Non-commutative Fitting invariants and annihilation of class groups

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Abstract

One can associate to each finitely presented module M over a commutative ring R an R-ideal $\mathrm{Fitt}_R(M)$ which is called the (zeroth) Fitting ideal of M over R and which is an important natural invariant of M. We generalize this notion to $\mathfrak{o}$-orders in separable algebras, where $\mathfrak{o}$ is a complete commutative noetherian local ring. As an application we construct annihilators of class groups assuming the validity of the Equivariant Tamagawa Number Conjecture for a certain motive attached to a Galois CM-extension of number fields.

Let R be a commutative ring with identity and M a finitely presented R-module. If we choose a presentation

$$R^a \xrightarrow{h} R^b \twoheadrightarrow M,$$

we can identify the homomorphism h with an $a \times b$ matrix with entries in R. If $a \geqslant b$, the (zeroth) Fitting ideal of M over R, denoted by $\mathrm{Fitt}_R(M)$, is defined to be the R-ideal generated by all $b \times b$ minors of the matrix corresponding to h. If $a < b$, one puts $\mathrm{Fitt}_R(M) = 0$. This notion was introduced by H. Fitting [Fi36] and became a very important tool in commutative algebra. For example, it can be used to detect annihilators, since $\mathrm{Fitt}_R(M)$ is always contained in the R-annihilator of M. We refer the reader to [No76] for a self-contained account of the theory.

Let A be a separable algebra over a field K and Λ an $\mathfrak{o}$-order in A, where $\mathfrak{o}$ is a complete commutative noetherian local ring with field of quotients K. We will assume once and for all that Λ is finitely generated as an $\mathfrak{o}$-module. We denote by $\zeta(A)$ resp. $\zeta(\Lambda)$ the center of A resp. Λ. Given a Λ-left[1] module M which admits a finite presentation

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M$$

we will define the Fitting invariant $\mathrm{Fitt}_\Lambda(h)$ of h over Λ to be an equivalence class of a certain $\zeta(\Lambda)$-submodule of $\zeta(A)$ using reduced norms. We will call $\mathrm{Fitt}_\Lambda(h)$ a Fitting invariant of M over Λ. In general, this notion depends on the chosen presentation h, but the assumption on $\mathfrak{o}$ being a complete commutative noetherian local ring allows us to obtain a relationship between two Fitting invariants of M; for this, we will make use of the fact that each finitely generated Λ-module has a projective cover.

As in the commutative case, Fitting invariants have interesting properties, especially concerning annihilation. We will see that there is a natural choice among all Fitting invariants of M if M admits a finite presentation such that $a = b$. Thus we obtain a well defined object $\mathrm{Fitt}_\Lambda(M)$ in this case. We define a partial order on Fitting invariants and, if the integral closure of $\mathfrak{o}$ in K is finitely generated as an $\mathfrak{o}$-module, we obtain a distinguished Fitting invariant $\mathrm{Fitt}_\Lambda^{\max}(M)$ of M

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[1] if nothing else is stated, all occurring modules are regarded as left modules

over Λ which is maximal with respect to this order. A first attempt to Fitting invariants over not necessarily commutative group rings of a finite group was given by A. Parker in his Ph.D. Thesis [Pa] essentially assuming that $a = b$. Note that this is the case which arises most often in arithmetic contexts. We will see that our definition is compatible with Parker's.
Now let L/K be a Galois CM-extension of number fields with Galois group G and p an odd prime such that the p-power roots of unity of L are cohomologically trivial as a G-module. Assuming the validity of the Equivariant Tamagawa Number Conjecture (ETNC) for the corresponding motive attached to L/K, we will construct elements of $\zeta(\mathbb{Z}_p G)$ which annihilate the p-part of the ideal class group cl_L of L. On the one hand, this generalizes a result of Greither [Gr07], where G is assumed to be abelian. On the other hand, the ETNC predicts more annihilators than the (unconditional) annihilators constructed by Burns and Johnston [BJ]. But that is what one expects, since the assumptions made in loc.cit. are adapted to ensure the validity of the Strong Stark Conjecture which is considerably weaker than the ETNC.

1. Preliminaries

1.0.1 *K-theory* Let Λ be a left noetherian ring with 1 and $\mathrm{PMod}(\Lambda)$ the category of all finitely generated projective Λ-modules. We write $K_0(\Lambda)$ for the Grothendieck group of $\mathrm{PMod}(\Lambda)$, and $K_1(\Lambda)$ for the Whitehead group of Λ which is the abelianized infinite general linear group. If S is a multiplicatively closed subset of the center of Λ which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all finitely generated S-torsion Λ-modules of finite projective dimension by $K_0 S(\Lambda)$. Writing Λ_S for the ring of quotients of Λ with denominators in S, we have the following Localization Sequence (cf. [CR87], p. 65)

$$K_1(\Lambda) \to K_1(\Lambda_S) \xrightarrow{\partial} K_0 S(\Lambda) \xrightarrow{\rho} K_0(\Lambda) \to K_0(\Lambda_S). \qquad (1)$$

In the special case where Λ is an $\mathfrak{o}$-order and S is the set of all nonzerodivisors of $\mathfrak{o}$, we also write $K_0 T(\Lambda)$ instead of $K_0 S(\Lambda)$. Moreover, we denote the relative K-group corresponding to a ring homomorphism $\Lambda \to \Lambda'$ by $K_0(\Lambda, \Lambda')$ (cf. [Sw68]). Then we have a Localization Sequence (cf. [CR87], p. 72)

$$K_1(\Lambda) \to K_1(\Lambda') \xrightarrow{\partial_{\Lambda,\Lambda'}} K_0(\Lambda, \Lambda') \to K_0(\Lambda) \to K_0(\Lambda').$$

It is also shown in [Sw68] that we have an isomorphism $K_0(\Lambda, \Lambda_S) \simeq K_0 S(\Lambda)$.

1.0.2 *Reduced norms* Let A be a semi-simple K-algebra and Λ an $\mathfrak{o}$-order in A, where $\mathfrak{o}$ is a noetherian domain with field of quotients K. We decompose A into its simple components

$$A = A_1 \oplus \ldots \oplus A_t,$$

i.e. each A_i is a simple K-algebra and $A_i = Ae_i = e_i A$ with central primitive idempotents e_i, $1 \leqslant i \leqslant t$. Each A_i is isomorphic to an algebra of $n_i \times n_i$ matrices over a skewfield D_i, and $K_i := \zeta(A_i) = \zeta(D_i)$ is a finite field extension of K. Moreover, we denote the Schur index of D_i by s_i, i.e. $[D_i : K_i] = s_i^2$. The reduced norm map

$$\mathrm{nr}_A : A \longrightarrow \zeta(A) = K_1 \oplus \ldots \oplus K_t$$

is defined componentwise (cf. [Re75], Ch. 9b) and extends to matrix rings over A in the obvious way and hence induces a map $K_1(A) \longrightarrow \zeta(A)^\times$ which we also denote by nr_A. If K is a global field, the image $\mathrm{nr}_A(K_1(A))$ is described explicitly by the Hasse-Schilling-Maass Theorem (cf. [Re75], Th. 33.15) and we will denote $\mathrm{nr}_A(K_1(A))$ for any A by $\zeta(A)^{\times +}$.
Let L be a subfield of either $\mathbb{C}$ or $\mathbb{C}_p$ for some prime p and let G be a finite group. In the case where A is the group ring LG the reduced norm map is always injective. If in addition $L = \mathbb{R}$, there exists

a canonical map $\hat{\partial}_G : \zeta(\mathbb{R}G)^\times \to K_0(\mathbb{Z}G, \mathbb{R}G)$ such that the restriction of $\hat{\partial}_G$ to $\zeta(\mathbb{R}G)^{\times +}$ equals $\partial_{\mathbb{Z}G,\mathbb{R}G} \circ \mathrm{nr}_{\mathbb{R}G}^{-1}$. This map is called the extended boundary homomorphism and was introduced by Burns and Flach [BF01].
We return to the more general case above, but we assume in addition that $\mathfrak{o}$ is integrally closed. We can choose a maximal $\mathfrak{o}$-order Λ' in A which contains Λ; the reduced norm maps Λ in general not into $\zeta(\Lambda)$, but into $\zeta(\Lambda') = \mathfrak{o}_1 \oplus \ldots \oplus \mathfrak{o}_t$, where $\mathfrak{o}_i$ denotes the integral closure of $\mathfrak{o}$ in K_i. This turns out to be the reason that we can not expect to define a Fitting invariant contained in $\zeta(\Lambda)$. Moreover, it leads us to the following definition. We denote the set of all $m \times n$ matrices with entries in a ring R by $M_{m\times n}(R)$ and in the case $m = n$ the group of all invertible elements of $M_{n\times n}(R)$ by $\mathrm{Gl}_n(R)$.

Definition 1.1. Let $\mathfrak{o}$ be a noetherian domain and let N and M be two $\zeta(\Lambda)$-submodules of an $\mathfrak{o}$-torsionfree $\zeta(\Lambda)$-module. Then N and M are called $\mathrm{nr}(\Lambda)$-*equivalent* if there exists an integer n and an invertible matrix $U \in \mathrm{Gl}_n(\Lambda)$ such that $N = \mathrm{nr}_A(U) \cdot M$. We denote the corresponding equivalence class by $[N]_{\mathrm{nr}(\Lambda)}$.

Remark 1.
- i) Of course, we can replace $U \in \mathrm{Gl}_n(\Lambda)$ by $U \in K_1(\Lambda)$ in the above definition.
- ii) If A is commutative or if $\mathfrak{o}$ is integrally closed and Λ is maximal, $\mathrm{nr}(\Lambda)$-equivalence is just equality.

Example. Let p be a prime and K be a finite extension of $\mathbb{Q}_p$. We denote by $\mathfrak{o}_K$ the ring of integers of K and choose a proper subring $\mathfrak{o}$ of $\mathfrak{o}_K$ of finite index m. Let A (resp. Λ') be the algebra of all 2×2-matrices with entries in K (resp. $\mathfrak{o}_K$) such that Λ' is a maximal $\mathfrak{o}_K$-order as well as a maximal $\mathfrak{o}$-order in A. We define an $\mathfrak{o}_K$-submodule of $\mathfrak{o}_K$ by

$$\Gamma := \{x \in \mathfrak{o}_K | x \cdot \mathfrak{o}_K \subset \mathfrak{o}\}.$$

Since Γ contains $m \cdot \mathfrak{o}_K$, it is of finite index in $\mathfrak{o}_K$, and hence

$$\Lambda := \left\{ \begin{pmatrix} r & \gamma \\ \gamma' & x \end{pmatrix} \mid r \in \mathfrak{o}, \gamma, \gamma' \in \Gamma, x \in \mathfrak{o}_K \right\}$$

is an $\mathfrak{o}$-order in A contained in Λ'. For any $x \in \mathfrak{o}_K$, $x \notin \mathfrak{o}$ the element $\lambda_x = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in \Lambda$ has reduced norm $x \notin \zeta(\Lambda) = \mathfrak{o}$. Finally, λ_x is invertible in Λ if and only if x is a unit of K. In this case, the $\mathfrak{o}$-submodule of $\mathfrak{o}_K$ generated by x is $\mathrm{nr}(\Lambda)$-equivalent to $\mathfrak{o}$. To give an explicit example, choose $p = 2$, $K = \mathbb{Q}_2(\sqrt{2})$ and $\mathfrak{o} = \mathbb{Z}_2[2\sqrt{2}]$. Then $x = 1 + \sqrt{2}$ is a unit of K which does not lie in $\mathfrak{o}$.

We can define a partial order on $\mathrm{nr}(\Lambda)$-equivalence classes:

Definition 1.2. Let N and M be two finitely generated $\mathfrak{o}$-torsionfree $\zeta(\Lambda)$-modules. Then we say that N is $\mathrm{nr}(\Lambda)$-contained in M (and write $[N]_{\mathrm{nr}(\Lambda)} \subset [M]_{\mathrm{nr}(\Lambda)}$) if for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$ there exists $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N' \subset M'$.

