Noncommutative Fitting invariants and non-abelian Stark-type conjectures

Habilitationsschrift

Andreas Nickel

Universität Bielefeld

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Introduction

This habilitation thesis consists of six related articles [1, 2, 3, 4, 5, 6]. Two of them [2, 6] are joint work with Henri Johnston. The general motivation originates from Stark-type conjectures which predict a deep relationship between certain analytic and arithmetic objects attached to (finite Galois extensions of) number fields.

More precisely, Stark-type conjectures give conjectural information about the leading term in the Taylor expansion of Artin L-series at integer values r. These conjectures have been introduced (in the cases $r = 0, 1$) by Stark [St71, St75, St76, St80] in the 1970s and later extended by Tate [Ta84]. An analogue at negative integers has been formulated by Gross in 1979, but published much later [Gr05]. Further refinements for abelian extensions are, amongst others, due to Brumer [Ta84, Chapitre IV], Rubin [Ru96], Popescu [Po02], Coates and Sinnott [CS74] and Snaith [Sn06].

It is often known and in general expected that all these conjectures are implied by the equivariant Tamagawa number conjecture (ETNC) as formulated by Burns and Flach [BF01] (in the case of so-called Tate motives). Hence this conjecture might be seen as the central conjecture in the field. The ETNC can also be related to classical Galois module theory, and in particular to the much studied 'Ω-conjectures' of Chinburg [Ch83, Ch85]. Note that the ETNC is a refinement of the Tamagawa number conjecture of Bloch and Kato [BK90], and so certain cases of the ETNC are also known as the Bloch-Kato conjecture.

The main focus in this thesis is on non-abelian Galois extensions. The articles [1, 2] provide the necessary tools of noncommutative algebra which are frequently used in the other four articles. The articles [3, 4] generalize conjectures attached to abelian extensions of number fields (as Brumer's conjecture, the Brumer-Stark conjecture, the Coates-Sinnott conjecture and Snaith's conjecture) to arbitrary finite Galois extensions of number fields. We also study the interplay with other conjectures and in particular with the ETNC, thereby giving new insights in the abelian case as well. Finally, in [5, 6] we prove refined non-abelian Stark-type conjectures in many cases. This includes conjectures of the author [3, 4] and of Burns [Bu11] as well as new cases of the ETNC for Tate motives. This will also lead to new results for abelian extensions.

The analytic class number formula

Let L be a number field. The Dedekind zeta function of L is first defined for complex numbers s with real part $\Re(s) > 1$ by the absolutely convergent series

$$\zeta_L(s) = \sum_{\mathfrak{A} \subset \mathcal{O}_L} N(\mathfrak{A})^{-s},$$

where the sum runs through all non-zero ideals $\mathfrak{A}$ of the ring of integers $\mathcal{O}_L$ of L, and $N(\mathfrak{A}) = [\mathcal{O}_L : \mathfrak{A}]$ denotes the absolute norm of the ideal $\mathfrak{A}$. In the case $L = \mathbb{Q}$ this definition reduces to that of the famous Riemann zeta function. Erich Hecke first proved that $\zeta_L(s)$ extends to a meromorphic function defined for all $s \in \mathbb{C}$ with only one simple pole at $s = 1$. The residue is given by the analytic class number formula which we now recall.

Let r_1 and r_2 be the number of real embeddings and the number of pairs of complex embeddings of L, respectively. We consider the following arithmetic invariants of the number field L:

- the class number h_L which is the order of the class group cl_L of L;
- the regulator R_L of L, a positive real number related to units;
- the number w_L of roots of unity in L;
- the discriminant d_L of the extension $L/\mathbb{Q}$.

The analytic class number formula then states that

$$\lim_{s \to 1}(s-1)\zeta_L(s) = \frac{2^{r_1} \cdot (2\pi)^{r_2} \cdot h_L \cdot R_L}{w_L \cdot \sqrt{|d_L|}}$$

This formula might be seen as the prototype of the results and conjectures which we treat in this thesis: The left hand side is of an analytic nature, whereas the right hand side is made up of arithmetic invariants of L.

