular,

$$SKu'(L/K) \subseteq \zeta(\mathcal{M}(G))$$

if G is monomial.

Proof. Since H_J is normal in G , the idempotent $|H_J|^{-1}N_{H_J}$ is central in $\mathbb{Q}G$ and lies in $\mathcal{M}(G)$. If χ is an irreducible character of G , the χ -component of $\prod_{\mathfrak{p} \in J} \text{nr}(N_{I_{\mathfrak{p}}})$ is zero if H_J is not contained in the kernel of χ . Hence we have an equality

$$\prod_{\mathfrak{p} \in J} \text{nr}(N_{I_{\mathfrak{p}}}) \cdot \theta_{S_J}^T(L/K, r) = \prod_{\mathfrak{p} \in J} \text{nr}(N_{I_{\mathfrak{p}}}) \cdot |H_J|^{-1}N_{H_J} \cdot \tilde{\theta}_{S_J}^T(L_J/K, r), \quad (12)$$

where $\tilde{\theta}_{S_J}^T(L_J/K, r)$ denotes any lift of $\theta_{S_J}^T(L_J/K, r)$ in $\zeta(\mathcal{M}(G))$; note that this is possible, since $\theta_{S_J}^T(L_J/K, r)$ lies in $\zeta(\mathcal{M}(G_J))$ by Theorem 4.2 as $\text{Hyp}(S_J, T)$ is satisfied for L_J/K . Hence the righthand side of the above equation also lies in $\zeta(\mathcal{M}(G))$. The second part of the theorem is clear by the definition of $SKu'(L/K)$ and the fact that each quotient of a monomial group is again monomial. \square

Corollary 4.5. *Let L/K be an abelian Galois extension of number fields with Galois group G and let J be a subset of S_{ram} . Then*

$$\prod_{\mathfrak{p} \in J} \text{nr}(N_{I_{\mathfrak{p}}}) \cdot \theta_{S_J}^T(r) \in \mathbb{Z}G,$$

whenever $r \leq 0$ and $\text{Hyp}(S_{\text{ram}} \cup S_{\infty}, T)$ is satisfied.

Proof. If G is abelian, the righthand side of equation (12) equals

$$\prod_{\mathfrak{p} \in J} N_{I_{\mathfrak{p}}} \cdot |H_J|^{-1}N_{H_J} \cdot \tilde{\theta}_{S_J}^T(L_J/K, r) = z \cdot N_{H_J} \cdot \tilde{\theta}_{S_J}^T(L_J/K, r),$$

where $z = |H_J|^{-1} \cdot \prod_{\mathfrak{p} \in J} |I_{\mathfrak{p}}|$ is an integer. The assertion follows, since $\theta_{S_J}^T(L_J/K, r)$ lies in $\mathbb{Z}G_J$ by Theorem 2.4 and Theorem 2.8. \square

Remark 4.6. *These results may tempt us to state a conjecture in complete analogy to Conjecture 2.2 also in the case $r < 0$. We have not done so, since the author is not aware of a convincing reason, why this should be true in general.*

4.3 An example: Frobenius groups

We recall the definition and some basic facts about Frobenius groups and then use them to provide many examples, where we can show that Stickelberger elements indeed lie in the integrality ring.

Definition 4.7. A **Frobenius group** is a finite group G with a proper nontrivial subgroup H such that $H \cap gHg^{-1} = \{1\}$ for all $g \in G - H$, in which case H is called a **Frobenius complement**.

Theorem 4.8. A Frobenius group G contains a unique normal subgroup N , known as the Frobenius kernel, such that G is a semidirect product $N \rtimes H$. Furthermore:

- (i) $|N|$ and $[G : N] = |H|$ are relatively prime.
- (ii) The Frobenius kernel N is nilpotent.
- (iii) If $\chi \in \text{Irr}(G)$ such that $N \not\leq \ker \chi$ then $\chi = \text{Ind}_N^G(\psi)$ for some $1 \neq \psi \in \text{Irr}(N)$.

Proof. For (i) and (iii) see [CR81, §14A]. For (ii) see [Ro96, 10.5.6]. \square

Corollary 4.9. Suppose that $G \simeq N \rtimes H$ is a Frobenius group with monomial Frobenius complement H . Then G is also monomial.

Proof. Let $\chi \in \text{Irr}(G)$. If $N \leq \ker \chi$ then χ is inflated from some $\varphi \in \text{Irr}(G/N)$. Otherwise $N \not\leq \ker \chi$ and so χ is induced from some $\psi \in \text{Irr}(N)$ by Theorem 4.8(iii). The Frobenius complement $H \simeq G/N$ is monomial by assumption, and the Frobenius kernel N is nilpotent by Theorem 4.8(ii) and thus is monomial. However, induction is transitive and inflation commutes with induction, so in both cases χ is induced from a linear character. \square

The following terminology has been introduced in [JN].

Definition 4.10. Let $\mathcal{M}_p(G)$ be a maximal \mathbb{Z}_p -order such that $\mathbb{Z}_p[G] \subseteq \mathcal{M}_p(G) \subseteq \mathbb{Q}_p[G]$ and let N be a normal subgroup of G . Define the **N -hybrid order** of $\mathbb{Z}_p[G]$ and $\mathcal{M}_p(G)$ to be $\mathcal{M}_p(G, N) = \mathbb{Z}_p[G]\varepsilon_N \oplus \mathcal{M}_p(G)(1 - \varepsilon_N)$. We say that $\mathbb{Z}_p[G]$ is **N -hybrid** if $\mathbb{Z}_p[G] = \mathcal{M}_p(G, N)$ for some choice of $\mathcal{M}_p(G)$.