To check antisymmetry, let $[N]_{\mathrm{nr}(\Lambda)} \subset [M]_{\mathrm{nr}(\Lambda)} \subset [N]_{\mathrm{nr}(\Lambda)}$. Then there is an $U \in K_1(\Lambda)$ and $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N \subset M' \subset \mathrm{nr}(U) \cdot N$. Assume that one of the inclusions is proper and hence $\mathrm{nr}(U)^i \cdot N \subsetneq \mathrm{nr}(U)^{i+1} \cdot N$ for all $i \in \mathbb{N}$. But since $\mathrm{nr}(U)$ is integral over $\mathfrak{o}$, there is a natural number n such that $\mathrm{nr}(U)^n \cdot N \subset \bigcup_{i=0}^{n-1} \mathrm{nr}(U)^i \cdot N$, a contradiction. Hence $[N]_{\mathrm{nr}(\Lambda)} = [M]_{\mathrm{nr}(\Lambda)}$.

Remark 2. It suffices to check the above property for one $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$. To see this assume that $N_0 \subset M_0$ for some $M_0 \in [M]_{\mathrm{nr}(\Lambda)}$ and let $N' \in [N]_{\mathrm{nr}(\Lambda)}$ be arbitrary. Then $N' = \mathrm{nr}_A(U) \cdot N_0$ for some $U \in K_1(\Lambda)$ and hence $N' \subset \mathrm{nr}_A(U) \cdot M_0 \in [M]_{\mathrm{nr}(\Lambda)}$.

Remark 3. Let $e \in A$ be a central idempotent. Suppose that N and M are two $\mathfrak{o}$-torsionfree $\zeta(\Lambda)$-modules which are $\mathrm{nr}(\Lambda)$-equivalent. Then eN and eM are $\mathrm{nr}(\Lambda e)$-equivalent $\zeta(\Lambda e)$-modules, since for $U \in K_1(\Lambda)$ we have $Ue \in K_1(\Lambda e)$ and $\mathrm{nr}_A(U)e = \mathrm{nr}_{Ae}(Ue)$. Hence $e[N]_{\mathrm{nr}(\Lambda)} := [eN]_{\mathrm{nr}(\Lambda e)}$ is well defined.

We will say that x is contained in $[N]_{\mathrm{nr}(\Lambda)}$ if there is a $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$ such that $x \in N_0$. Accordingly, we say that $x_1, \ldots, x_n$ generate $[N]_{\mathrm{nr}(\Lambda)}$ if they generate N_0 for some $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$.

2. Projective resolutions

Let Λ be a semiperfect ring with radical $\mathfrak{r} := \mathrm{rad}(\Lambda)$, i.e. $\Lambda/\mathfrak{r}$ is a semi-simple artinian ring and every idempotent in $\Lambda/\mathfrak{r}$ is the image of an idempotent in Λ. Let M be a finitely generated Λ-module; then M has a projective cover, say $f_0 : P_0 \twoheadrightarrow M$ (cf. [CR81], Ch. 6C for basic facts on projective covers) and we will assume that the kernel of f_0 is again finitely generated. For instance, this happens if Λ is left artinian or if Λ is an $\mathfrak{o}$-algebra, finitely generated as an $\mathfrak{o}$-module, where $\mathfrak{o}$ is a complete commutative noetherian local ring. Now we can choose a projective cover $f_1 : P_1 \twoheadrightarrow \ker(f_0)$ and proceeding in this way yields a projective covering $\mathcal{P}_M$ of M:

$$\mathcal{P}_M : \ldots \to P_i \xrightarrow{f_i} P_{i-1} \xrightarrow{f_{i-1}} \ldots \xrightarrow{f_1} P_0 \overset{f_0}{\twoheadrightarrow} M. \tag{2}$$

Note that this covering is unique up to isomorphism and that each f_i maps into $\mathfrak{r}P_{i-1}$. We call a complex trivial if it is the direct sum of complexes of the form

$$\ldots \to 0 \to 0 \to P \xrightarrow{\mathrm{id}_P} P \to 0 \to 0 \to \ldots$$

with projective P. We now prove the following result which is a generalization of [Ei95], Th. 20.2.

Proposition 2.1. *Let Λ be a semiperfect ring and M a finitely generated Λ-module which admits a projective covering $\mathcal{P}_M$. Then any projective resolution of M is isomorphic to the direct sum of $\mathcal{P}_M$ and a trivial complex.*

Proof. Let $\mathcal{F}_M : \ldots \to Q_n \to \ldots \xrightarrow{h_1} Q_0 \overset{h_0}{\twoheadrightarrow} M$ be a projective resolution of M. Since P_0 and Q_0 are projective, there are homomorphisms $g_0 : Q_0 \to P_0$ and $s_0 : P_0 \to Q_0$ such that $f_0 g_0 = h_0$ and $h_0 s_0 = f_0$. We observe that

$$f_0(1 - g_0 s_0) = f_0 - h_0 s_0 = 0.$$

Hence $1 - g_0 s_0$ maps P_0 into $\ker(f_0) \subset \mathfrak{r}P_0$; thus Nakayama's Lemma implies $g_0 s_0(P_0) = P_0$ and hence we find a map $t_0 : P_0 \to P_0$ such that $g_0 s_0 t_0 = \mathrm{id}_{P_0}$. We see that g_0 is surjective (which was clear from the outset by definition of a projective cover) and replacing s_0 by $s_0 t_0$ we may assume that s_0 is a section of g_0. Proceeding inductively we obtain epimorphisms $g_i : Q_i \twoheadrightarrow P_i$ and sections s_i of g_i such that $h_i s_i = s_{i-1} f_i$. Therefore $\mathcal{F}_M$ is isomorphic to the direct sum of $\mathcal{P}_M$ and a complex with trivial homology which consists only of projective modules. But such a complex is isomorphic to a trivial complex, as can be seen by adjusting the proof of [Ei95], Lemma 20.1. One only has to replace 'free' by 'projective'. □

3. Non-commutative Fitting invariants

Let A be a separable K-algebra and Λ an $\mathfrak{o}$-order in A, where $\mathfrak{o}$ is a complete commutative noetherian local ring with field of quotients K. Let M be a finitely presented Λ-module and choose a presentation

$$\Lambda^a \xrightarrow{h} \Lambda^b \overset{\pi}{\twoheadrightarrow} M. \tag{3}$$

We identify the homomorphism h with the corresponding matrix in $M_{a\times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b\times b$ submatrices of h if $a \geqslant b$. In the case $a = b$ we call (3) a quadratic presentation.

Definition 3.1. We define the Fitting invariant of h over Λ to be

$$\mathrm{Fitt}_\Lambda(h) = \begin{cases} [0]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a < b \\ \left[\langle \mathrm{nr}_A(H) | H \in S(h)\rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a \geqslant b. \end{cases}$$

We call $\mathrm{Fitt}_\Lambda(h)$ a Fitting invariant of M over Λ. If M is a Λ-module which admits a quadratic presentation h we put $\mathrm{Fitt}_\Lambda(M) := \mathrm{Fitt}_\Lambda(h)$.

Hence if $a \geqslant b$, the Fitting invariant of h over Λ is the $\mathrm{nr}(\Lambda)$-equivalence class of the $\zeta(\Lambda)$-submodule of $\zeta(A)$ generated by the reduced norms of all $b \times b$ submatrices of h. Even if the above definition does in general not only depend on the isomorphism class of M, we often suppress the dependency on the presentation h and write $\mathcal{F}_\Lambda(M)$ (or simply $\mathcal{F}(M)$ if Λ is clear from the context) instead of $\mathrm{Fitt}_\Lambda(h)$.
Now let $E_n(\Lambda)$ denote the subgroup of $\mathrm{Gl}_n(\Lambda)$ of all matrices which have reduced norm equal to 1. We have the following

Theorem 3.2. i) *If h_1 and h_2 are two finite presentations of M, then there exist $n \in \mathbb{N}$, a matrix $X \in \mathrm{Gl}_n(\Lambda)$ and finite presentations h_1' and h_2' of M such that* $\mathrm{Fitt}_\Lambda(h_i) = \mathrm{Fitt}_\Lambda(h_i')$ *for $i = 1, 2$ and $h_1' \circ X = h_2'$. If $\mathfrak{o}$ has finite Krull dimension, we can choose $X \in E_n(\Lambda)$.*

ii) *If h_1 and h_2 are quadratic presentations, we have* $\mathrm{Fitt}_\Lambda(h_1) = \mathrm{Fitt}_\Lambda(h_2)$. *In particular,* $\mathrm{Fitt}_\Lambda(M)$ *is well defined.*

iii) *If $\mathfrak{o}$ is integrally closed and Λ is a maximal order, then* $\mathrm{Fitt}_\Lambda(M)$ *is maximal among all Fitting invariants of M over Λ.*

Proof. Since $\mathfrak{o}$ is a complete commutative noetherian local ring, Λ is semiperfect and each finitely generated Λ-module M has a projective covering (2). Given a finite presentation $h_1 : \Lambda^a \to \Lambda^b$ of M, Proposition 2.1 implies that there are projective Λ-modules Q_0 and Q_1 and isomorphisms $\psi_1 : \Lambda^a \simeq P_1 \oplus Q_0 \oplus Q_1$ and $\phi_1 : \Lambda^b \simeq P_0 \oplus Q_0$ such that $h_1 = \phi_1^{-1}(f_1 \oplus \mathrm{id}_{Q_0}|0)\psi_1$.
Now let $h_2 : \Lambda^{a_2} \to \Lambda^{b_2}$ be a second finite presentation of M. If $b_2 < b$, we may replace h_2 by $h_2 \oplus \mathrm{id} : \Lambda^{a_2} \oplus \Lambda^{b-b_2} \to \Lambda^{b_2} \oplus \Lambda^{b-b_2}$ without changing the Fitting invariant of h_2. Note that $h_2 \oplus \mathrm{id}$ is quadratic if h_2 is. So we may assume $b_2 = b$. If likewise $a_2 < a$ (which can not happen if h_1 and h_2 are quadratic), we replace h_2 by $(h_2|0) : \Lambda^{a_2} \oplus \Lambda^{a-a_2} \to \Lambda^{b_2}$ such that we may also assume that $a_1 = a$. As above, there exist isomorphisms $\psi_2 : \Lambda^a \simeq P_1 \oplus Q_0 \oplus Q_1$ and $\phi_2 : \Lambda^b \simeq P_0 \oplus Q_0$ such that $h_2 = \phi_2^{-1}(f_1 \oplus \mathrm{id}_{Q_0}|0)\psi_2$. We finally get $h_1 \circ X = U \circ h_2$, where $X := \psi_1^{-1}\psi_2 \in \mathrm{Gl}_a(\Lambda)$ and $U := \phi_1^{-1}\phi_2 \in \mathrm{Gl}_b(\Lambda)$. By $\mathrm{nr}(\Lambda)$-equivalence we have $\mathrm{Fitt}_\Lambda(U \circ h_2) = \mathrm{Fitt}_\Lambda(h_2)$ and if h_1 and h_2 are quadratic, also $\mathrm{Fitt}_\Lambda(h_1 \circ X) = \mathrm{Fitt}_\Lambda(h_1)$. For arbitrary h_1 and h_2 we have to show that we may assume $\mathrm{nr}(X) = 1$ if $\mathfrak{o}$ has finite Krull dimension. Since Λ has finite stable range by a Theorem of Bass (cf. [CR87], Th. 41.25) in this case, we may write $X = E \cdot \tilde{X}$, where E is a product of elementary matrices and $\tilde{X}$ is of shape

$$\begin{pmatrix} Y_1 & * \\ 0 & Y_2 \end{pmatrix}.$$

Here, Y_1 is a upper triangular matrix whose diagonal consists of ones and $Y_2 \in \mathrm{Gl}_d(\Lambda)$ for some $d \in \mathbb{N}$. Enlarging a if necessary we may assume that there are at least d columns of zeros on the righthand side of the matrix corresponding to h_2. Since the inverse of $\tilde{X}$ is of shape $\begin{pmatrix} Y_1^{-1} & * \\ 0 & Y_2^{-1} \end{pmatrix}$, the equation $h_1 \circ E = h_2 \circ \tilde{X}^{-1}$ shows that we may replace Y_2 by the $d \times d$ identity matrix and end up with $\mathrm{nr}X = \mathrm{nr}\tilde{X} = \mathrm{nr}Y_1 = 1$ as desired.