The Dedekind zeta function satisfies a functional equation relating its values at s and $1-s$. More precisely, let us define

$$\begin{aligned} \Gamma_{\mathbb{R}}(s) &:= \pi^{-s/2}\Gamma(s/2) \\ \Gamma_{\mathbb{C}}(s) &:= 2(2\pi)^{-s}\Gamma(s), \end{aligned}$$

where $\Gamma(s)$ denotes the usual Gamma function. Then the function

$$\Lambda_L(s) := |d_L|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_L(s)$$

satisfies the functional equation

$$\Lambda_L(s) = \Lambda_L(1-s).$$

Unwinding some familiar properties of the Gamma function one finds that $\zeta_L(s)$ has a zero of order $r := r_1 + r_2 - 1$ at $s = 0$, and the leading term $\zeta_L^*(0)$ of the Taylor expansion at $s = 0$ equals

$$\zeta_L^*(0) = \lim_{s \to 0} s^{-r}\zeta_L(s) = -\frac{h_L \cdot R_L}{w_L}.$$

Note that by Dirichlet's unit theorem the integer r is equal to the rank of the unit group $\mathcal{O}_L^\times$ of $\mathcal{O}_L$. Another interesting observation is that

$$\frac{w_L}{R_L}\zeta_L^*(0) \in \mathbb{Z}$$

and annihilates the class group.

Artin L-series

Now let L/K be a finite Galois extension of number fields with Galois group G. Then G acts on the class group cl_L of L and we wish to consider the Galois module structure of cl_L. For instance, one may ask if we can use Artin L-series rather than just the Dedekind zeta function to construct non-trivial annihilators of the class group.

For this let S be a finite set of places of K containing the set S_∞ of all archimedean places. We recall the definition of the S-truncated Artin L-series attached to a complex valued character χ of G. For every (finite) place v of K we choose a place w of L above v and write G_w and I_w for the decomposition group at w and inertia subgroup at w, respectively. We choose a lift $\phi_w \in G_w$ of the Frobenius automorphism at w. Let V_χ be a $\mathbb{C}[G]$-module with character χ. Then ϕ_w acts on $V_\chi^{I_w}$, the submodule of V_χ that is invariant under the action of I_w. For complex s with $\Re(s) > 1$ the S-truncated Artin L-series is given by the Euler product

$$L_S(s, \chi) = \prod_{v \notin S} \det(1 - N(v)^{-s}\phi_w \mid V_\chi^{I_w})^{-1}.$$

Note that a finite place v corresponds to a prime ideal $\mathfrak{p}_v$ in the ring of integers $\mathcal{O}_K$ of K, and $N(v) := N(\mathfrak{p}_v)$. In particular, if $S = S_\infty$ and $L = K$ (and χ is the trivial character), this recovers the Euler product of the Dedekind zeta function $\zeta_L(s)$.

An example

As an example suppose that K is totally real and that L is a totally complex quadratic extension of K. Then G is a cyclic group of order two and we let $j \in G$ be its generator (complex conjugation). There is only one non-trivial irreducible complex character χ, and χ maps j to -1. Let us denote the roots of unity in L by μ_L and put $Q := [\mathcal{O}_L^\times : \mathcal{O}_K^\times \mu_L]$. Then Q equals 1 or 2 by [Wa82, Theorem 4.12] and

$$\frac{R_L}{R_K} = \frac{2^{r_2 - 1}}{Q}$$

by [Wa82, Proposition 4.16]. It now follows easily from the class number formulae for L and K and some basic properties of Artin L-series that the value of $L_{S_\infty}(\chi, s)$ at $s = 0$ is given by

$$L_{S_\infty}(\chi, 0) = \frac{2^{r_2} \cdot h_L^-}{w_L \cdot Q}, \qquad (1)$$

where $h_L^- := h_L/h_K$ denotes the relative class number. Ignoring 2-parts, the class group of L decomposes into a 'plus' and a 'minus' part (denoted by $\mathrm{cl}_L^\pm$) upon which j acts as 1 and -1, respectively. Then $h_L^- = |\mathrm{cl}_L^-|$ (up to powers of 2) and formula (1) tells us that

$$\frac{1-j}{2} \cdot w_L \cdot L_{S_\infty}(\chi, 0) \in \mathbb{Z}[\tfrac{1}{2}][G]$$

and annihilates the class group (away from its 2-part). Note that on plus parts we merely say that zero is an annihilator.