Theorem 4.11. Let L/K be a finite Galois extension of number fields with $\text{Gal}(L/K) \simeq G \times A$, where $G \simeq N \rtimes H$ is a Frobenius group and A is abelian. Suppose that the Frobenius complement H is abelian. Then for every prime $p \nmid |N|$ and for every (p, r) -admissible sets S and T we have

$$\theta_S^T(r) \in \mathcal{I}_p(G \times A) = \zeta(\mathbb{Z}_p[G \times A]).$$

Proof. We first observe that $(G \times A)/N \simeq H \times A$ is abelian. Thus N contains the commutator subgroup G' of $G \times A$ and so $|G'|$ is not divisible by p . Hence $\mathcal{I}_p(G \times A) = \zeta(\mathbb{Z}_p[G \times A])$ by Proposition 1.5. As H is abelian, it is monomial and hence G is also monomial by Corollary 4.9. Thus Theorem 4.2 implies that

$$\theta_S^T(r) \in \zeta(\mathcal{M}_p(G))[A]. \tag{13}$$

The group ring $\mathbb{Z}_p[G]$ is N -hybrid by [JN, Proposition 2.13] and so there is a ring isomorphism

$$\zeta(\mathbb{Z}_p[G \times A]) \simeq \mathbb{Z}_p[G/N \times A] \oplus (1 - \varepsilon_N)\zeta(\mathcal{M}_p(G))[A]. \quad (14)$$

If we compare (13) and (14), we see that it suffices to show that $\theta_S^T(r)\varepsilon_N$ belongs to $\zeta(\mathbb{Z}_p[G \times A])\varepsilon_N \simeq \mathbb{Z}_p[G/N \times A]$. However, $G/N \times A$ is abelian and $\theta_S^T(r)\varepsilon_N$ naturally identifies with the corresponding Stickelberger element attached to the (abelian) subextension L^N/K . Hence the result now follows from Theorem 4.2 with $C = G/N \times A$ and $H = 1$. \square

For an odd prime l we denote by D_{2l} the dihedral group of order $2l$. If $q = l^n$ is a prime power, we let $\text{Aff}(q)$ be the group of affine transformations on \mathbb{F}_q , the finite field with q elements. Hence $\text{Aff}(q)$ is isomorphic to a semidirect product $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$ with the natural action. In particular, we have $\text{Aff}(3) \simeq D_6 \simeq S_3$, the symmetric group on three letters. Note that D_{2l} and $\text{Aff}(q)$ are Frobenius groups, and that in both cases the Frobenius kernel coincides with the commutator subgroup.

Corollary 4.12. *Let l be an odd prime. Let L/K be a finite Galois extension of number fields with $\text{Gal}(L/K) \simeq G \times A$, where A is abelian and G is isomorphic to either D_{2l} or $\text{Aff}(q)$ where $q = l^n$ for some n . Then for every (p, r) -admissible sets S and T we have*

$$\theta_S^T(r) \in \mathcal{I}(G \times A).$$

Proof. If $p \neq l$ is a prime, then $\theta_S^T(r) \in \mathcal{I}_p(G \times A)$ follows from Theorem 4.11. If $p = l$, then by [JN, Proposition 6.7] (if $G \simeq \text{Aff}(q)$) and [JN, Proposition 6.9] (if $G \simeq D_{2l}$) we have $\mathcal{I}_p(G) = \zeta(\mathcal{M}_p(G))$. Then Corollary 1.4 implies that $\mathcal{I}_p(G \times A) = \zeta(\mathcal{M}_p(G))[A]$ and so the result follows from Theorem 4.2. \square

5 The ETNC in almost tame extensions

5.1 The conjecture

Let us fix a finite Galois extension L/K of number fields with Galois group G and a finite set S of places of K which contains $S_{\text{ram}} \cup S_\infty$. Let ΔS be the kernel of the augmentation map $\mathbb{Z}S(L) \rightarrow \mathbb{Z}$ which maps each $\mathfrak{P} \in S(L)$ to 1 and let

$$\lambda_S : \mathbb{R} \otimes E_S \rightarrow \mathbb{R} \otimes \Delta S, \quad u \mapsto - \sum_{\mathfrak{P} \in S(L)} \log |u|_{\mathfrak{P}} \mathfrak{P}$$

be the negative of the usual Dirichlet map. Note that λ_S is in fact an isomorphism of $\mathbb{R}G$ -modules. Furthermore, let $\tau_S \in \text{Ext}_G^2(\Delta S, E_S)$ be Tate's canonical class (cf. [Ta66]); then τ_S is given by a 2-extension

$$E_S \twoheadrightarrow A \rightarrow B \twoheadrightarrow \Delta S,$$

where A and B are finitely generated cohomologically trivial $\mathbb{Z}G$ -modules. Thus we may view $A \rightarrow B$ as a perfect complex with A placed in degree 0 and the pair $(A \rightarrow B, \lambda_S^{-1})$ is a trivialized complex. In [Bu01] the author defines the following element of $K_0(\mathbb{Z}G, \mathbb{R})$:

$$T\Omega(L/K, 0) := \psi_G^*(\chi_{\mathbb{Z}G, \mathbb{R}}(A \rightarrow B, \lambda_S^{-1}) + \hat{\delta}_G(L_S^*(0)^\#)).$$

Here, ψ_G^* is a certain involution on $K_0(\mathbb{Z}G, \mathbb{R})$ which is not important for our purposes, since we will be only interested in the nullity of $T\Omega(L/K, 0)$. In fact, the ETNC for the motive $h^0(\text{Spec}(L))$ with coefficients in $\mathbb{Z}G$ in this context simply asserts the following.

Conjecture 5.1. *The element $T\Omega(L/K, 0) \in K_0(\mathbb{Z}G, \mathbb{R})$ is zero.*

Note that this statement is also equivalent to the Lifted Root Number Conjecture formulated by Gruenberg, Ritter and Weiss [GRW99] (cf. [Bu01, Theorem 2.3.3]). By [Bu01, Theorem 2.2.4] one knows that $T\Omega(L/K, 0)$ lies in $K_0(\mathbb{Z}G, \mathbb{Q})$ if and only if Stark's conjecture holds for all irreducible characters of G . In this case the ETNC decomposes into local conjectures at each prime p by means of the isomorphism

$$K_0(\mathbb{Z}G, \mathbb{Q}) \simeq \bigoplus_{p \nmid \infty} K_0(\mathbb{Z}_p G, \mathbb{Q}_p).$$

Let $T\Omega(L/K, 0)_p$ be the image of $T\Omega(L/K, 0)$ in $K_0(\mathbb{Z}_p G, \mathbb{Q}_p)$. If we further assume that $T\Omega(L/K, 0)_p$ is torsion, then it is well known (see [GRW99, §8] for example) that $T\Omega(L/K, 0)_p$ vanishes if and only if $T\Omega(L'/K', 0)_p$ vanishes for all intermediate Galois extensions L'/K' whose Galois group is p -elementary, i.e. G is the direct product of a p -group and a cyclic group of order prime to p . We will prove an analogous result on minus parts in the next paragraph.