Now suppose that $\mathfrak{o}$ is integrally closed, Λ is maximal and $\mathrm{Fitt}_\Lambda(M) = \mathrm{Fitt}_\Lambda(\psi)$ for some quadratic presentation ψ of M. Let h be an arbitrary finite presentation of M as in (3). We have to show that $\mathrm{Fitt}_\Lambda(h) \subset \mathrm{Fitt}_\Lambda(\psi)$. We have proven that we may assume that $h = (\psi|0) \circ X$ for some $X \in \mathrm{Gl}_a(\Lambda)$. Hence each $H \in S_b(h)$ is the product $\psi \circ \tilde{X}$ for some $b \times b$ submatrix $\tilde{X}$ of X. Thus $\mathrm{nr}_A(H) = \mathrm{nr}_A(\psi) \cdot \mathrm{nr}_A(\tilde{X}) \in \mathrm{Fitt}_\Lambda(\psi)$, since $\mathrm{nr}_A(\tilde{X}) \in \zeta(\Lambda)$. □

EXAMPLES.

i) Consider the algebra A of 2×2 matrices with entries in $\mathbb{Q}_p$ and the maximal order $\Lambda = M_{2\times 2}(\mathbb{Z}_p)$. The Fitting invariant of the trivial module is clearly $\mathrm{Fitt}_\Lambda(0) = \zeta(\Lambda) = \mathbb{Z}_p$. The map $\psi : \Lambda^2 \to \Lambda$ given by
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \lambda_1 := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \lambda_2 := \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$
is also surjective and thus a finite presentation of 0. But
$$\mathrm{Fitt}_\Lambda(\psi) = \langle \mathrm{nr}(\lambda_1), \mathrm{nr}(\lambda_2) \rangle_{\mathbb{Z}_p} = \langle 15, 24 \rangle_{\mathbb{Z}_p} = 3\mathbb{Z}_p$$
which differs from $\mathrm{Fitt}_\Lambda(0)$ if $p = 3$. We choose $X = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in E_2(\Lambda)$ to see that $\tilde{\psi} = \psi \circ X$ has Fitting invariant $\mathrm{Fitt}_\Lambda(\tilde{\psi}) = \mathbb{Z}_p$.

ii) In general, the Fitting invariant $\mathrm{Fitt}_\Lambda(M)$ is not maximal. Choose an order Λ and $\lambda \in \Lambda$ such that $\mathrm{nr}(\lambda) \notin \zeta(\Lambda)$ (compare the example following remark 1). Then the map $\psi : \Lambda^2 \to \Lambda$ given by $\binom{1}{0} \mapsto 1$, $\binom{0}{1} \mapsto \lambda$ is a finite presentation of 0, and $\mathrm{Fitt}_\Lambda(0)$ is properly contained in $\mathrm{Fitt}_\Lambda(\psi)$. Otherwise both invariants would be $\mathrm{nr}(\Lambda)$-equivalent and hence $\mathrm{Fitt}_\Lambda(\psi)$ generated by one single x, say. But since $\mathrm{nr}(1) = 1$, the generator x has to be a unit of $\zeta(\Lambda)$ and hence $\langle x \rangle = \zeta(\Lambda)$ which does not contain $\mathrm{nr}(\lambda)$.

Now let M be a finitely presented Λ-module and $h_1 : \Lambda^{a_1} \to \Lambda^{b_1}$ and $h_2 : \Lambda^{a_2} \to \Lambda^{b_2}$ two finite presentations of M. By the above theorem we may assume $a_1 = a_2 =: a$ and $b_1 = b_2 =: b$ without changing the Fitting invariants of h_1 and h_2. Moreover, there is an $X \in \mathrm{Gl}_a(\Lambda)$ such that $h_1 \circ X = h_2$. Hence
$$\Lambda^a \oplus \Lambda^a \overset{(h_1|h_2)}{\longrightarrow} \Lambda^b \twoheadrightarrow M$$
is also a finite presentation of M such that $\mathrm{Fitt}_\Lambda((h_1|h_2))$ contains both $\mathrm{Fitt}_\Lambda(h_1)$ and $\mathrm{Fitt}_\Lambda(h_2)$. Now assume that the integral closure $\mathfrak{o}'$ of $\mathfrak{o}$ in K is finitely generated as an $\mathfrak{o}$-module and choose a maximal $\mathfrak{o}'$-order Λ' in A containing Λ. Since $\mathfrak{o}$ is noetherian and $\zeta(\Lambda')$ is finitely generated as an $\mathfrak{o}$-module, the following definition is well defined:

DEFINITION 3.3. Assume that the integral closure of $\mathfrak{o}$ in K is finitely generated as $\mathfrak{o}$-module and let M be a finitely presented Λ-module. Then we define $\mathrm{Fitt}_\Lambda^{\max}(M)$ to be the Fitting invariant of M over Λ which is maximal among all Fitting invariants of M over Λ.

Remark 4. i) Now Theorem 3.2 (iii) states that if $\mathfrak{o}$ is integrally closed and Λ is maximal, we have $\mathrm{Fitt}_\Lambda(M) = \mathrm{Fitt}_\Lambda^{\max}(M)$ for any M which admits a quadratic presentation.

ii) If M admits a quadratic presentation, it is often more natural to work with $\mathrm{Fitt}_\Lambda(M)$ rather than with $\mathrm{Fitt}_\Lambda^{\max}(M)$. Compare for example Proposition 5.3 and 6.3 below.

We set $M_K := K \otimes_{\mathfrak{o}} M$ and define
$$\Upsilon(M) := \{i \in \{1, \ldots, t\} | e_i M_K = 0\},$$
$$e = e(M) := \sum_{i \in \Upsilon(M)} e_i.$$

By remark 3 above, multiplication on $\mathrm{Fitt}_\Lambda(h)$ by an idempotent e' of A is well defined and it is easy to see that $e'\mathrm{Fitt}_\Lambda(h) = \mathrm{Fitt}_{\Lambda e'}(\Lambda e' \otimes_\Lambda h)$.

Lemma 3.4. *Let M be a finitely presented Λ-module and let $e = e(M)$. If h is a finite presentation of M, we have*

$$\mathrm{Fitt}_\Lambda(h) = e\mathrm{Fitt}_\Lambda(h) = \mathrm{Fitt}_{\Lambda e}(\Lambda e \otimes_\Lambda h).$$

Proof. Put $e' = 1-e$ such that we have a decomposition $A = Ae \oplus Ae'$. We show that $e'\mathrm{Fitt}_\Lambda(h) = 0$. Ae' decomposes into simple components $A_i = Ae_i$ with center K_i such that $e'\mathrm{Fitt}_\Lambda(h)$ can be computed via

$$(A_i)^a \xrightarrow{1\otimes he_i} (A_i)^b \twoheadrightarrow e_i M_K.$$

Since $e_i M_K \neq 0$ for each $i \notin \Upsilon(M)$, $1 \otimes he_i$ is not surjective for any simple component A_i and so does $1 \otimes H_i$ for each $H_i \in S_b(he_i)$. But this means that left multiplication on $M_{b\times b}(A_i)$ by H_i is also not surjective and hence $\mathrm{nr}_{A_i}(H_i)$ vanishes, since the K_i-determinant of this multiplication is a power of $\mathrm{nr}_{A_i}(H_i)$. □

Remark 5. In the case $\Lambda = \mathfrak{o}G$, where $\mathfrak{o}$ is the localization or the completion of $\mathbb{Z}$ at a prime p and G is a finite group, Parker [Pa] defines $\mathrm{Fitt}_\Lambda(M) \in \zeta(\Lambda e)^\times$ to be the reduced norm of h assuming the existence of an exact sequence

$$(\Lambda e)^n \overset{h}{\rightarrowtail} (\Lambda e)^n \twoheadrightarrow \Lambda e \otimes_\Lambda M$$

for some n. This definition is well defined modulo $\mathrm{nr}_A(K_1(\Lambda e))$ (cf. loc.cit., Lemma 3.2.1). Hence our definition is compatible with Parker's. Moreover, Lemma 3.4 generalizes loc.cit., Lemma 3.3.2.

We summarize some first properties of Fitting invariants in the following Proposition whose third item generalizes [Pa], Prop. 3.3.3.

Proposition 3.5. *Let M_1, M_2, M_3 be finitely presented Λ-modules and let $\mathrm{Fitt}_\Lambda(h_1)$ resp. $\mathrm{Fitt}_\Lambda(h_3)$ be Fitting invariants of M_1 resp. M_3 over Λ.*

- i) *If $M_1 \twoheadrightarrow M_2$ is an epimorphism, then there exists a finite presentation h_2 of M_2 such that $\mathrm{Fitt}_\Lambda(h_1) \subset \mathrm{Fitt}_\Lambda(h_2)$.*
- ii) *If $M_2 = M_1 \times M_3$, then $\mathrm{Fitt}_\Lambda(h_1 \oplus h_3) = \mathrm{Fitt}_\Lambda(h_1) \cdot \mathrm{Fitt}_\Lambda(h_3)$ is a Fitting invariant of M_2. If in addition M_1 and M_3 admit quadratic presentations, so does M_2 and we have $\mathrm{Fitt}_\Lambda(M_2) = \mathrm{Fitt}_\Lambda(M_1) \cdot \mathrm{Fitt}_\Lambda(M_3)$.*
- iii) *If $M_1 \xrightarrow{\iota} M_2 \twoheadrightarrow M_3$ is an exact sequence of Λ-modules, then there is a Fitting invariant $\mathrm{Fitt}_\Lambda(h_2)$ for M_2 over Λ such that*

$$\mathrm{Fitt}_\Lambda(h_1) \cdot \mathrm{Fitt}_\Lambda(h_3) \subset \mathrm{Fitt}_\Lambda(h_2).$$

If ι is injective and h_3 is a quadratic presentation, we can force the above inclusion to be an equality. If in addition M_1 and M_3 admit quadratic presentations, so does M_2 and we have

$$\mathrm{Fitt}_\Lambda(M_1) \cdot \mathrm{Fitt}_\Lambda(M_3) = \mathrm{Fitt}_\Lambda(M_2).$$

Proof. Let $\Lambda^a \xrightarrow{h_1} \Lambda^b \overset{\pi_1}{\twoheadrightarrow} M_1$ be a finite presentation of M_1. Denote the epimorphism $M_1 \twoheadrightarrow M_2$ by μ and define $\pi_2 = \mu \circ \pi_1 : \Lambda^b \twoheadrightarrow M_2$. Let C be the cokernel of the inclusion $\ker(\pi_1) \rightarrowtail \ker(\pi_2)$ and choose an epimorphism $\Lambda^c \twoheadrightarrow C$ for some $c \in \mathbb{N}$. This map factors through $\ker(\pi_2)$ and we denote the corresponding map by $g : \Lambda^c \to \ker(\pi_2)$. This yields a finite presentation of M_2:

$$\Lambda^a \oplus \Lambda^c \xrightarrow{(h_1|g)} \Lambda^b \overset{\pi_2}{\twoheadrightarrow} M_2$$

We conclude that $S_b(h_1) \subset S_b((h_1|g))$ and get (i). (ii) is clear once we observe that the reduced norm of a matrix H vanishes if H is of shape