Equivariant L-values

We now return to the more general situation, where L/K is an arbitrary finite Galois extension of number fields with Galois group G. Let us denote the center of a ring Λ by $\zeta(\Lambda)$. There is a canonical isomorphism $\zeta(\mathbb{C}[G]) \simeq \prod_\chi \mathbb{C}$, where the sum runs through all absolutely irreducible characters of G. We define the S-truncated equivariant Artin L-series to be the meromorphic $\zeta(\mathbb{C}[G])$-valued function

$$L_S(s) := (L_S(\chi, s))_\chi.$$

The value $L_S(0)$ is non-zero only when K is totally real and L is totally complex. For simplicity, we will also assume that L is a CM-field. This means that L is a quadratic extension of a totally real field L^+. Then L^+ is the unique maximal real subfield of L and complex conjugation induces an automorphism j on L that is central in G. If T is a second finite set of places of K such that $S \cap T = \emptyset$, we define $\delta_T(s) := (\delta_T(s,\chi))_\chi$, where $\delta_T(s,\chi) = \prod_{v \in T} \det(1 - N(v)^{1-s}\phi_w^{-1} | V_\chi^{I_w})$. We put

$$\theta_S^T := \delta_T(0) \cdot L_S(0)^\sharp \in \zeta(\mathbb{C}[G]),$$

where $^\sharp : \mathbb{C}[G] \to \mathbb{C}[G]$ denotes the anti-involution that sends each $g \in G$ to its inverse. If T is empty, we will drop it from the notation. By a result of Siegel [Si70] one knows that

$$\theta_S^T \in \zeta(\mathbb{Q}[G]), \tag{2}$$

and this is equivalent to Stark's conjecture for those irreducible characters χ of G for which $L_S(\chi, 0)$ does not vanish.

Brumer's conjecture

Now suppose that S contains all places of K that ramify in L. Then it was independently shown in [Ba77, Ca79, DR80] that for *abelian* G one has

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S \subseteq \mathbb{Z}[G], \tag{3}$$

where $\mathrm{Ann}_R(M)$ denotes the R-annihilator ideal of an R-module M. Now we are ready to state Brumer's conjecture.

Conjecture (Brumer)**.** *Let L/K be an abelian CM-extension of number fields and let S be a finite set of places of K containing all archimedean places and all places that ramify in L. Then one has an inclusion*

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{cl}_L).$$

A few remarks are in order:

(1) It is not hard to see that Brumer's conjecture is equivalent to the assertion that θ_S^T annihilates the class group for all finite sets T of places of K for which the following hypothesis $Hyp(S,T)$ is satisfied: We have that $S \cap T = \emptyset$ and no non-trivial root of unity in L is congruent to 1 modulo all primes of L lying above those in T.

(2) In the case $K = \mathbb{Q}$ and a cyclotomic extension L of $\mathbb{Q}$ Brumer's conjecture is just Stickelberger's theorem from the late 19th century.

(3) As an abelian group, the class group decomposes into its p-Sylow subgroups $\mathrm{cl}_L(p)$. Therefore Brumer's conjecture naturally decomposes into 'p-parts'.

(4) There is a large body of evidence in support of Brumer's conjecture; see for instance the work of Wiles [Wi90b] and of Greither [Gr00]. In particular, Greither [Gr07] has shown that the appropriate special case of the ETNC implies the p-part of Brumer's conjecture whenever the p-part of the roots of unity is a cohomologically trivial G-module. For instance, this holds whenever p does not divide the order of the Galois group G or if L contains no primitive p-th root of unity.