5.2 A reduction step on minus parts

Let L/K be a Galois CM-extension with Galois group G and let p be an odd prime. Let $j \in G$ denote complex conjugation. There are canonical isomorphisms

$$\begin{aligned} K_0(\mathbb{Z}_p G, \mathbb{Q}_p) &\simeq K_0(\mathbb{Z}_p G_+, \mathbb{Q}_p) \oplus K_0(\mathbb{Z}_p G_-, \mathbb{Q}_p) \\ DT(\mathbb{Z}_p G) &\simeq DT(\mathbb{Z}_p G_+) \oplus DT(\mathbb{Z}_p G_-). \end{aligned}$$

Moreover, we naturally have $\mathbb{Z}_p G_+ \simeq \mathbb{Z}_p G^+$, where $G^+ := G/\langle j \rangle$ is the Galois group of the extension L^+/K of totally real fields.

Since Stark's conjecture is known for odd characters [Ta84, Theorem 1.2, p. 70], the element $T\Omega(L/K, 0)$ has a well defined image $T\Omega(L/K, 0)_p^-$ in $K_0(\mathbb{Z}_p G_-, \mathbb{Q}_p)$. In the proof of the following proposition we will also view $T\Omega(L/K, 0)_p^-$ as an element in $K_0(\mathbb{Z}_p G, \mathbb{Q}_p)$ or rather $DT(\mathbb{Z}_p G)$ by requiring that its plus part is trivial.

Proposition 5.2. *Let L/K be a Galois CM-extension with Galois group G and let p be an odd prime. Assume that $T\Omega(L/K, 0)_p^-$ belongs to $DT(\mathbb{Z}_p G_-)$. Then $T\Omega(L/K, 0)_p^-$ vanishes if and only if $T\Omega(L'/K', 0)_p^-$ vanishes for all intermediate Galois CM-extensions L'/K' whose Galois group is either p -elementary or a direct product of a p -elementary group and a cyclic group of order 2 (generated by j).*

Proof. Let L'/K' be an arbitrary intermediate Galois extension of L/K with Galois group H . Then there are subgroups G_1 and G_2 of G with G_2 normal in G_1 such that $H \simeq G_1/G_2$. There are canonical restriction and quotient maps

$$DT(\mathbb{Z}_p G) \xrightarrow{\text{res}_{G_1}^G} DT(\mathbb{Z}_p G_1) \xrightarrow{\text{quot}_{G_2}^{G_1}} DT(\mathbb{Z}_p H).$$

We denote the image of $T\Omega(L/K, 0)_p^-$ in $DT(\mathbb{Z}_p H)$ by w_H . Functoriality of $T\Omega(L/K, 0)$ [Bu01, Proposition 2.1.4] implies that $w_H = T\Omega(L'/K', 0)_p^-$ whenever L'/K' is a CM-extension. If $w_G = T\Omega(L/K, 0)_p^-$ vanishes, then clearly $w_H = 0$ for all subquotients H of G . Conversely, [GRW99, Proposition 9] says that $w_G = 0$ if $w_H = 0$ whenever H is a p -elementary group. Fix such a p -elementary subquotient H . If j lies in G_2 , then L' is totally real and $w_H = \text{quot}_{H}^{G_1^+}(\text{res}_{G_1^+}^{G^+} w_{G^+})$, where $G_1^+ := G_1/\langle j \rangle$. However, w_{G^+} is trivial and thus also $w_H = 0$. Hence we may assume that $j \notin G_2$ so that L' is a CM-field. If j lies in G_1 , then K' is totally real and $w_H = 0$ by assumption. If $j \notin G_1$, we let \tilde{G}_1 be the minimal subgroup of G that contains G_1 and j . As j is central in G , we have $\tilde{G}_1 = G_1 \times \langle j \rangle$ and G_2 is still normal in \tilde{G}_1 . Then $\tilde{H} := \tilde{G}_1/G_2 \simeq H \times \langle j \rangle$ is a subquotient of G that corresponds to an intermediate CM-extension. Moreover, we have $w_{\tilde{H}} = 0$ by assumption and thus $w_H = \text{res}_H^{\tilde{H}}(w_{\tilde{H}}) = 0$ as desired. \square

5.3 A reformulation in terms of Fitting invariants

We will say that the CM-extension L/K is *almost tame* above p if j lies in $G_{\mathfrak{p}}$ for every prime \mathfrak{p} of K above p which is wildly ramified in L/K .

We have the following relation to the integrality conjecture 2.2 (cf. [Ni11d, proof of Theorem 5.1 and Corollary 5.6]):

Theorem 5.3. *Let p be an odd prime and let L/K be a Galois CM-extension. Assume that $T\Omega(L/K, 0)_p^-$ vanishes. If the p -part of the roots of unity of L is a cohomologically trivial G -module or if L/K is almost tame above p , then the p -part of Conjecture 2.2 holds, i.e. $SKu_p(L/K) \subseteq \mathcal{I}_p(G)$.*

Thus the main result of this section (Theorem 5.8 below) may be seen as a partial converse of Theorem 5.3.

Now let T consist of a prime $\mathfrak{p}_0 \nmid p$ and all finite places of K which ramify in L/K and do not lie above p ; we may choose \mathfrak{p}_0 such that $E_S^{T_{\text{nr}}}$ is torsionfree for every finite set S of places of K which contains S_∞ and is disjoint to T . Of course, T_{nr} consists of the single prime \mathfrak{p}_0 . We denote the set of all wildly ramified primes above p by $S_{p,w}$ and put $S_1 := S_{p,w} \cup S_\infty$. In particular, the sets S_1 and T are $(p, 0)$ -admissible.