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_3 \end{pmatrix},$$

where $H_i \in M_{a_i \times b_i}(\Lambda)$ for $i = 1, 3$ and either $a_1 > b_1$ or $a_3 > b_3$. To see this, let E be a splitting field of A and write $1 \otimes H \in A_E$ as a direct sum of matrices with entries in E. Then each of these matrices is of the same shape as H with a_i, b_i replaced by some multiple of it. The column vectors are linearly dependent such that the determinant of each of these matrices vanishes and hence also $\mathrm{nr}(H) = 0$.
Now we pass to (iii). By (i) we may assume that ι is injective. We choose finite presentations $\Lambda^{a_i} \xrightarrow{h_i} \Lambda^{b_i} \overset{\pi_i}{\twoheadrightarrow} M_i$ for $i = 1, 3$ and construct a finite presentation of M_2 in the following way. The epimorphism π_3 factors through M_2 via a map f_1 and we define $\pi_2 = (\iota \circ \pi_1 | f_1) : \Lambda^{b_1} \oplus \Lambda^{b_3} \twoheadrightarrow M_2$. In a similar manner we construct $h_2 = (h_1 | f_2)$, where f_2 realizes the factorization of h_3 through $\ker(\pi_2)$. Then h_2 corresponds to a matrix of shape

$$\begin{pmatrix} h_1 & * \\ 0 & h_3 \end{pmatrix}.$$

From this we get the desired inclusion. Moreover, if we can choose $a_3 = b_3$, the matrix corresponding to h_3 is quadratic. Hence each $H_2 \in S_{b_2}(h_2)$ has either reduced norm equal to zero or is of shape

$$\begin{pmatrix} H_1 & * \\ 0 & h_3 \end{pmatrix}$$

for some $H_1 \in S_{b_1}(h_1)$. This completes the proof. $\square$

Now let C be a finitely generated $\mathfrak{o}$-torsion Λ-module of projective dimension at most 1 and denote by $[C]$ the corresponding class in $K_0T(\Lambda)$. Then C admits a quadratic presentation if and only if $\rho([C]) = 0$. To see the non-trivial implication choose an epimorphism $\pi : \Lambda^n \twoheadrightarrow C$. The kernel of π is projective, and it is stably isomorphic to Λ^n if and only if $\rho([C]) = 0$. Now replace π by $\pi' = (\pi|0) : \Lambda^n \oplus \Lambda^m \twoheadrightarrow C$ for suitable $m \in \mathbb{N}$ such that $\ker(\pi') = P \oplus \Lambda^m$ is free.
Now assume that C admits a quadratic presentation $\psi : \Lambda^n \rightarrowtail \Lambda^n$. Then the class $[1 \otimes \psi]$of $1 \otimes \psi \in \mathrm{Gl}_n(A)$ in $K_1(A)$ is a preimage of $[C]$ and $\mathrm{Fitt}_\Lambda(C)$ is generated by $\mathrm{nr}_A([1 \otimes \psi])$. Proposition 3.5 (iii) implies that the relative Fitting invariant introduced just below is well defined.

Definition 3.6. Assume that C and C' are two finitely generated $\mathfrak{o}$-torsion Λ-modules of projective dimension at most 1. If $\rho([C] - [C']) = 0$, we choose $x \in K_1(A)$ such that $\partial(x) = [C] - [C']$ and define

$$\mathrm{Fitt}_\Lambda(C : C') := \left[\langle \mathrm{nr}_A(x) \rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)}.$$

Remark 6. If Λ is a group ring $\mathfrak{o}G$ of a finite group G, where $\mathfrak{o}$ is a complete discrete valuation ring, then Swan's Theorem implies that $\rho([C]) = 0$ for any finitely generated $\mathfrak{o}$-torsion Λ-module C of projective dimension at most 1 (cf. [CR81] Th. 32.1). There is also a more general result due to A. Hattori [Ha65], see [CR81], Th. 32.5. A similar statement holds if Λ is the complete group algebra $\mathbb{Z}_p[[G]]$ of a profinite group G which has a p-Sylow subgroup of finite index, see Lemma 6.2 below.

We immediately get

Proposition 3.7. *Let $C' \rightarrowtail C \twoheadrightarrow C''$ be an exact sequence of finitely generated $\mathfrak{o}$-torsion Λ-modules of projective dimension at most 1. If C' (resp. C'') admits a quadratic presentation, we have*

$$\mathrm{Fitt}_\Lambda(C') = \mathrm{Fitt}_\Lambda(C : C'') \text{ (resp. } \mathrm{Fitt}_\Lambda(C'') = \mathrm{Fitt}_\Lambda(C : C')).$$

If C admits a quadratic presentation, then so does $C' \oplus C''$ and we have

$$\mathrm{Fitt}_\Lambda(C) = \mathrm{Fitt}_\Lambda(C' \oplus C'').$$

4. Fitting invariants and Annihilation

Let A be a separable K-algebra and Λ an $\mathfrak{o}$-order in A, where $\mathfrak{o}$ is an integrally closed complete commutative noetherian local ring with field of quotients K. We choose a maximal order Λ' containing Λ. As A decomposes into its simple components $A_i = Ae_i$, $1 \leqslant i \leqslant t$, we have

$$\Lambda' = \Lambda'_1 \oplus \ldots \oplus \Lambda'_t,$$

where $\Lambda'_i = \Lambda' e_i$. Now let $H \in M_{b\times b}(\Lambda)$ and decompose H into

$$H = \sum_{i=1}^{t} H_i \in M_{b\times b}(\Lambda') = \bigoplus_{i=1}^{t} M_{b\times b}(\Lambda'_i).$$

Define $H_i^* = \mathrm{nr}_{A_i}(H_i)H_i^{-1}$ if H_i is invertible over A_i, and $H_i^* = 0$ otherwise. A non-commutative analog of the adjoined matrix is

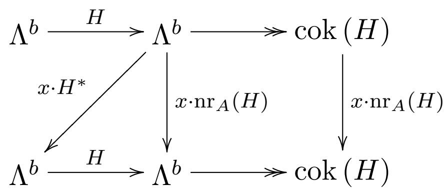
$$H^* := \sum_{i=1}^{t} H_i^*.$$

LEMMA 4.1. *We have $H^* \in M_{b\times b}(\Lambda')$ and $H^*H = HH^* = \mathrm{nr}_A(H)\cdot 1_{b\times b}$.*

Proof. The reduced characteristic polynomial f_i of H_i has coefficients in $\zeta(\Lambda_i) = \mathfrak{o}_i$. Since the constant term of f_i equals $\pm\mathrm{nr}(H_i)$, multiplying the equation $f_i(H_i) = 0$ by H_i^{-1} actually shows that H_i^* is a polynomial in H_i with coefficients in $\mathfrak{o}_i$. The second assertion is clear. □

THEOREM 4.2. *Let M be a finitely presented Λ-module and $h : \Lambda^a \to \Lambda^b$ be a finite presentation of M. Let $x \in \zeta(\Lambda')$ and $H \in S_b(h)$ such that $x \cdot H^* \in M_{b\times b}(\Lambda)$. Then $x\cdot\mathrm{nr}_A(H) \in \zeta(\Lambda)$ annihilates M. In particular, $x \cdot y \in \mathrm{Ann}_\Lambda(M)$ for all $y \in \mathrm{Fitt}_\Lambda(h)$ if $x \cdot \mathrm{nr}_A(U) \cdot H^* \in M_{b\times b}(\Lambda)$ for any $H \in S_b(h)$ and $U \in K_1(\Lambda)$.*

Proof. Since $\mathrm{Fitt}_\Lambda(h)$ is generated by $\mathrm{nr}_A(H)$, $H \in S_b(H)$, it suffices to show that $x \cdot \mathrm{nr}_A(H)$ annihilates M. The cokernel of H surjects onto M and hence the assertion follows from the commutative diagram



□

Now Lemma 4.1 implies

COROLLARY 4.3. *If Λ is maximal or if A is commutative, then M is annihilated by each of its Fitting invariants.*

Let G be a finite group of order n and $\Lambda = \mathfrak{o}G$ the group ring of G, where $\mathfrak{o}$ is a complete discrete valuation ring with field of quotients K. Recall that $KG = \oplus_{i=1}^{t} M_{n_i\times n_i}(D_i)$, where D_i is a skew field of degree s_i^2 over its center K_i with ring of integers $\mathfrak{o}_i$. The central conductor of Λ' over Λ is defined to be $\mathcal{F} := \{x \in \zeta(\Lambda') : x\Lambda' \subset \Lambda\}$ and is explicitly given by

$$\mathcal{F} = \bigoplus_{i=1}^{t} \frac{n}{n_i s_i}\mathcal{D}^{-1}(\mathfrak{o}_i/\mathfrak{o}),$$

where $\mathcal{D}^{-1}(\mathfrak{o}_i/\mathfrak{o})$ denotes the inverse different of $\mathfrak{o}_i$ over $\mathfrak{o}$ (cf. [CR81], Th. 27.13).

Corollary 4.4. *Let $\mathfrak{o}$ be a complete discrete valuation ring and $\Lambda = \mathfrak{o}G$ be the group ring of a finite group G. If $\mathrm{Fitt}_\Lambda(h)$ is a Fitting invariant of a finitely presented Λ-module M, then*

$$\mathcal{F} \cdot \mathrm{Fitt}_\Lambda(h) \subset \mathrm{Ann}_\Lambda(M).$$

5. Group rings

In this section we specialize to the case $A = \mathbb{Q}_pG$ and $\Lambda = \mathbb{Z}_pG$, where G is a finite group. In this case each simple component A_i of A corresponds to an irreducible character χ_i and $\zeta(A_i) = K_i = \mathbb{Q}_p(\chi_i)$, where $\mathbb{Q}_p(\chi_i) = \mathbb{Q}_p(\chi_i(g) : g \in G)$. Note that $\chi_i(1) = n_i s_i$.

5.0.3 *χ-twists* We largely adopt the treatment of [Bu08], §1. Fix an irreducible character χ and let E_χ be the minimal subfield of $\mathbb{C}_p$ over which χ can be realized and which is both Galois and of finite degree over $\mathbb{Q}_p$. We put

$$\mathrm{pr}_\chi := \sum_{g \in G} \chi(g^{-1})g, \quad e_\chi := \frac{\chi(1)}{|G|}\mathrm{pr}_\chi.$$

Hence e_χ is a central primitive idempotent of $E_\chi G$ and pr_χ is the associated projector. We write $\mathfrak{o}_\chi$ for the ring of integers of E_χ and choose a maximal $\mathfrak{o}_\chi$-order $\mathfrak{M}$ in $E_\chi G$ which contains $\mathfrak{o}_\chi G$. We fix an indecomposable idempotent f_χ of $e_\chi\mathfrak{M}$ and define an $\mathfrak{o}_\chi$-torsionfree right $\mathfrak{o}_\chi G$-module by setting $T_\chi := f_\chi\mathfrak{M}$. Note that this slightly differs from the definition in [Bu08], but follows the notation of [BJ]. T_χ is free of rank $\chi(1)$ over $\mathfrak{o}_\chi$ and the associated right $E_\chi G$-module $E_\chi \otimes_{\mathfrak{o}_\chi} T_\chi$ has character χ. For any left $\mathbb{Z}_pG$-module M we set $M[\chi] := T_\chi \otimes_{\mathbb{Z}_p} M$, upon which G acts on the left by $t \otimes m \mapsto tg^{-1} \otimes g(m)$ for $t \in T_\chi$, $m \in M$ and $g \in G$. For any integer i we write $H^i(G, M)$ for the Tate cohomology in degree i of M with respect to G. Moreover, we write M^G resp. M_G for the maximal submodule resp. the maximal quotient module of M upon which G acts trivially. We obtain a left exact functor $M \mapsto M^\chi$ and a right exact functor $M \mapsto M_\chi$ from the category of left G-modules to the category of $\mathfrak{o}_\chi$-modules by setting $M^\chi := M[\chi]^G$ and $M_\chi := M[\chi]_G = T_\chi \otimes_{\mathbb{Z}_pG} M$. The action of $N_G := \sum_{g \in G} g$ induces a homomorphism $M_\chi \to M^\chi$ with kernel $H^{-1}(G, M[\chi])$ and cokernel $H^0(G, M[\chi])$. Hence $M_\chi \simeq M^\chi$ whenever M and hence also $M[\chi]$ is a c.t. (short for cohomologically trivial) G-module.
For any Λ-module M we denote the Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ of M by $M^\vee$ which is equipped with the natural G-action $(gf)(m) = f(g^{-1}m)$ for $f \in M^\vee$, $g \in G$ and $m \in M$. We have

$$(M^\vee)^\chi = (T_\chi \otimes_{\mathbb{Z}_p} \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p))^G = (\mathrm{Hom}(T_{\check{\chi}} \otimes_{\mathbb{Z}_p} M, \mathbb{Q}_p/\mathbb{Z}_p))^G = (M_{\check{\chi}})^\vee, \qquad (4)$$

where $\check{\chi}$ denotes the character contragredient to χ. If M is finite, we have $\mathrm{Ann}_\Lambda(M^\vee) = \mathrm{Ann}_\Lambda(M)^\sharp$, where we denote by $^\sharp : A \to A$ the involution induced by $g \mapsto g^{-1}$. We use these observations to prove another annihilation result:

Proposition 5.1. *Let $\Lambda = \mathbb{Z}_pG$ and M be a finitely presented Λ-module. Choose a maximal order Λ' containing Λ and let $\mathrm{Fitt}_{\Lambda'}(h)$ be a Fitting invariant of $\Lambda' \otimes_\Lambda M$. Moreover, let $x = \sum_{i=1}^t x_i \in \mathrm{Fitt}_{\Lambda'}(h) \subset \bigoplus_{i=1}^t \mathfrak{o}_i$ and $y_i \in \mathcal{D}^{-1}(\mathfrak{o}_{\chi_i}/\mathbb{Z}_p)$, $1 \leqslant i \leqslant t$. Then*

$$\sum_{i=1}^t \sum_{\omega \in \mathrm{Gal}(E_{\chi_i}/\mathbb{Q}_p)} y_i^\omega x_i^\omega \mathrm{pr}_{\chi_i^\omega} \in \mathrm{Ann}_\Lambda(M).$$

Proof. By Theorem 3.2 we may assume that h is a quadratic presentation of $\Lambda' \otimes_\Lambda M$. Moreover, Lemma 3.4 implies that we may assume that M is finite. Let us fix an integer i and abbreviate χ_i by χ. We tensor the finite presentation $\Lambda'^a \xrightarrow{h} \Lambda'^a \twoheadrightarrow \Lambda' \otimes_\Lambda M$ over Λ' with T_χ and obtain an exact sequence of $\mathfrak{o}_\chi$-modules

$$T_\chi^a \xrightarrow{h_\chi} T_\chi^a \twoheadrightarrow M_\chi.$$

If we write $\mathrm{nr}(h) = \sum_{i=1}^t \mathrm{nr}(h_i)$, we have an equality $\mathrm{nr}(h_i) = \det_{E_\chi}(h_\chi)$ and hence

$$(\mathrm{Fitt}_{\Lambda'}(h) \cap \mathfrak{o}_i)\mathfrak{o}_\chi = \mathrm{Fitt}_{\mathfrak{o}_\chi}(M_\chi) \subset \mathrm{Ann}_{\mathfrak{o}_\chi}(M_\chi). \tag{5}$$

But the right hand side equals $\mathrm{Ann}_{\mathfrak{o}_{\check\chi}}(M_\chi^\vee) = \mathrm{Ann}_{\mathfrak{o}_{\check\chi}}((M^\vee)^{\check\chi})$ by (4) above. Now [BJ], Lemma 11.1 and Lemma 11.2 imply that $\sum_{\omega\in\mathrm{Gal}(E_{\check\chi}/\mathbb{Q}_p)} y_j^\omega x_j^\omega \mathrm{pr}_{\check\chi^\omega} \in \mathrm{Ann}_\Lambda(M^\vee)$, where $\check\chi = \chi_j$. Applying the involution $\sharp$ yields the desired result, since clearly $\mathfrak{o}_i = \mathfrak{o}_j$ and $E_\chi = E_{\check\chi}$. □

Remark 7. The equality in (5) above shows that computing Fitting invariants over the maximal order Λ' is equivalent to computing the Fitting ideals $\mathrm{Fitt}_{\mathfrak{o}_\chi}(M_\chi)$ for all characters χ. Hence the authors of [BJ] implicitly compute Fitting invariants over the maximal order Λ' to derive annihilation results in the spirit of Brumer's conjecture.

Lemma 5.2. *Let $\Lambda = \mathbb{Z}_pG$ be a group ring of a finite group G.*

i) *If $x \in \zeta(\Lambda)$ is a nonzerodivisor, we have*

$$(\Lambda/(x))^\vee \simeq \Lambda/(x^\sharp)$$

ii) *If a Λ-homomorphism $\psi : \Lambda^n \to \Lambda^n$ induces $\overline{\psi} : (\Lambda/(x))^n \to (\Lambda/(x))^n$, then*

$$\begin{array}{ccc} ((\Lambda/(x))^\vee)^n & \xrightarrow{\overline{\psi}^\vee} & ((\Lambda/(x))^\vee)^n \\ \downarrow \simeq & & \downarrow \simeq \\ (\Lambda/(x^\sharp))^n & \xrightarrow{\overline{\psi}^{T,\sharp}} & (\Lambda/(x^\sharp))^n \end{array}$$

commutes.

Proof. In the special case where $x = p^m$ is a power of p, we have

$$(\Lambda/(p^m))^\vee = \mathrm{Hom}(\Lambda/(p^m), \mathbb{Z}_p/p^m\mathbb{Z}_p) \simeq \Lambda/(p^m),$$

where the isomorphism on the right hand side is explicitly given by $f \mapsto \sum_{g\in G} f(g)g$ for $f \in \mathrm{Hom}(\Lambda/(p^m), \mathbb{Z}_p/p^m\mathbb{Z}_p)$. A lengthy, but easy computation shows that the above diagram commutes in this case.

Passing to the general case, we first observe that $x^\sharp$ annihilates $(\Lambda/(x))^\vee$. Applying duals twice we see that $x^\sharp$ is indeed the exact annihilator. Thus it suffices to show that $(\Lambda/(x))^\vee$ is cyclic as Λ-module. Choose $m \in \mathbb{N}$ large enough such that p^m annihilates $(\Lambda/(x))$. Then there exists a nonzerodivisor $y \in \zeta(\Lambda)$ such that $p^m = x \cdot y$. This gives an exact sequence

$$\Lambda/(x) \overset{\cdot y}{\rightarrowtail} \Lambda/(p^m) \twoheadrightarrow \Lambda/(y).$$

The dual of this sequence induces a surjection $(\Lambda/(p^m))^\vee \twoheadrightarrow (\Lambda/(x))^\vee$. Since $(\Lambda/(p^m))^\vee$ is cyclic by the above special case, so does $(\Lambda/(x))^\vee$. Moreover, if ψ induces $\overline{\psi}^{T,\sharp}$ on $(\Lambda/(p^m))^n$, it also induces this map on $(\Lambda/(x^\sharp))^n$ via the epimorphism $(\Lambda/(p^m))^n \twoheadrightarrow (\Lambda/(x^\sharp))^n$. □

The second assertion of the following result is a non-commutative generalization of [CG98], Prop. 6. We also adopt some of the arguments in loc.cit.

Proposition 5.3. i) *Let C be a finite c.t. Λ-module and $c \in \zeta(A)$ be a generator of* $\mathrm{Fitt}_\Lambda(C)$. *Then $C^\vee$ is also c.t., c is a nonzerodivisor and* $\mathrm{Fitt}_\Lambda(C^\vee)$ *is generated by $c^\sharp$.*

ii) *If $M \rightarrowtail C \to C' \twoheadrightarrow M'$ is an exact sequence of finite Λ-modules, where C and C' are c.t., then there are Fitting invariants $\mathcal{F}(M^\vee)$ and $\mathcal{F}(M')$ of $M^\vee$ and M' over Λ such that*

$$\mathcal{F}(M^\vee)^\sharp \cdot \mathrm{Fitt}_\Lambda(C') = \mathcal{F}(M') \cdot \mathrm{Fitt}_\Lambda(C).$$

In particular, we have

$$\mathrm{Fitt}_\Lambda^{\max}(M^\vee)^\sharp \cdot \mathrm{Fitt}_\Lambda(C') = \mathrm{Fitt}_\Lambda^{\max}(M') \cdot \mathrm{Fitt}_\Lambda(C).$$

Proof. The Λ-module $C^\vee$ is c.t. by [NSW00], Corollary (1.7.6). Choose a quadratic presentation $\psi : \Lambda^n \to \Lambda^n$ of C such that $\mathrm{nr}(\psi) = c$. Since C is finite, ψ is injective and invertible over A and so does $\psi^{T,\sharp}$, the inverse given by $(\psi^{-1})^{T,\sharp}$. We will show that $\psi^{T,\sharp}$ is a finite presentation of $C^\vee$. Let x be a nonzerodivisor contained in the central conductor. Then xc annihilates C by Corollary 4.4 and the sequence

$$(\Lambda/(xc))^n \xrightarrow{\overline{\psi}} (\Lambda/(xc))^n \twoheadrightarrow C$$

is still exact. By Lemma 5.2 we have a dual exact sequence

$$C^\vee \rightarrowtail (\Lambda/(xc)^\sharp)^n \xrightarrow{\overline{\psi}^{T,\sharp}} (\Lambda/(xc)^\sharp)^n \twoheadrightarrow \mathrm{cok}\,(\overline{\psi}^{T,\sharp}). \tag{6}$$

Put $g := (\psi^{T,\sharp})^*$. Then $x^\sharp g$ has entries in Λ by Lemma 4.1 and we claim that

$$\ker(\overline{\psi}^{T,\sharp}) = \mathrm{im}\,(\overline{x^\sharp g}), \;\; \ker(\overline{x^\sharp g}) = \mathrm{im}\,(\overline{\psi}^{T,\sharp}).$$

If this is known, we get

$$C^\vee = \ker(\overline{\psi}^{T,\sharp}) = \mathrm{im}\,(\overline{x^\sharp g}) \simeq (\Lambda/(xc)^\sharp)^n/\ker(\overline{x^\sharp g}) = (\Lambda/(xc)^\sharp)^n/\mathrm{im}\,(\overline{\psi}^{T,\sharp}) = \mathrm{cok}\,(\overline{\psi}^{T,\sharp}).$$

Note that under this identification $\overline{v} \in \mathrm{cok}\,(\overline{\psi}^{T,\sharp}))$ corresponds to $\overline{x^\sharp g}(\overline{v}) \in \ker(\overline{\psi}^{T,\sharp})$. Now sequence (6) implies (i), since $\mathrm{cok}\,(\psi^{T,\sharp}) = \mathrm{cok}\,(\overline{\psi}^{T,\sharp})$.
We have to prove the two equalities above. For this let $\overline{v} \in (\Lambda/(xc)^\sharp)^n$; then $\overline{v}$ lies in the kernel of $\overline{\psi}^{T,\sharp}$ if and only if there exists a lift $v \in \Lambda^n$ of $\overline{v}$ and $w \in \Lambda^n$ such that $\psi^{T,\sharp}(v) = (xc)^\sharp \cdot w = x^\sharp \psi^{T,\sharp} g(w)$. Since $\psi^{T,\sharp}$ is injective, we can remove it from both sides, which gives $\overline{v} \in \mathrm{im}\,(\overline{x^\sharp g})$. Now assume that $\overline{v} \in \ker(\overline{x^\sharp g})$, i.e. there exists $w \in \Lambda^n$ such that $x^\sharp g(v) = (xc)^\sharp(w)$. Here we may add $\psi^{T,\sharp}$ to both sides and obtain $(xc)^\sharp v = (xc)^\sharp \psi^{T,\sharp}(w)$ and hence $v = \psi^{T,\sharp}(w)$. This proves the second equality.
For (ii) let us at first assume that $C = C'$. Let us denote the morphism $C \to C$ of the above sequence by α_C. By projectivity of Λ^n we can construct the following diagram

$$\begin{array}{ccccccc}
 & & \Lambda^n & \xrightarrow{d} & \Lambda^n & & \\
 & & \downarrow \psi & & \downarrow \psi & & \\
 & & \Lambda^n & \xrightarrow{\alpha} & \Lambda^n & & \\
 & & \downarrow & & \downarrow & & \\
M & \hookrightarrow & C & \xrightarrow{\alpha_C} & C & \twoheadrightarrow & M'
\end{array}$$

We see that the map $(\alpha|\psi) : \Lambda^n \oplus \Lambda^n \to \Lambda^n$ is a finite presentation of M' and thus $\mathrm{Fitt}_\Lambda((\alpha|\psi))$ is a Fitting invariant of M'. Writing $(\alpha|\psi) = \psi \cdot (\psi^{-1}\alpha|1)$ we see that $\mathrm{Fitt}_\Lambda((\alpha|\psi))$ is generated by $\mathrm{nr}(\psi) \cdot \mathrm{nr}(H)$, where H runs over the quadratic submatrices of $\psi^{-1}\alpha$.