(5) Note that the ETNC naturally decomposes into p-parts whenever its 'rationality part' is known. We will exclusively deal with the ETNC for the pair $(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}[G])$, where r is a non-positive integer. Then the rationality part is equivalent to Stark's conjecture for L/K if $r = 0$, and to Gross' conjecture if $r < 0$. The rationality part of the ETNC that is needed for Brumer's conjecture is known by the above result (2) of Siegel.

One may ask if an analogue of Brumer's conjecture can be formulated for arbitrary, not necessarily abelian Galois extensions. We formulate such a conjecture (and also an analogue of the stronger Brumer-Stark conjecture) in [3]. The statement of this conjecture must be more involved than in the abelian case because the inclusion (3) does not hold in general. Thus an element in $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S$ in general does not even act on the class group!

Fitting ideals

The ETNC is formulated for arbitrary, not necessarily abelian Galois extensions L/K. One might expect, and this turns out to be the case, that the ETNC predicts annihilators of the class group in the non-abelian situation as well. However, Greither's approach does not generalize directly because it makes heavily use of Fitting ideals, and the latter are only defined over commutative rings. We request such a notion for rings like the group ring $\mathbb{Z}_p[G]$ of an arbitrary finite group G.

We recall the classical notion of Fitting ideals over commutative rings due to Hans Fitting [Fi36]. Let R be a commutative ring with identity and let M be a finitely presented R-module. If we choose a finite presentation

$$R^a \xrightarrow{h} R^b \twoheadrightarrow M, \tag{4}$$

we may identify the homomorphism h with an $a \times b$ matrix with entries in R. If $a \geq b$, the (zeroth) Fitting ideal of M over R, denoted by $\mathrm{Fitt}_R(M)$, is defined to be the R-ideal generated by all $b \times b$ minors of the matrix corresponding to h. In the case $a < b$ one puts $\mathrm{Fitt}_R(M) = 0$. A key observation is that $\mathrm{Fitt}_R(M)$ only depends on M, and not on the particular choice of finite presentation h. This notion is now a very important tool in commutative algebra owing to several useful properties. For instance, one always has an inclusion

$$\mathrm{Fitt}_R(M) \subseteq \mathrm{Ann}_R(M), \tag{5}$$

and the Fitting ideal is often much easier to compute than the annihilator ideal. For example, it has good behavior with respect to quotients of R, as well as epimorphisms and direct sums of R-modules. For a full account of the theory we refer the reader to [No76].

Noncommutative Fitting invariants

The notion of noncommutative Fitting invariants is introduced in [1] and further developed in [2]. The class of rings we consider are $\mathfrak{o}$-orders in finite dimensional separable algebras, where $\mathfrak{o}$ is a commutative noetherian complete local domain. The group ring $\mathbb{Z}_p[G]$ may serve as a standard example (in this case, and assuming the restrictive hypothesis that $a = b$ in (4) noncommutative Fitting invariants have already been introduced by Parker [Pa07]). A second interesting example naturally arises in Iwasawa theory: If $G = H \rtimes \Gamma$ is the semi-direct product of a finite group H and a pro-finite group Γ isomorphic to $\mathbb{Z}_p$, then we may view the Iwasawa algebra (i.e. the completed group ring) $\mathbb{Z}_p[[G]]$ as an order over the power series ring $\mathbb{Z}_p[[T]]$ as follows. Let γ be a topological generator of Γ. Choose an integer $N > 0$ such that Γ^{p^N} is central in G. Then there is an isomorphism $\mathbb{Z}_p[[\Gamma^{p^N}]] \simeq \mathbb{Z}_p[[T]]$ induced by $\gamma^{p^N} \mapsto 1 + T$, and $\mathbb{Z}_p[[G]]$ becomes a $\mathbb{Z}_p[[T]]$-order via the embedding

$$\mathbb{Z}_p[[T]] \simeq \mathbb{Z}_p[[\Gamma^{p^N}]] \rightarrowtail \mathbb{Z}_p[[G]].$$