Theorem 5.4. *Let p be an odd prime and L/K a Galois CM-extension which is almost tame above p . Then the following are equivalent:*

- (i) $T\Omega(L/K, 0)_p^- = 0$;
- (ii) $\text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p)) = [\langle \theta_{S_1}^T(0) \rangle]_{\text{nr}(\mathbb{Z}_p G_-)}$;
- (iii) $\text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p)) \subseteq [\langle \theta_{S_1}^T(0) \rangle]_{\text{nr}(\mathbb{Z}_p G_-)}$.

Proof. First note that $A_L^T(p)$ is a cohomologically trivial G -module by [Ni11a, Theorem 1] such that the projective dimension of $A_L^T(p)$ as a $\mathbb{Z}_p G_-$ -module is at most 1. Thus $\text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p))$ is well defined. That (i) and (ii) are equivalent is just a reformulation

of [Ni11a, Theorem 2] in terms of Fitting invariants. To see this let E be a finite Galois extension of \mathbb{Q}_p with Galois group Γ such that every odd representation of G has a realization over E . Let $R_p(G)^-$ denote the subring of $R_p(G)$ generated by odd characters. Then remark 1.1 has an obvious analogue on minus parts, and in the notation of §1.1.3 we see that $\theta_{S_1}^T(0) \in \zeta(\mathbb{Q}_p G_-)^\times$ corresponds to the representing homomorphism $f_{\theta_{S_1}^T(0)} \in \text{Hom}_\Gamma(R_p(G)^-, E^\times)$ which on irreducible odd characters χ is given by

$$f_{\theta_{S_1}^T(0)} : \chi \mapsto L_{S_1}^T(0, \check{\chi}).$$

However, in the notation of [Ni11a] we have $f_{\theta_{S_1}^T(0)} = \Theta_{S_1}^T$.

Clearly (ii) implies (iii) and we are left with showing the converse. Let $z \in (\mathbb{Q}_p G_-)^\times$ be a generator of $\text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p))$ and let $f_z \in \text{Hom}_\Gamma(R_p(G)^-, E^\times)$ be the corresponding representing homomorphism. Write

$$z = \sum_{\chi} z_{\chi} e_{\chi} \in \zeta(EG_-) = \bigoplus_{\chi} E e_{\chi},$$

where the sum runs through all odd irreducible (\mathbb{C}_p -valued) characters of G . Then $f_z(\chi) = z_{\chi}$ for all odd irreducible χ by (4), and by [Ni11a, Proposition 5] we have

$$\prod_{\chi} z_{\chi}^{\chi(1)} \sim_p |A_L^T(p)|,$$

where \sim_p means “equality up to a p -adic unit”. By [Ni11a, Proposition 4] however, we also have

$$\prod_{\chi} (L_{S_1}^T(0, \check{\chi}))^{\chi(1)} \sim_p |A_L^T(p)|$$

so that z is also a generator of $[(\theta_{S_1}^T(0))]_{\text{nr}(\mathbb{Z}_p G_-)}$ by [Ni10, Proposition 5.4]. Hence (iii) implies (ii) and we are done. \square

5.4 Exceptional primes

For a natural number n let ζ_n be a primitive n th root of unity and let us denote the normal closure of L over \mathbb{Q} by L^{cl} ; note that L^{cl} is again a CM-field.

Definition 5.5. *We will call a prime p **exceptional** for L/K if at least one of the following holds:*

- (i) $p = 2$,
- (ii) there is a prime \mathfrak{p} in K above p which ramifies wildly in L and $j \notin G_{\mathfrak{p}}$, i.e. L/K is not almost tame above p ,
- (iii) $L^{\text{cl}} \subseteq (L^{\text{cl}})^+(\zeta_p)$.

Remark 5.6. 1. Note that there are only finitely many exceptional primes, since such a prime has to ramify in L^{cl}/\mathbb{Q} or equals 2.

2. If p is non-exceptional, then the negation of (ii) ensures that the p -minus ray class group $A_L^T(p)$ is a cohomologically trivial G -module. We have already used this in the proof of Theorem 5.4. In fact, there is a second technical point, where (ii) is needed. We will indicate this in the course of the proof.
3. The negation of (iii) will permit us to adjust a descent method introduced by Wiles [Wi90b] and further developed by Greither [Gr00] to the non-abelian situation. In particular, it is needed in the proof of Lemma 5.9.

Lemma 5.7. *Let L/K be a Galois CM-extension and let p be a prime. If p is non-exceptional for L/K , then it is non-exceptional for every intermediate Galois CM-extension L'/K' .*

Proof. Let p be non-exceptional for L/K and let L'/K' be an intermediate Galois CM-extension with Galois group H . Clearly $p \neq 2$ and so (i) does not hold. Suppose that there is a prime \mathfrak{p}' in K' above p such that \mathfrak{p}' is wildly ramified in L' . Then $\mathfrak{p} := \mathfrak{p}' \cap K$ is a prime in K above p which is wildly ramified in L . Thus $j \in G_{\mathfrak{P}}$ for every prime \mathfrak{P} in L above \mathfrak{p} as p is non-exceptional for L/K . We write $H = G_1/G_2$, where G_1 and G_2 are subgroups of G with G_2 normal in G_1 . As K' is totally real, we have $j \in G_1$ and thus $j \in G_{1,\mathfrak{P}} = G_1 \cap G_{\mathfrak{P}}$. Let \mathfrak{P}' be the prime in L' below \mathfrak{P} . Then the natural surjection $G_1 \rightarrow H$ maps $j \in G_1$ to $j \in H$ and $G_{1,\mathfrak{P}}$ onto $H_{\mathfrak{P}'}$. So $j \in H_{\mathfrak{P}'}$ and (ii) for L'/K' does not hold either.