Now we replace Λ by $\Lambda/(xc)$ and ψ by $\overline{\psi}$ as above. Dualizing the diagram yields

$$\begin{array}{ccccc}
C^\vee & \xrightarrow{\alpha_C^\vee} & C^\vee & \twoheadrightarrow & M^\vee \\
\downarrow & & \downarrow & & \\
(\Lambda/(xc)^\sharp)^n & \xrightarrow{\overline{\alpha}^{T,\sharp}} & (\Lambda/(xc)^\sharp)^n & & \\
\downarrow{\scriptstyle \overline{\psi}^{T,\sharp}} & & \downarrow{\scriptstyle \overline{\psi}^{T,\sharp}} & & \\
(\Lambda/(xc)^\sharp)^n & \xrightarrow{\overline{d}^{T,\sharp}} & (\Lambda/(xc)^\sharp)^n & & \\
\downarrow{\scriptstyle \overline{x^\sharp g}} & & \downarrow{\scriptstyle \overline{x^\sharp g}} & & \\
C^\vee & \xrightarrow{\tilde{\alpha}_C^\vee} & C^\vee & \twoheadrightarrow & \mathrm{cok}\,(\tilde{\alpha}_C^\vee)
\end{array}$$

An easy computation shows that indeed $\tilde{\alpha}_C^\vee = \alpha_C^\vee$ and thus the cokernel of the bottom sequence is again $M^\vee$. Hence $\mathrm{Fitt}_\Lambda((d^{T,\sharp}|\psi^{T,\sharp}))$ is a Fitting invariant of $M^\vee$. We may write $(d^{T,\sharp}|\psi^{T,\sharp}) = \psi^{T,\sharp} \cdot ((\psi^{-1}\alpha)^{T,\sharp}|1)$ such that $\mathrm{Fitt}_\Lambda((d^{T,\sharp}|\psi^{T,\sharp}))$ is generated by $\mathrm{nr}(\psi^{T,\sharp}) \cdot \mathrm{nr}(H)$, where H runs over the quadratic submatrices of $(\psi^{-1}\alpha)^{T,\sharp}$. Hence $\mathrm{Fitt}_\Lambda((d^{T,\sharp}|\psi^{T,\sharp}))^\sharp = \mathrm{Fitt}_\Lambda((\alpha|\psi))$ as desired.
For the general case we choose quadratic presentations $\phi : \Lambda^n \rightarrowtail \Lambda^n$ of C and $\phi' : \Lambda^m \rightarrowtail \Lambda^m$ of C'. As above we can lift $\alpha_C : C \to C'$ to a homomorphism $\alpha : \Lambda^n \to \Lambda^m$, and in turn α to a homomorphism $d : \Lambda^n \to \Lambda^m$ such that $\phi' \circ d = \alpha \circ \phi$. Again $\mathcal{F}(M') := \mathrm{Fitt}_\Lambda((\alpha|\phi'))$ is a Fitting invariant of M' over Λ. Now we add C' to the two leftmost terms of the sequence and C to the two rightmost terms such that we obtain the following diagram:

$$\begin{array}{ccccccc}
 & & \Lambda^n \oplus \Lambda^m & \xrightarrow{d'} & \Lambda^n \oplus \Lambda^m & & \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \\
 & & \Lambda^n \oplus \Lambda^m & \xrightarrow{\alpha'} & \Lambda^n \oplus \Lambda^m & & \\
 & & \downarrow & & \downarrow & & \\
M \oplus C' & \hookrightarrow & C \oplus C' & \longrightarrow & C \oplus C' & \twoheadrightarrow & C \oplus M'
\end{array}$$

Here, the $(n+m) \times (n+m)$-matrices are given as

$$\alpha' = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}, \; d' = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}, \; \psi = \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}.$$

The above shows that $\mathrm{Fitt}_\Lambda(((d')^{T,\sharp}|\psi^{T,\sharp}))^\sharp = \mathrm{Fitt}_\Lambda((\alpha'|\psi))$. But the latter Fitting invariant equals

$$\mathrm{Fitt}_\Lambda\left(\begin{pmatrix} 0 & 0 & \phi & 0 \\ \alpha & 0 & 0 & \phi' \end{pmatrix}\right) = \mathrm{Fitt}_\Lambda(\phi)\mathrm{Fitt}_\Lambda((\alpha|\phi')) = \mathrm{Fitt}_\Lambda(C)\mathcal{F}(M').$$

Likewise we find that $\mathrm{Fitt}_\Lambda(((d')^{T,\sharp}|\psi^{T,\sharp}))$ is the product of $\mathrm{Fitt}_\Lambda(C')^\sharp$ and a Fitting invariant of $M^\vee$ over Λ.
For the last assertion, we observe that we can choose a quadratic presentation ϕ of C and a lift α of α_C such that α has shape $(0|\tilde{\alpha})$, where $\tilde{\alpha}$ is a finite presentation of M'. To see this, let $\pi : \Lambda^m \twoheadrightarrow M'$ be the epimorphism which is the composition of $\pi_m : \Lambda^m \twoheadrightarrow C'$ and $\pi' : C' \twoheadrightarrow M'$. Choose $\tilde{\alpha} : \Lambda^n \to \Lambda^m$ such that $\mathrm{im}\,(\tilde{\alpha}) = \ker(\pi)$. Then $\pi_m\tilde{\alpha}$ surjects onto $\ker(\pi')$ and thus factors through C. This factorization together with an epimorphism $\Lambda^k \twoheadrightarrow M$ gives a surjection $\Lambda^k \oplus \Lambda^n \twoheadrightarrow C$

such that $\alpha = (0|\tilde{\alpha})$ is a lift of α_C. Now let $h \in M_{a\times b}(\Lambda)$ be a finite presentation of M' such that $\mathrm{Fitt}_\Lambda(h) = \mathrm{Fitt}_\Lambda^{\max}(M')$. By Theorem 3.2 we may assume that $a = n$, $b = m$ and $h \circ X = \tilde{\alpha}$ for an appropriate matrix $X \in \mathrm{Gl}_a(\Lambda)$. Hence the images of h and $\tilde{\alpha}$ coincide such that we can choose h for an $\tilde{\alpha}$. We see that $\mathcal{F}(M') = \mathrm{Fitt}_\Lambda((0|h|\phi'))$ is an appropriate Fitting invariant of M' such that the above proof works. But $\mathcal{F}(M')$ contains and thus equals $\mathrm{Fitt}_\Lambda(h)$ by maximality. We have proven that there is a Fitting invariant $\mathcal{F}(M^\vee)$ of $M^\vee$ such that

$$\mathcal{F}(M^\vee)^\sharp \cdot \mathrm{Fitt}_\Lambda(C') = \mathrm{Fitt}_\Lambda^{\max}(M') \cdot \mathrm{Fitt}_\Lambda(C).$$

Dualizing the above sequence likewise implies the existence of a Fitting invariant $\mathcal{F}(M')$ of M' such that

$$\mathcal{F}(M')^\sharp \cdot \mathrm{Fitt}_\Lambda(C^\vee) = \mathrm{Fitt}_\Lambda^{\max}(M^\vee) \cdot \mathrm{Fitt}_\Lambda((C')^\vee).$$

The first part of the Proposition now implies that

$$\begin{aligned} \mathrm{Fitt}_\Lambda^{\max}(M^\vee)^\sharp &= \mathcal{F}(M')\mathrm{Fitt}_\Lambda(C)\mathrm{Fitt}_\Lambda(C')^{-1} \\ &\subset \mathrm{Fitt}_\Lambda^{\max}(M')\mathrm{Fitt}_\Lambda(C)\mathrm{Fitt}_\Lambda(C')^{-1} \\ &= \mathcal{F}(M^\vee)^\sharp \end{aligned}$$

and hence $\mathcal{F}(M^\vee) = \mathrm{Fitt}_\Lambda^{\max}(M^\vee)$ as desired. □

If C is a finite c.t. $\mathbb{Z}_pG$-module for an abelian group G, one knows that $|\mathbb{Z}_pG/\mathrm{Fitt}_{\mathbb{Z}_pG}(C)| = |C|$. This is very useful if we want to compute $\mathrm{Fitt}_{\mathbb{Z}_pG}(C)$, since it suffices to compute an ideal I contained in $\mathrm{Fitt}_{\mathbb{Z}_pG}(C)$ such that $|\mathbb{Z}_pG/I| = |C|$. An analogous statement in the non-abelian case is the following

Proposition 5.4. *Let $\Lambda = \mathbb{Z}_pG$ and C be a finite c.t. Λ-module. Let E be a splitting field of $A = \mathbb{Q}_pG$ with ring of integers $\mathfrak{o}_E$ and $c \in \zeta(A)^\times \cap \mathrm{Fitt}_\Lambda(C)$. Write $1 \otimes c = \sum_{\chi\in\mathrm{Irr}\,(G)} c_\chi e_\chi \in \zeta(E\otimes A) = \bigoplus_{\chi\in\mathrm{Irr}\,(G)} Ee_\chi$ Then c is a generator of $\mathrm{Fitt}_\Lambda(C)$ if and only if there is an $\alpha \in \mathfrak{o}_E^\times$ such that $\prod_{\chi\in\mathrm{Irr}\,(G)} c_\chi^{\chi(1)} = \alpha \cdot |C|$.*

Proof. If c is a generator of $\mathrm{Fitt}_\Lambda(C)$, the desired formula follows immediately from [Ni], Prop. 5. Conversely, let c' be a generator of $\mathrm{Fitt}_\Lambda(C)$. Then $c = \lambda \cdot c'$ for some $\lambda \in \zeta(\Lambda) \cap \zeta(A)^\times$. If we write $1 \otimes \lambda = \sum_\chi \lambda_\chi e_\chi \in \zeta(E \otimes A)$, the above product formula implies that $\prod_\chi \lambda_\chi^{\chi(1)}$ is a unit of $\mathfrak{o}_E$. Hence each λ_χ is a unit and thus λ is a unit of $\zeta(\Lambda')$, where Λ' is a maximal order in A. But since $\zeta(\Lambda) \cap \zeta(\Lambda')^\times = \zeta(\Lambda)^\times$, we are done. □

In the case of commutative rings, Fitting ideals behave well under base change. We provide some base change results for the case at hand. Let us begin with the more general situation, where $\mathfrak{o}$ is a complete commutative noetherian local ring and Λ is an $\mathfrak{o}$-order in the separable K-algebra A.

Lemma 5.5. *If $e \in A$ is a central idempotent and $\mathcal{F}(M)$ is a Fitting invariant of a finitely presented Λ-module M, then $e\mathcal{F}(M)$ is a Fitting invariant of the Λe-module $\Lambda e \otimes_\Lambda M$.*

Proof. Obvious from the definitions. □

If $\Lambda = \mathfrak{o}G$ is a group ring of a finite group G and H is normal in G, then $e := |H|^{-1}N_H$ is a central idempotent and $\Lambda e \simeq \mathfrak{o}[G/H]$. Thus base change behaves well if we factor G by a normal subgroup.

Corollary 5.6. *Let $\Lambda = \mathfrak{o}G$ be a group ring of a finite group G and I a two-sided ideal of Λ such that $\overline{\Lambda} := \Lambda/I$ is commutative. If $\mathcal{F}(M)$ is a Fitting invariant of the finitely presented Λ-module M, then $\mathcal{F}(M)$ has a well defined image in $\overline{\Lambda}$ which is the Fitting ideal of $\overline{\Lambda} \otimes_\Lambda M$ over $\overline{\Lambda}$.*

Proof. Since $\overline{\Lambda}$ is abelian, the ideal I contains $J := \Delta(G, G')$, where G' denotes the commutator subgroup of G and $\Delta(G, G')$ is the kernel of the natural epimorphism $\mathfrak{o}G \twoheadrightarrow \mathfrak{o}[G/G']$. Hence we may first base change to Λ/J by Lemma 5.5. Since Λ/J is commutative and Fitting ideals behave well under base change, we are done. □

6. Complete group algebras

In this section let Λ be the complete group algebra $\mathbb{Z}_p[[G]]$, where G is a profinite group which contains a finite normal subgroup H such that $G/H \simeq \Gamma$ for a pro-p-group Γ, isomorphic to $\mathbb{Z}_p$; thus G can be written as a semi-direct product $H \rtimes \Gamma$. We fix a topological generator γ of Γ and choose a natural number n such that γ^{p^n} is central in G. Since also $\Gamma^{p^n} \simeq \mathbb{Z}_p$, there is an isomorphism $\mathbb{Z}_p[[\Gamma^{p^n}]] \simeq \mathbb{Z}_p[[T]]$ induced by $\gamma^{p^n} \mapsto 1+T$. Here, $\mathfrak{o} := \mathbb{Z}_p[[T]]$ denotes the power series ring in one variable over $\mathbb{Z}_p$. If we view Λ as an $\mathfrak{o}$-module, there is a decomposition

$$\Lambda = \bigoplus_{i=0}^{p^n-1} \mathfrak{o}\gamma^i[H].$$

Hence Λ is finitely generated as an $\mathfrak{o}$-module and an $\mathfrak{o}$-order in the separable $K := Quot(\mathfrak{o})$-algebra $A = \mathcal{Q}(G) := \bigoplus_i K\gamma^i[H]$. Note that A is obtained from Λ by inverting all regular elements. As in the case of group rings we denote by $\sharp : A \to A$ the involution induced by mapping each $g \in G$ to g^{-1}. Moreover, we denote the Iwasawa adjoint of a finitely generated $\mathfrak{o}$-torsion Λ-module M by $\alpha(M)$.