Now let Λ be an $\mathfrak{o}$-order in a finite dimensional separable F-algebra A, where $\mathfrak{o}$ is a commutative noetherian complete local domain with field of fractions F. The idea is to replace determinants by reduced norms. We give two examples, why this is not straightforward. The reduced norm is a map $\mathrm{nr} : A \to \zeta(A)$ and extends to matrix rings over A. As it takes values in the center of A, it *seems* to be natural to define the Fitting invariant of a finitely presented (left) Λ-module M to be an ideal in $\zeta(\Lambda)$ generated by reduced norms.

(1) Let $\Lambda = M_2(\mathbb{Z}_p)$ be the ring of 2×2 matrices with entries in $\mathbb{Z}_p$ and consider the finitely presented module $M = 0$. There is a natural finite presentation

$$\Lambda \xrightarrow{\mathrm{id}} \Lambda \twoheadrightarrow 0.$$

Using this presentation we obtain $\mathrm{Fitt}_\Lambda(\mathrm{id}) = \zeta(\Lambda) = \mathbb{Z}_p$. However, if we consider the presentation

$$\begin{array}{ccc} \Lambda e_1 \oplus \Lambda e_2 & \xrightarrow{h} & \Lambda \twoheadrightarrow 0 \\ e_1 & \mapsto & \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \\ e_2 & \mapsto & \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \end{array}$$

then $\mathrm{Fitt}_\Lambda(h) = \langle 15, 24 \rangle_{\mathbb{Z}_p} = 3\mathbb{Z}_p$. We see that $\mathrm{Fitt}_\Lambda(\mathrm{id}) \neq \mathrm{Fitt}_\Lambda(h)$ if $p = 3$, and thus the naive generalization is not well defined.

(2) Let p be an odd prime and let $D_{2p} = \langle x, y \mid x^p = 1 = y^2, yx = x^{-1}y \rangle$ be the dihedral group of order $2p$. Consider the group ring $\Lambda = \mathbb{Z}_p[D_{2p}]$. Then a computation shows that

$$\mathrm{nr}(x + y) = \frac{1}{p} \sum_{g \in D_{2p}} g \notin \zeta(\mathbb{Z}_p[D_{2p}]).$$

So the image of the reduced norm does not even lie in the center of Λ in general.

We define in [1] the Fitting invariant of a finite presentation h to be a certain equivalence class of $\zeta(\Lambda)$-modules generated by reduced norms. We define a partial order on this equivalence classes and show that for a finitely presented Λ-module M there is a unique maximal Fitting invariant which we denote $\mathrm{Fitt}_{\Lambda}^{\max}(M)$. We show that this notion enjoys many (but not all) of the useful properties of the commutative case. To obtain annihilators from $\mathrm{Fitt}_{\Lambda}^{\max}(M)$, one has to multiply by a certain ideal $\mathcal{H}(\Lambda)$ of $\zeta(\Lambda)$. If Λ is maximal or commutative, then $\mathcal{H}(\Lambda) = \zeta(\Lambda)$, but in general $\mathcal{H}(\Lambda)$ is a proper ideal of $\zeta(\Lambda)$. We call $\mathcal{H}(\Lambda)$ the *denominator ideal* of Λ.

In [2] we determine all p-adic group rings $\mathbb{Z}_p[G]$ for which $\mathcal{H}(\mathbb{Z}_p[G]) = \zeta(\mathbb{Z}_p[G])$; namely, this is the case if and only if p does not divide the order of the commutator subgroup of G. A similar result holds for Iwasawa algebras. We also develop general lower bounds for denominator ideals in certain cases. Even in the group ring case, however, it remains an open problem to determine $\mathcal{H}(\Lambda)$ explicitly.