Now suppose that (iii) holds for p and L'/K' . Then we have

$$(L')^{\text{cl}} \subseteq ((L')^{\text{cl}})^+(\zeta_p) \subseteq (L^{\text{cl}})^+(\zeta_p)$$

and thus also

$$(L')^{\text{cl}}(L^{\text{cl}})^+ \subseteq (L^{\text{cl}})^+(\zeta_p).$$

However, the field $(L')^{\text{cl}}$ is a CM-field and therefore not contained in $(L^{\text{cl}})^+$. Hence the first inclusion in

$$(L^{\text{cl}})^+ \subseteq (L^{\text{cl}})^+(L')^{\text{cl}} \subseteq L^{\text{cl}}$$

is proper. We find that $(L^{\text{cl}})^+(L')^{\text{cl}} = L^{\text{cl}}$ is contained in $(L^{\text{cl}})^+(\zeta_p)$, a contradiction. \square

5.5 The main theorem

We now prove the following theorem which is Theorem 0.3 of the introduction.

Theorem 5.8. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ -invariant attached to L and p vanishes, then the p -minus part of the ETNC for the pair $(h^0(\text{Spec}(L)), \mathbb{Z}G)$ is true.*

Proof. We first observe that we may assume that G is monomial. In fact, we already know that $T\Omega(L/K, 0)_p^-$ is torsion, since the strong Stark conjecture holds by [Ni11a, Corollary 2]. Then Proposition 5.2 implies that we may assume that G is either p -elementary or a direct product of a p -elementary group and a cyclic group of order 2. In both cases, the group G is nilpotent and thus monomial. Note that p is still a non-exceptional prime by Lemma 5.7. We will henceforth assume that G is monomial. In particular, we may use the results of [Ni11b], where the identity (7) is implicitly assumed to hold.

Let L_∞ and K_∞ be the cyclotomic \mathbb{Z}_p -extensions of L and K , respectively. We denote the Galois group of K_∞/K by Γ_K . Hence Γ_K is isomorphic to \mathbb{Z}_p , and we fix a topological generator γ_K . Accordingly, we set $\Gamma_L = \text{Gal}(L_\infty/L)$ with a topological generator γ_L . Furthermore, we denote the n -th layer in the cyclotomic extension L_∞/L by L_n such that L_n/L is cyclic of order p^n . We put

$$\mathcal{X}_T^- := \varprojlim_{\leftarrow} A_{L_n}^T(p).$$

We denote the Galois group of L_∞/K by \mathcal{G} ; hence $\mathcal{G} = H \rtimes \Gamma$, where H is a subgroup of G and Γ is topologically generated by a preimage γ of γ_K under the canonical epimorphism $\mathcal{G} \rightarrow \mathcal{G}/H = \Gamma_K$. We denote the Iwasawa algebra $\mathbb{Z}_p[[\mathcal{G}]]$ by $\Lambda(\mathcal{G})$. Then \mathcal{X}_T^- is a finitely generated R -torsion $\Lambda(\mathcal{G})_- := \Lambda(\mathcal{G})/(1+j)$ -module, where $R := \mathbb{Z}_p[[\Gamma']]$ with $\Gamma' \simeq \mathbb{Z}_p$ central in \mathcal{G} . Note that R is isomorphic to $\mathbb{Z}_p[[T]]$, the power series ring in one variable over \mathbb{Z}_p .

The vanishing of the Iwasawa μ -invariant implies that the μ -invariant of \mathcal{X}_T^- also vanishes; hence the projective dimension of \mathcal{X}_T^- over $\Lambda(\mathcal{G})_-$ is at most 1 by [Ni11b, Proposition 4.1]. Then by [Ni10, Lemma 6.2] the $\Lambda(\mathcal{G})_-$ -module \mathcal{X}_T^- admits a quadratic presentation and thus $\text{Fitt}_{\Lambda(\mathcal{G})_-}(\mathcal{X}_T^-)$ is well defined. This Fitting invariant is computed via the equivariant Iwasawa main conjecture (which is a theorem under our current hypotheses by Ritter and Weiss [RW11] and Kakde [Ka13]) in [Ni11b, Theorem 4.4]. Since the precise statement of this theorem would force us to introduce a lot of further notation, we only state the following consequence [Ni11b, Lemma 6.3] of this theorem which will be sufficient for our purposes:

$$\text{Fitt}_{\mathbb{Z}_p G_-}((\mathcal{X}_T^-)_{\Gamma_L}) = [(\theta_{S_p}^T(0))]_{\text{nr}(\mathbb{Z}_p G_-)}, \quad (15)$$

where $(\mathcal{X}_T^-)_{\Gamma_L}$ denotes the Γ_L -coinvariants of \mathcal{X}_T^- .

We now adopt a method originally introduced by Wiles [Wi90b] and further developed (in an equivariant way) by Greither [Gr00] and the author [Ni11a]. In fact, the following is carried out in some detail in [Ni11b]; but there, the full integrality conjecture 2.2 is assumed to hold for an infinite class of field extensions. Since we only will use our results established in §4, we have to take care if everything still works.

Lemma 5.9. *Let $N > 0$ be a natural number. Then there are infinitely many primes $r \in \mathbb{Z}$ such that*

(i) $r \equiv 1 \pmod{p^N}$.

(ii) $j \in G_{\mathfrak{R}}$ for all primes \mathfrak{R} in L above r .

(iii) The Frobenius automorphism Frob_p at p in $\text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ generates $\text{Gal}(k_r/\mathbb{Q})$, where k_r denotes the unique subfield of $\mathbb{Q}(\zeta_r)$ of degree p^N over \mathbb{Q} .