Let $\mathfrak{m} := (p, T)$ be the maximal ideal of $\mathfrak{o}$. Since $\gamma^{p^n} = 1 + T \equiv 1 \bmod \mathfrak{m}$, we have

$$\overline{\Lambda} := \Lambda/\mathfrak{m}\Lambda = \sum_i \mathbb{F}_p\gamma^i[H] = \mathbb{F}_p[H \rtimes C_{p^n}],$$

where C_{p^n} denotes the cyclic group of order p^n. Note that $\mathfrak{m}$ is contained in the radical of Λ.

Lemma 6.1. *Let $f \in \mathbb{Z}_p[T]$ be a Weierstraß polynomial and $M = \Lambda/(f)$. Then $f^\sharp(T) = (1 + T)^{\deg(f)} f((1+T)^{-1} - 1)$ is also a Weierstraß polynomial and $\alpha(M) = \Lambda/(f^\sharp)$.*

Proof. As in the proof of Lemma 5.2 the exact annihilator of $\alpha(M)$ is $f^\sharp$ such that we only have to show that $\alpha(M)$ is cyclic as Λ-module. The Iwasawa μ-invariant of M is zero such that $\alpha(M) = \mathrm{Hom}(\varinjlim_n M/p^nM, \mathbb{Q}_p/\mathbb{Z}_p)$. Applying the Pontryagin dual to the exact sequence

$$M/pM \rightarrowtail \varinjlim_n M/p^nM \xrightarrow{\cdot p} \varinjlim_n M/p^nM$$

implies that $\alpha(M)/p\alpha(M) \simeq (M/pM)^\vee$. Since p lies in the radical of Λ, it suffices to show that $(M/pM)^\vee$ is cyclic. But since f is a nonzerodivisor, the ring $M/pM = \Lambda/(p, f)$ is Gorenstein of dimension zero. Therefore the socle of M/pM is cyclic which is equivalent to $(M/pM)^\vee$ being cyclic modulo the radical. Now we are done via Nakayama's Lemma. □

Lemma 6.2. *Let C be a finitely generated R-torsion Λ-module of projective dimension at most 1. Then C admits a quadratic presentation.*

Proof. Let us first assume that G is abelian. Then G is the direct product of its p-Sylow subgroup G_p and a finite group H' prime to p such that there is a decomposition $\Lambda = \bigoplus_\chi \mathbb{Z}_p[\chi]G_p$, where the sum runs through all irreducible characters χ of H' module Galois conjugation over $\mathbb{Q}_p$. Now let $P \rightarrowtail \Lambda^n \twoheadrightarrow C$ be a projective resolution of C. Then $P = \bigoplus_\chi (\mathbb{Z}_p[\chi]G_p)^{n_\chi}$ with appropriate $n_\chi \in \mathbb{N}$ by [NSW00], Corollary 5.2.19. But since C is R-torsion, all these n_χ coincide. For the general case we can adjust the proof of [RW04], Lemma 13 to show that the map $\rho : K_0T(\Lambda) \to K_0(\Lambda)$ is zero if this is the case for abelian G. Note that the authors of loc.cit. so to speak show Lemma 6.2 for a special element of $K_0T(\Lambda)$. □

We have the following non-commutative version of [Gr04] Prop. 1 and 2.

Proposition 6.3. i) *Let C be a finitely generated R-torsion Λ-module of projective dimension at most 1 which has no $\mathbb{Z}_p$-torsion and let c be a generator of $\mathrm{Fitt}_\Lambda(C)$. Then $c^\sharp$ is a generator of $\mathrm{Fitt}_\Lambda(\alpha(C))$.*

ii) *Let $M \rightarrowtail C \xrightarrow{\gamma} C' \twoheadrightarrow M'$ be an exact sequence of finitely generated R-torsion Λ-modules which have no $\mathbb{Z}_p$-torsion and such that the projective dimension of C and C' is at most 1. Then there are Fitting invariants $\mathcal{F}(\alpha(M))$ and $\mathcal{F}(M')$ of $\alpha(M)$ and M' over Λ such that*

$$\mathcal{F}(\alpha(M))^\sharp \mathrm{Fitt}_\Lambda(C') = \mathrm{Fitt}_\Lambda(C)\mathcal{F}(M').$$

In particular, we have

$$\mathrm{Fitt}_\Lambda^{\max}(\alpha(M))^\sharp \mathrm{Fitt}_\Lambda(C') = \mathrm{Fitt}_\Lambda(C)\mathrm{Fitt}_\Lambda^{\max}(M').$$

Proof. Choose a Weierstraß polynomial $f \in \mathbb{Z}_p[T]$ such that f annihilates C. If $\psi : \Lambda^n \rightarrowtail \Lambda^n$ is a quadratic presentation of C, then $(\Lambda/(f))^n \xrightarrow{\overline{\psi}} (\Lambda/(f))^n \twoheadrightarrow C$ is still exact. Now Lemma 6.1 implies that applying α yields an exact sequence

$$\alpha(C) \rightarrowtail (\Lambda/(f^\sharp))^n \xrightarrow{\overline{\psi}^{T,\sharp}} (\Lambda/(f^\sharp))^n \twoheadrightarrow \mathrm{cok}\,(\overline{\psi}^{T,\sharp}).$$

As in the proof of Proposition 5.3 there is an isomorphism $\alpha(C) \simeq \mathrm{cok}\,(\overline{\psi}^{T,\sharp}) = \mathrm{cok}\,(\psi^{T,\sharp})$ which implies (i). For (ii) we can conclude as in the proof of Proposition 5.3. □

Now we assume that $\Gamma' \simeq \mathbb{Z}_p$ is normal in G such that $\Gamma' \cap H = 1$. We fix a topological generator γ' of Γ' and put $G' := G/\Gamma'$. We observe that the natural epimorphism $G \twoheadrightarrow G'$ induces an embedding $H \rightarrowtail G'$ such that H is normal in G'. Note that this naturally arises in Iwasawa theory: If G' is the Galois group of a finite extension of number fields L/K, and Γ, resp. Γ' are the Galois groups of the cyclotomic $\mathbb{Z}_p$-extensions K_∞ resp. L_∞ of K resp. L, then $G := \mathrm{Gal}(L_\infty/K)$ is the semi-direct product of a normal subgroup H of G' and Γ such that Γ' is normal in G.
We recall some results concerning the algebra $A = \mathcal{Q}(G)$ due to Ritter and Weiss [RW04]. Let E be a splitting field of $\mathbb{Z}_pG'$ and fix an irreducible (E-valued) character χ of G' and an EG'-module V_χ with character χ. We can view V_χ as a representation of G, where $g \in G$ acts on V_χ as g mod Γ'. Hence χ is also an irreducible character of G. Let η be an irreducible constituent of $\mathrm{res}_H^G \chi$ and set

$$St(\eta) := \{g \in G : \eta^g = \eta\},\ e_\eta = \frac{\eta(1)}{|H|}\sum_{h\in H}\eta(h^{-1})h,\ e_\chi = \sum_{\eta | \mathrm{res}_H^G \chi} e_\eta.$$

By [RW04], corollary to Proposition 6, e_χ is a primitive central idempotent of $\mathcal{Q}^E(G) := E \otimes_{\mathbb{Q}_p} \mathcal{Q}(G)$. By loc.cit., Proposition 5 there is a distinguished element $\gamma_\chi \in \zeta(\mathcal{Q}^E(G)e_\chi)$ which generates a pro-cyclic p-subgroup Γ_χ of $(\mathcal{Q}^E(G)e_\chi)^\times$. Moreover, γ_χ induces an isomorphism $\mathcal{Q}^E(\Gamma_\chi) \xrightarrow{\simeq} \zeta(\mathcal{Q}^E(G)e_\chi)$ by loc.cit., Proposition 6. The authors define the following map

$$j_\chi : \zeta(\mathcal{Q}^E(G)) \twoheadrightarrow \zeta(\mathcal{Q}^E(G)e_\chi) \simeq \mathcal{Q}^E(\Gamma_\chi) \to \mathcal{Q}^E(\Gamma),$$

where the last arrow is induced by mapping γ_χ to γ^{w_χ}, where $w_\chi = [G : St(\eta)]$. It is shown that for any matrix $\Phi \in M_{n\times n}(\mathcal{Q}(G))$ we have

$$j_\chi(\mathrm{nr}(\Phi)) = \det{}_{\mathcal{Q}^E(\Gamma)}(\Phi | \mathrm{Hom}_{EH}(V_\chi, \mathcal{Q}^E(G)^n)). \tag{7}$$

Here, Φ acts on $f \in \mathrm{Hom}_{EH}(V_\chi, \mathcal{Q}^E(G)^n)$ via right multiplication, and γ acts on the left via $(\gamma f)(v) = \gamma \cdot f(\gamma^{-1}v)$ for all $v \in V_\chi$. For any group G let us denote the canonical augmentation map $E[[G]] \twoheadrightarrow E$ by aug_G. We now prove the following base change result:

Theorem 6.4. *Let M be a finitely presented Λ-module and $\mathcal{F}(M)$ a Fitting invariant of M over Λ. Assume that $\mathcal{F}(M)$ is generated by $\phi_i = \mathrm{nr}(\Phi_i)$, $i = 1, \ldots, k$. Then $j_\chi(\phi_i)$ actually lies in $E[[\Gamma]]$ and the elements*

$$\sum_{\chi \in \mathrm{Irr}\,(G')} \mathrm{aug}_{\,\Gamma}(j_\chi(\phi_i))e_\chi, \quad i = 1, \ldots, k$$

lie in the center of $\mathbb{Q}_pG'$ and generate a Fitting invariant of $M/(\gamma' - 1)$ over $\mathbb{Z}_pG'$.