We denote by $\mathcal{I}(\Lambda)$ the $\zeta(\Lambda)$-submodule of $\zeta(A)$ generated by the elements $\mathrm{nr}(H)$, $H \in M_n(\Lambda)$, $n \in \mathbb{N}$. Then $\mathcal{I}(\Lambda)$ is in fact a commutative ring and $\mathcal{H}(\Lambda) \cdot \mathcal{I}(\Lambda) \subseteq \zeta(\Lambda)$.

A non-abelian generalization of Brumer's conjecture

We now give the non-abelian analogue of Brumer's conjecture. For this we note that (in contrast to Fitting invariants) one can define $\mathcal{H}(\Lambda)$ and $\mathcal{I}(\Lambda)$ in greater generality; in particular, one can define $\mathcal{H}(\mathbb{Z}[G])$ and $\mathcal{I}(\mathbb{Z}[G])$ for every finite group G.

Conjecture (Non-abelian Brumer). *Let L/K be a Galois CM-extension of number fields with Galois group G and let S be a finite set of places of K containing all archimedean places and all places that ramify in L. Then for all sets T such that $Hyp(S,T)$ is satisfied we have $\theta_S^T \in \mathcal{I}(\mathbb{Z}[G])$ and*

$$\mathcal{H}(\mathbb{Z}[G]) \cdot \theta_S^T \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{cl}_L).$$

We show that this conjecture is implied by the ETNC (in many cases) following Greither's approach, but using our more general notion of Fitting invariant.

Choose a maximal $\mathbb{Z}$-order $\mathfrak{M}(G)$ in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$. Then $\zeta(\mathfrak{M}(G))$ and the central conductor

$$\mathcal{F}(\mathbb{Z}[G]) := \{x \in \zeta(\mathfrak{M}(G)) \mid x\mathfrak{M}(G) \subseteq \mathbb{Z}[G]\}$$

do not depend on the choice of maximal order. We have inclusions

$$\mathcal{I}(\mathbb{Z}[G]) \subseteq \zeta(\mathfrak{M}(G)), \quad \mathcal{F}(\mathbb{Z}[G]) \subseteq \mathcal{H}(\mathbb{Z}[G]).$$

Replacing $\mathcal{I}(\mathbb{Z}[G])$ by $\zeta(\mathfrak{M}(G))$ and replacing $\mathcal{H}(\mathbb{Z}[G])$ by $\mathcal{F}(\mathbb{Z}[G])$ we therefore obtain a weak version of the (non-abelian) Brumer conjecture. This weak version has meanwhile been studied in some detail by Nomura [No].

We point out that Burns [Bu11] has recently presented a universal theory of refined Stark conjectures. In particular, the Galois group G may be non-abelian, and he uses leading terms rather than values of Artin L-functions to construct conjectural nontrivial annihilators of the class group. His conjecture thereby further extends our non-abelian generalization of Brumer's conjecture. He also shows that his conjecture is implied by the ETNC.

Negative integers

Using L-values at integers $r < 0$, one can define higher Stickelberger elements $\theta_S^T(r)$. Coates and Sinnott [CS74] conjectured that these elements can be used to construct annihilators of the higher K-groups $K_{-2r}(\mathcal{O}_L)$. As Burns has done in the case $r = 0$, we use leading terms rather than values at negative integers to formulate in [4] a conjecture on the annihilation of higher K-groups. This simultaneously generalizes the Coates-Sinnott conjecture and a conjecture of Snaith [Sn06].

The Quillen-Lichtenbaum conjecture relates K-groups to étale cohomology, predicting that for all odd primes p, integers $r < 0$ and $i = 0, 1$ the canonical p-adic Chern class maps

$$K_{i-2r}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow H^{2-i}_{\text{ét}}(\mathcal{O}_L[1/p], \mathbb{Z}_p(1-r))$$

constructed by Soulé [So79] are isomorphisms. Following fundamental work of Voevodsky and Rost, Weibel [We09] has completed the proof of the Milnor-Bloch-Kato conjecture which relates Milnor K-theory to étale cohomology and implies the Quillen-Lichtenbaum conjecture. In this way, one obtains a cohomological version of the conjecture on the annihilation of higher K-groups.