Proof. This is [Ni11b, Lemma 6.5], but the proof of [Gr00, Proposition 4.1] carries over unchanged to the present situation. \square

Let $N \in \mathbb{N}$ be a positive integer and choose a prime r as in Lemma 5.9 which does not ramify in L^{cl}/\mathbb{Q} . We put $L' := Lk_r$, $K' = Kk_r$ and $G' = \text{Gal}(L'/K) = G \times C_N$, where $C_N \simeq \text{Gal}(k_r/\mathbb{Q})$ is cyclic of order p^N , generated by Frob_p . Note that p is still

a non-exceptional prime for L'/K and that G' is also monomial. Moreover, we define $T' := T \cup S_r$, where S_r denotes the set of places in K above r . We have an exact sequence

$$(\mathbb{Z}_p \otimes (\mathcal{O}_{L'} / \prod_{\mathfrak{A}|r} \mathfrak{A})^\times)^- \twoheadrightarrow A_{L'}^{T'}(p) \twoheadrightarrow A_{L'}^T(p).$$

Since $j \in G_{\mathfrak{A}}$ for all $\mathfrak{A} | r$, we may conclude as in [Ni11a, p. 28] to deduce that the leftmost term is trivial. We obtain an isomorphism

$$A_{L'}^{T'}(p) \simeq A_{L'}^T(p) \quad (16)$$

and hence $A_{L'}^T(p)$ is cohomologically trivial as G' -module by [Ni11a, Theorem 1]. As in [Ni11a, p. 28] the restriction map induces an isomorphism

$$(A_{L'}^T(p))_{C_N} \simeq A_{L'}^T(p). \quad (17)$$

More precisely, the cokernel of the restriction map $A_{L'}^T(p) \rightarrow A_{L'}^T(p)$ identifies with a quotient of C_N by class field theory. However, j acts by conjugation and thus trivially on C_N . Hence the cokernel is trivial. The composite map

$$A_{L'}^T(p) \xrightarrow{\text{res}} A_{L'}^T(p) \longrightarrow A_{L'}^T(p)$$

is given by the norm of the cyclic group C_N . As $A_{L'}^T(p)$ is cohomologically trivial, the kernel of the norm is precisely $(\text{Frob}_p - 1)A_{L'}^T(p)$. This gives the desired isomorphism (17).

The sets S_1 and T' are $(p, 0)$ -admissible and thus the Stickelberger element $\theta_{S_1}^{T'}(L'/K, 0)$ lies in $\zeta(\mathcal{M}_p(G))[C_N]$ by Theorem 4.2. However, we have

$$\theta_{S_1}^{T'}(L'/K, 0) = \left(\frac{1-j}{2} \delta_{S_r}(0) \right) \theta_{S_1}^T(L'/K, 0) \quad (18)$$

and we claim that

$$\frac{1-j}{2} \delta_{S_r}(0) \in \text{nr}((\mathbb{Z}_p G'_-)^{\times}). \quad (19)$$

In fact, as mentioned above, we have

$$(\mathbb{Z}_p \otimes (\mathcal{O}_{L'} / \prod_{\mathfrak{A}|r} \mathfrak{A})^\times)^- = 0.$$

Hence $\frac{1-j}{2} \delta_{S_r}(0)$ is a generator of $\text{Fitt}_{(\mathbb{Z}_p G'_-)}(0)$ and therefore lies in $\text{nr}((\mathbb{Z}_p G'_-)^{\times})$. However, we have $\text{nr}((\mathbb{Z}_p G'_-)^{\times}) \subseteq \mathcal{I}_p(G') \subseteq \zeta(\mathcal{M}_p(G))[C_N]$ by Lemma 1.3, and thus

$$\theta_{S_1}^T(L'/K, 0) \in \zeta(\mathcal{M}_p(G))[C_N] \quad (20)$$

for all N . We define an element $\alpha_p \in \mathcal{I}_p(G)$ by

$$\alpha_p = \prod_{\mathfrak{p} \in S_p \setminus S_1} \text{nr}(1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{A}}^{-1})$$

such that we have an equality $\theta_{S_1}^T(L'/K, 0) \cdot \alpha_p = \theta_{S_p}^T(L'/K, 0)$. Similarly, we define $\alpha'_p \in \mathcal{I}_p(G')$ such that $\theta_{S_1}^T(L'/K, 0) \cdot \alpha'_p = \theta_{S_p}^T(L'/K, 0)$. Now choose a second natural number $M \leq N$ and put

$$\nu := \sum_{i=0}^{p^M-1} \text{Frob}_p^{ip^{N-M}} \in \mathbb{Z}_p C_N \subseteq \zeta(\mathbb{Z}_p G').$$

The following result is [Ni11b, Lemma 6.7].

Lemma 5.10. *Let f be the least common multiple of the residual degrees $f_{\mathfrak{p}}(K/\mathbb{Q})$ of all $\mathfrak{p} \in S_p$. If $N - M \geq v_p(|G| \cdot f)$, then $|G| \cdot \alpha'_p$ is a nonzerodivisor in $\zeta(\mathbb{Z}_p G')/\nu$.*

We now observe that enlarging L to L' does not affect the vanishing of μ by [NSW00, Theorem 11.3.8]. Choose natural numbers $M \leq N$ such that $r = r(N)$ fulfills all the above conditions and $N - M \geq v_p(|G| \cdot f)$, where f was defined in Lemma 5.10. Let $\mathcal{G}' = \text{Gal}(L'_\infty/K)$ and let $\mathcal{X}_{T'}^-$ be the projective limit of the minus p -ray class groups $A_{L'_n}^{T'}(p)$. Then $\mathcal{X}_{T'}^-$ has projective dimension at most one as before and the EIMC for the extension $(L'_\infty)^+/K$ implies the following analogue of equation (15):

$$\text{Fitt}_{\mathbb{Z}_p G'_-}((\mathcal{X}_{T'}^-)_{\Gamma_{L'}}) = [\langle \theta_{S_p}^{T'}(L'/K, 0) \rangle]_{\text{nr}(\mathbb{Z}_p G'_-)}. \quad (21)$$

For each prime \mathfrak{p} of K let $\mathfrak{P}' \subset L'$ be a prime above \mathfrak{p} . By [Ni11b, Proposition 4.7], we have a right exact sequence

$$\left(\bigoplus_{\mathfrak{p} \in S_p} \text{ind}_{G'_{\mathfrak{P}'}}^{G'} \mathbb{Z}_p \right)^- \rightarrow (\mathcal{X}_{T'}^-)_{\Gamma_{L'}} \rightarrow A_{L'}^{T'}(p). \quad (22)$$

Note that $(\text{ind}_{G'_{\mathfrak{P}'}}^{G'} \mathbb{Z}_p)^- = 0$ whenever $\mathfrak{p} \in S_{p,w} = S_p \cap S_1$, since j lies in the decomposition group $G'_{\mathfrak{P}'}$ in this case; it is here, where we use that (ii) of Definition 5.5 does not hold for p . Therefore the Fitting invariant of the leftmost term is generated by α'_p . By (18) and (19) the Stickelberger elements $\theta_{S_p}^{T'}(L'/K, 0)$ and $\theta_{S_p}^T(L'/K, 0)$ only differ by the norm of a unit. Hence $\theta_{S_p}^T(L'/K, 0)$ is also a generator of $\text{Fitt}_{\mathbb{Z}_p G'_-}((\mathcal{X}_{T'}^-)_{\Gamma_{L'}})$ by (21). The above sequence (22) gives rise to the following inclusion of Fitting invariants (cf. [Ni10, Proposition 3.5 (iii)]):

$$\text{Fitt}_{\mathbb{Z}_p G'_-} \left(\left(\bigoplus_{\mathfrak{p} \in S_p} \text{ind}_{G'_{\mathfrak{P}'}}^{G'} \mathbb{Z}_p \right)^- \right) \cdot \text{Fitt}_{\mathbb{Z}_p G'_-}(A_{L'}^{T'}(p)) \subseteq \text{Fitt}_{\mathbb{Z}_p G'_-}((\mathcal{X}_{T'}^-)_{\Gamma_{L'}}).$$

If we choose a generator c' of $\text{Fitt}_{\mathbb{Z}_p G'_-}(A_{L'}^{T'}(p))$, there exists $x \in \zeta(\mathbb{Z}_p G')$ such that

$$\alpha'_p c' = x \cdot \theta_{S_p}^T(L'/K, 0) = x \cdot \alpha'_p \theta_{S_1}^T(L'/K, 0).$$

It follows from Lemma 1.3 and (20) that multiplication by $|G|$ yields an equality in $\zeta(\mathbb{Z}_p G')$ such that Lemma 5.10 gives

$$|G| \cdot c' \equiv |G| \cdot x \cdot \theta_{S_1}^T(L'/K, 0) \pmod{\nu}. \quad (23)$$

Let $\text{aug} : \mathbb{Z}_p G' \rightarrow \mathbb{Z}_p G$ be the natural augmentation map. Since Fitting invariants behave well under base change (cf. [Ni10, Lemma 5.5]), the element $c := \text{aug}(c')$ generates the Fitting invariant of $A_L^T(p)$ by (16) and (17). Since $\text{aug}(\theta_{S_1}^T(L'/K, 0)) = \theta_{S_1}^T(L/K, 0)$ and $\text{aug}(\nu) = p^M$, the congruence (23) implies

$$c \equiv \text{aug}(x) \cdot \theta_{S_1}^T(L/K, 0) \pmod{p^{M-m} \mathcal{I}_p(G)},$$

where p^m is the exact p -power dividing $|G|$. This gives an inclusion

$$\text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p)) \subseteq [\langle \theta_{S_1}^T(L/K, 0) \rangle]_{\text{nr}(\mathbb{Z}_p G_-)},$$

as we may choose M arbitrarily large. Now we are done via Theorem 5.4. \square

We end this section with giving the proofs of some of the corollaries mentioned in the introduction.

Proof of Corollary 0.4. This is an immediate consequence of Theorem 5.8 and Theorem 5.3. \square

Proof of Corollary 0.6. The central conjecture (Conjecture 2.4.1) of [BB07] states that a certain element $T\Omega(L/K, 1) \in K_0(\mathbb{Z}G, \mathbb{R})$ vanishes. By [BB07, Theorem 5.2] one has an equality

$$\psi_G^*(T\Omega(L/K, 0)) - T\Omega(L/K, 1) = T\Omega^{loc}(L/K, 1),$$

and the vanishing of the righthand side is equivalent to a conjecture of Bley and Burns [BB03] by [BB07, Remark 5.4]. However, this conjecture is known if L/K is at most tamely ramified by [BB03, Corollary 6.3 (i)]. Finally, if we suppose that Leopoldt's conjecture holds, then [BB10, Theorem 1.1 and Corollary 1.2] imply the desired relation to the ETNC for the pair $(h^0(\text{Spec}(L))(1), \mathbb{Z}G)$. However, it is sufficient for our purposes that Leopoldt's conjecture holds on minus parts and this is in fact true (see also [CJ13, Remark 1.4]):

Recall that by [NSW00, Theorem 10.3.6] Leopoldt's conjecture for odd p is equivalent to the assertion that the canonical homomorphism

$$\Delta : \mathcal{O}_L^\times \otimes \mathbb{Z}_p \longrightarrow \prod_{\mathfrak{p} \in S_p(L)} \widehat{\mathcal{O}}_{\mathfrak{p}}^\times$$

is injective, where $\widehat{\mathcal{O}}_{\mathfrak{p}}^\times$ denotes the pro- p -completion of the group of units of the local field $L_{\mathfrak{p}}$ (hence $\widehat{\mathcal{O}}_{\mathfrak{p}}^\times$ is canonically isomorphic to the group of principal units). By [NSW00, Lemma 10.3.13 and Lemma 8.7.7] the kernel of Δ is torsion-free. However, the minus part of $\mathcal{O}_L^\times \otimes \mathbb{Z}_p$ are the p -power roots of unity in L . It follows that Δ is injective on minus parts as desired. \square

6 The non-abelian Brumer-Stark conjecture

As before, let L/K be a Galois CM-extension with Galois group G . The following conjecture has been formulated in [Ni11d] and is a non-abelian generalization of Brumer's conjecture.