We show in [4] that the cohomological version of the conjecture is implied by the ETNC for the pair $(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}[G])$. For instance, this leads to an unconditional proof of a maximal order variant of (a non-abelian generalization of) the Coates-Sinnott conjecture.

Iwasawa theory

Fix an odd prime p. In [5] we generalize and further extend work of Greither and Popescu [GP] to the non-abelian setting. More precisely, we use Iwasawa theory to show that the p-parts of our non-abelian generalizations of the Brumer, the Brumer-Stark and the Coates-Sinnott conjecture hold whenever the set S contains all p-adic places and Iwasawa's μ-invariant vanishes. The latter condition is conjecturally always true and holds, for instance, when L is an arbitrary Galois p-extension of an absolutely abelian number field. In the case of the Coates-Sinnott conjecture we in fact show a stronger statement which turns out to be a reformulation of the appropriate special case of the ETNC.

The main ingredient of the proof is the equivariant Iwasawa main conjecture for totally real fields. Building on the fundamental results of Wiles [Wi90a], this is now a theorem if $\mu = 0$ thanks to recent work of Ritter and Weiss [RW11] and of Kakde [Ka13]. However, the formulations of the main conjecture used by Greither and Popescu, by Ritter and Weiss and by Kakde are quite different. In fact, the formulation of Greither and Popescu is restricted to abelian extensions. We give a non-abelian analogue and then show that all three formulations are equivalent. This fills an annoying gap in the literature. Note that the comparison between Kakde's approach and the approach of Ritter and Weiss has independently been studied in detail by Venjakob [Ve13].

Unconditional results

Finally, in [6] we (unconditionally!) prove many new cases of the ETNC, of Burns' conjecture and further Stark-type conjectures. The main tool in this paper is relative

K-theory and a subtle analysis of the structure of p-adic group rings. Our results build on work of Burns and Greither [BG03] (see also Huber and Kings [HK03], Benois and Nguyen Quang Do [BN02]), Flach [Fl11], Burns and Flach [BF06], Ritter and Weiss [RW97], Bley [Bl06] and others.

We list some of the results that we achieve in [6].

(1) Let $L/\mathbb{Q}$ be a Galois extension with $G = \mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{Aff}(q)$, where $q = l^n$ is a prime power and $\mathrm{Aff}(q)$ denotes the group of affine transformations on $\mathbb{F}_q$, the finite field with q elements. Then Burns' conjecture holds for $L/\mathbb{Q}$ (up to a factor 2 if $l = 2$). In fact, the p-part of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}[G])$ holds for all $p \neq l$ if $r = 0$ and for all odd p if $r < 0$ is odd and L is totally real.

(2) Fix a natural number n. Then there is an infinite family of Galois extensions L/K with Galois group G cyclic of order n and $K/\mathbb{Q}$ non-abelian (in fact non-Galois) such that the ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ holds. Note that the only known examples L/K with $K/\mathbb{Q}$ non-abelian for which the ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ is known to hold have been either trivial, quadratic or biquadratic.

(3) We give an explicit infinite family of finite non-abelian groups (dihedral groups of order $2p$, where $p \equiv 1 \pmod 4$ is a prime) with the property that for each member G there are infinitely many extensions $L/\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$ such that the ETNC for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[G])$ holds. Up until now, the only known family of finite non-abelian groups with this property has been that containing the single group Q_8, the quaternion group of order 8 (this is a result of Burns and Flach [BF03]).

(4) Burns' conjecture holds for an arbitrary Galois extension L/K with Galois group isomorphic to S_3, the symmetric group on 3 letters.

(5) Let $L/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(L/\mathbb{Q}) \simeq D_{12}$, the dihedral group of order 12. Then Burns' conjecture holds.

We finally like to point out that noncommutative Fitting invariants have even more applications to the ETNC. In particular, we refer the interested reader to the author's article [Ni11]. The author decided not to include this article in this thesis as it is the second part (non-abelian analogue) of his dissertation thesis.

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