Conjecture 6.1. *Let S be a finite set of places of K containing $S_{\text{ram}} \cup S_\infty$. Then $\mathfrak{A}_S \theta_S(0) \subseteq \mathcal{I}(G)$ and for each $x \in \mathcal{H}(G)$ we have*

$$x \cdot \mathfrak{A}_S \theta_S(0) \subseteq \text{Ann}_{\mathbb{Z}G}(\text{cl}_L).$$

Remark 6.2. • *If G is abelian, the inclusion $\mathfrak{A}_S \theta_S(0) \subseteq \mathcal{I}(G) = \mathbb{Z}G$ holds by Theorem 2.4 and, since $\mathcal{H}(G) = \mathbb{Z}G$ in this case, Conjecture 6.1 recovers Brumer's conjecture.*

- *Replacing the class group cl_L by its p -parts $\text{cl}_L(p)$ for each rational prime p , Conjecture 6.1 naturally decomposes into local conjectures at each prime p . Note that it is then possible to replace $\mathcal{H}(G)$ by $\mathcal{H}_p(G)$ by [Ni11d, Lemma 1.4].*
- *Burns [Bu11] has also formulated a conjecture which generalizes many refined Stark conjectures to the non-abelian situation. In particular, it implies this generalization of Brumer's conjecture (cf. [Bu11, Proposition 3.5.1]).*

For $\alpha \in L^\times$ we define

$$S_\alpha := \{\mathfrak{p} \subset K : \mathfrak{p} | N_{L/K}(\alpha)\}$$

and we call α an *anti-unit* if $\alpha^{1+j} = 1$. Let $\omega_L := \text{nr}(|\mu_L|)$. The following is a non-abelian generalization of the Brumer-Stark conjecture (cf. [Ni11d, Conjecture 2.6]).

Conjecture 6.3. *Let S be a finite set of places of K containing $S_{\text{ram}} \cup S_\infty$. Then $\omega_L \cdot \theta_S(0) \in \mathcal{I}(G)$ and for each $x \in \mathcal{H}(G)$ and each fractional ideal \mathfrak{a} of L , there is an anti-unit $\alpha = \alpha(x, \mathfrak{a}, S) \in L^\times$ such that*

$$\mathfrak{a}^{x \cdot \omega_L \cdot \theta_S(0)} = (\alpha)$$

and for each finite set T of primes of K such that $\text{Hyp}(S \cup S_\alpha, T)$ is satisfied there is an $\alpha_T \in E_{S_\alpha}^T$ such that

$$\alpha^{z \cdot \delta_T(0)} = \alpha_T^{z \cdot \omega_L} \quad (24)$$

for each $z \in \mathcal{H}(G)$.

Remark 6.4. • *If G is abelian, we have $\mathcal{I}(G) = \mathcal{H}(G) = \mathbb{Z}G$ and $\omega_L = |\mu_L|$. Hence it suffices to treat the case $x = z = 1$. Then [Ta84, Proposition 1.2, p. 83] states that the condition (24) on the anti-unit α is equivalent to the assertion that the extension $L(\alpha^{1/\omega_L})/K$ is abelian.*

- *As above, we obtain local conjectures for each prime p .*
- *The non-abelian Brumer-Stark conjecture (at p) implies the non-abelian Brumer conjecture (at p) by [Ni11d, Lemma 2.12].*

We now prove the remaining results mentioned in the introduction which are mainly concerned with the above two conjectures.

Proof of Corollary 0.5. Theorem 5.8 and [Ni11d, Theorem 5.3] imply that L/K fulfills the (non-abelian) strong Brumer-Stark property at p . This in turn implies (ii) by [Ni11d, Proposition 3.8] and (i) by [Ni11d, Lemma 2.9]. Since the condition $L^{\text{cl}} \not\subseteq (L^{\text{cl}})^+(\zeta_p)$ forces $\zeta_p \notin L$, the p -part of the roots of unity is trivial and thus cohomologically trivial as G -module. Therefore Theorem 5.8 and [Bu11, Theorem 4.1.1] imply (iii). Finally, as already mentioned above, the vanishing of $T\Omega(L/K, 0)$ is equivalent to the Lifted Root Number Conjecture of Gruenberg, Ritter and Weiss as formulated in [GRW99] (cf. [Bu01, Theorem 2.3.3]). Thus Theorem 5.8 also implies (iv). \square

Proof of Theorem 0.7. It suffices to prove the Brumer-Stark conjecture at each odd prime p . We first assume that p is unramified. As $L = L^{\text{cl}}$ in this case, the prime p is non-exceptional by Remark 5.6. In particular, L/\mathbb{Q} is almost tame above p and the result follows from Corollary 0.5 if μ vanishes. If μ does not vanish (which conjecturally will never be the case) then $p \nmid [L : \mathbb{Q}]$ by assumption. Hence the p -minus part of the ETNC is equivalent to the p -part of the strong Stark conjecture for odd characters which is a theorem by [Ni11a, Corollary 2]. Hence the Brumer-Stark conjecture at p holds by [Ni11d, Theorem 5.2]. Now assume that p ramifies in L . Since p is the only p -adic place of the rationals, we have $S_p \subseteq S_{\text{ram}}$, and the result follows from [Ni13, Corollary 4.6], where the identity (7) is implicitly assumed to hold (it is only here, where we have to restrict to monomial extensions). \square

7 Negative integers

For completeness, we include the following result which is an easy consequence of [Nil1c, Theorem 4.1] and [Bu, Corollary 2.10].

Theorem 7.1. *Let L/K be a Galois extension of number fields with Galois group G and let $r < 0$. Assume that L is totally real if r is odd (resp. that L/K is CM if r is even). If p is an odd prime such that the p -part (resp. minus p -part) of the ETNC for the pair $(h^0(\text{Spec}(L))(r), \mathbb{Z}G)$ holds, then the p -part of Conjecture 2.6 is true. In particular, this applies if the Iwasawa μ -invariant attached to L and p vanishes.*

Corollary 7.2. *Let L/K be a Galois extension of number fields with Galois group G and let $r < 0$. Let p be an odd prime and assume that G has a unique 2-Sylow subgroup. Then the p -part of Conjecture 2.6 holds provided that the Iwasawa μ -invariant attached to p and the maximal real subfield of L vanishes.*

Proof. This immediately follows from Theorem 7.1 and Proposition 3.1. □

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Andreas Nickel anickel3@math.uni-bielefeld.de

Universität Bielefeld, Fakultät für Mathematik, Postfach 100131, 33501 Bielefeld, Germany