Proof. Let χ be an (E-valued) irreducible character of G' and put $m := \chi(1)$. For any $\overline{f} \in \mathrm{Hom}_{EG'}(V_\chi, EG')$ we will define an $f \in \mathcal{V}_\chi := \mathrm{Hom}_{EH}(V_\chi, \mathcal{Q}^E(G))$ such that f takes values in $E[[G]]$ and $f(v) \bmod \Gamma' = \overline{f}(v)$ for any $v \in V_\chi$. Let $\overline{\gamma} \in G'$ be the image of γ in G'. Then we can decompose G' into

$$G' = \bigcup_{i=0}^{[G':H]-1} H\overline{\gamma}^i.$$

Hence any $g' \in G'$ can be uniquely written as $g' = h_{g'} \cdot \overline{\gamma}^{i(g')}$ with $h_{g'} \in H$ and $0 \leqslant i(g') < [G' : H]$. If $\overline{f}(v) = \sum_{g' \in G'} x_{g'}g' \in EG'$, we define $f(v) := \sum_{g' \in G'} x_{g'}h_{g'}\gamma^{i(g')}$ which lies in $\mathrm{Hom}_{EH}(V_\chi, E[[G]])$, since $h_{hg'} = h \cdot h_{g'}$ for any $h \in H$, $g' \in G'$. Clearly $f \bmod \Gamma' = \overline{f}$. The E-vector space $\mathrm{Hom}_{EG'}(V_\chi, EG')$ has dimension m, and we fix an E-basis $\overline{f}_1, \ldots, \overline{f}_m$. We claim that $f_1, \ldots, f_m$ is a $\mathcal{Q}^E(\Gamma)$-basis of $\mathcal{V}_\chi$. Since the dimension of $\mathcal{V}_\chi$ as $\mathcal{Q}^E(\Gamma)$-vector space is m by [RW04], Proposition 6 resp. its proof, it suffices to show that $f_1, \ldots, f_m$ are linearly independent over $\mathcal{Q}^E(\Gamma)$. Assume that there are $\lambda_i \in \mathcal{Q}^E(\Gamma)$, not all of them equal to zero, such that $\sum_{i=1}^m \lambda_i f_i = 0$. We may assume that $\lambda_i \in \mathfrak{o}_E[[\Gamma]]$ for all i, and identifying $\mathfrak{o}_E[[\Gamma]]$ with the power series ring $\mathfrak{o}_E[[T]]$, we may also assume that there is at least one λ_i which is not divisible by T. Since T corresponds to $1 - \gamma \in \mathfrak{o}_E[[\Gamma]]$, this means that $\mathrm{aug}_{\,\Gamma}(\lambda_i) \neq 0$. But if $\overline{\lambda}_i := \lambda_i \bmod \Gamma' = \sum_{j=1}^{[G':H]} \alpha_{ij}\overline{\gamma}^j$, $\alpha_{ij} \in \mathfrak{o}_E$, we have

$$\begin{aligned} 0 = \sum_{i=1}^m \overline{\lambda_i f_i} &= \sum_{i=1}^m \sum_{j=1}^{[G':H]} \alpha_{ij}\left(\overline{\gamma}^j \overline{f}_i\right) \\ &= \sum_{i=1}^m \sum_{j=1}^{[G':H]} \alpha_{ij}\overline{f}_i \end{aligned}$$

which implies that $\mathrm{aug}_{\,\Gamma}(\lambda_i) = \mathrm{aug}_{\,G'}(\overline{\lambda}_i) = \sum_{j=1}^{[G':H]} \alpha_{ij} = 0$ for any i, a contradiction.
Recall that $f_i \in \mathrm{Hom}_{EH}(V_\chi, E[[G]])$ for any i and that $\mathrm{Hom}_{EH}(V_\chi, E[[G]])$ is a left $E[[\Gamma]]$-module and a right $E[[G]]$-module, as $(\gamma f)(v) = \gamma f(\gamma^{-1}v)$ and $(f\alpha)(v) = f(v) \cdot \alpha$ for $f \in \mathrm{Hom}_{EH}(V_\chi, E[[G]])$, $v \in V_\chi$ and $\alpha \in E[[G]]$. Moreover, $\gamma^{w_\chi} f = f\gamma_\chi$ by the proof of [RW04], Proposition 6. Now let $\mathcal{A}_\chi$ be the $E[[G]]$-submodule of $\mathcal{V}_\chi$ generated by $\gamma^j f_i$, $j = 0, \ldots, w_\chi - 1$, $i = 1, \ldots, m$. Then $\mathcal{A}_\chi$ is a free $E[[\Gamma]]$-module of rank m and we choose a basis $g_1, \ldots, g_m$. Writing g_i as an $E[[G]]$-linear combination of the $\gamma^j f_i$ we find that $\overline{g}_i$ lies in $\mathrm{Hom}_{EG'}(V_\chi, EG')$. On the other hand, we can write any f_i as an $E[[\Gamma]]$-linear combination of $g_1, \ldots, g_m$, and hence $\overline{f}_i$ can be written as an E-linear combination of the $\overline{g}_j$. Thus $\overline{g}_1, \ldots, \overline{g}_m$ is also an E-basis of $\mathrm{Hom}_{EG'}(V_\chi, EG')$.
Now let $\alpha \in \Lambda$ be arbitrary and write $\overline{\alpha}$ for the image of α in $\mathbb{Z}_pG'$. For any x let r_x denote right multiplication by x. Then $r_\alpha \circ g_i = g_i\alpha = \sum_{j=1}^m \beta_{ij}g_j$ for some $\beta_{ij} \in E[[\Gamma]]$ such that

$$j_\chi(\mathrm{nr}(\alpha)) = \det{}_{\mathcal{Q}^E(\Gamma)}(\beta_{ij})$$

by (7). But clearly $r_{\overline{\alpha}} \circ \overline{g}_i = \sum_j \overline{\beta}_{ij}\overline{g}_j$ and hence the χ-part of $\mathrm{nr}(\overline{\alpha})$ equals

$$\det{}_E(\overline{\alpha}|\mathrm{Hom}_{EG'}(V_\chi, EG')) = \det{}_E(\overline{\beta}_{ij}) = \mathrm{aug}_{\,\Gamma}(j_\chi(\mathrm{nr}(\alpha))) \tag{8}$$

and a similar equation holds for $\alpha \in M_{n \times n}(\Lambda)$. Now let $\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M$ be a finite presentation of M such that $\mathrm{Fitt}_\Lambda(h) = \mathcal{F}(M)$. Tensoring with $\mathbb{Z}_pG'$ over Λ yields a finite presentation $\overline{h}$ of $M/(\gamma'-1)$ over $\mathbb{Z}_pG'$. Moreover, $\mathcal{F}(M)$ is generated by $\psi_j = \mathrm{nr}(H_j)$, where $H_j \in S_b(h)$, $1 \leqslant j \leqslant k'$, while the elements $\mathrm{nr}(\overline{H}_j)$ generate a Fitting invariant $\mathcal{F}(M/(\gamma'-1))$ of $M/(\gamma'-1)$ over $\mathbb{Z}_pG'$. Now equation (8) implies the Theorem in the case $\phi_i = \psi_i$. For the general case let us abbreviate the map $\sum_{\chi \in \mathrm{Irr}\,(G')} \mathrm{aug}_\Gamma \circ j_\chi$ by π. We claim that π maps $\zeta(\Lambda)$ into $\zeta(\mathbb{Z}_pG')$. Since $\mathrm{aug}_\Gamma \circ j_\chi$ just maps γ_χ to one and γ_χ acts trivially on V_χ by [RW04], Prop. 5, the image of $\lambda \in \zeta(\Lambda)$ under this map acts on V_χ as λ itself. Likewise $\overline{\lambda} e_\chi \in \zeta(\mathbb{Q}_pG')$ acts on V_χ as λ, since χ is a character of G'. Hence $\pi(\lambda) - \overline{\lambda}$ acts as zero on V_χ for each χ and lies in the center of $\mathbb{Q}_pG'$; thus $\pi(\lambda) = \overline{\lambda} \in \zeta(\mathbb{Z}_pG')$. Now let ϕ_i, $1 \leqslant i \leqslant k$ be arbitrary generators of $\mathcal{F}(M)$. Then we may write $\phi_i = \sum_{j=1}^{k'} \lambda_{ij}\psi_j$ with $\lambda_{ij} \in \zeta(\Lambda)$ and obtain $\pi(\phi_i) = \sum_j \pi(\lambda_{ij})\pi(\psi_j)$ which lies in $\mathcal{F}(M/(\gamma'-1))$ by the claim. By a dual argument each ψ_j lies in the $\zeta(\mathbb{Z}_pG')$-module generated by $\pi(\phi_i)$, $1 \leqslant i \leqslant k$. Hence these elements also generate $\mathcal{F}(M/(\gamma'-1))$. □

7. An application: Annihilation of class groups

Let us fix a finite Galois CM-extension L/K of number fields with Galois group G, i.e. L is a CM-field, K is totally real and complex conjugation induces an unique automorphism j of L which lies in the center of G. For any prime $\mathfrak{p}$ of K we fix a prime $\mathfrak{P}$ of L above $\mathfrak{p}$ and write $G_\mathfrak{P}$ resp. $I_\mathfrak{P}$ for the decomposition group resp. inertia subgroup of L/K at $\mathfrak{P}$. Moreover, we denote the residual group at $\mathfrak{P}$ by $\overline{G_\mathfrak{P}} = G_\mathfrak{P}/I_\mathfrak{P}$ and choose a lift $\phi_\mathfrak{P} \in G_\mathfrak{P}$ of the Frobenius automorphism $\overline{\phi_\mathfrak{P}} \in \overline{G_\mathfrak{P}}$. We fix an odd prime p and put $\Lambda := \mathbb{Z}_pG/(1+j)$ which is a $\mathbb{Z}_p$-order in the separable algebra $A = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$. For any $\mathbb{Z}_pG$-module M we define $M^- = \Lambda \otimes_{\mathbb{Z}_pG} M$. Since p is odd, taking minus parts is an exact functor. If M is a $\mathbb{Z}G$-module, we define M^- to be $\mathbb{Z}[\frac{1}{2}]G/(1+j) \otimes_{\mathbb{Z}G} M$. This notation is nonstandard, but practical: for example, taking minus parts is an exact functor, since we invert 2.

For any subgroup H of G, let $N_H := \sum_{h \in H} h$. We define central idempotents of $\mathbb{Q}_pG_\mathfrak{P}$ by

$$e'_\mathfrak{P} := |I_\mathfrak{P}|^{-1} N_{I_\mathfrak{P}},\ e''_\mathfrak{P} = 1 - e'_\mathfrak{P}.$$

We define a $\mathbb{Z}_pG_\mathfrak{P}$-module $U_\mathfrak{P}$ by

$$U_\mathfrak{P} := \langle N_{I_\mathfrak{P}}, 1 - e'_\mathfrak{P}\phi_\mathfrak{P}^{-1} \rangle_{\mathbb{Z}_pG_\mathfrak{P}} \subset \mathbb{Q}_pG_\mathfrak{P}.$$

Note that $U_\mathfrak{P} = \mathbb{Z}_pG_\mathfrak{P}$ if $\mathfrak{p}$ is unramified in L/K. If S is a finite set of places of K containing all the infinite places S_∞, and χ is a (complex) character of G, we denote the S-truncated Artin L-function attached to χ and S by $L_S(s,\chi)$ and define $L_S^*(0,\chi)$ to be the leading coefficient of the Taylor expansion of $L_S(s,\chi)$ at $s=0$. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}G) = \prod_{\chi \in \mathrm{Irr}\,(G)} \mathbb{C}$, where $\mathrm{Irr}\,(G)$ denotes the set of irreducible complex characters of G. We define the equivariant Artin L-function to be the meromorphic $\zeta(\mathbb{C}G)$-valued function

$$L_S(s) := (L_S(s,\chi))_{\chi \in \mathrm{Irr}\,(G)}.$$

We put $L_S^*(0) = (L_S^*(0,\chi))_{\chi \in \mathrm{Irr}\,(G)}$ and abbreviate $L_{S_\infty}(s)$ to $L(s)$. Note that if $x = (x_\chi)_\chi \in \zeta(\mathbb{C}G)$, then $x^\sharp = (x_{\overline{\chi}})_\chi$. In [Bu01] the author defines the following element of $K_0(\mathbb{Z}G, \mathbb{R})$:

$$T\Omega(L/K,0) := \psi_G^*(\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1}) + \hat{\partial}_G(L_S^*(0)^\sharp)).$$

Here, ψ_G^* is a certain involution on $K_0(\mathbb{Z}G,\mathbb{R})$ which is not important for our purposes, since we will be only interested in the nullity of $T\Omega(L/K,0)$. Furthermore, $\tau_S \in \mathrm{Ext}_G^2(E_{S_L}, \Delta S_L)$ is Tate's canonical class (cf. [Ta66]), where S_L denotes the set of places of L which lie above those in S, E_{S_L} are the S_L-units of L and ΔS_L is the kernel of the augmentation map $\mathbb{Z}S_L \twoheadrightarrow \mathbb{Z}$ which maps each $\mathfrak{P} \in S_L$ to 1. Finally, λ_S denotes the negative of the usual Dirichlet map, so $\lambda_S : \mathbb{R} \otimes E_{S_L} \to \mathbb{R} \otimes \Delta S_L$, $u \mapsto -\sum_{\mathfrak{P} \in S_L} \log|u|_\mathfrak{P}\mathfrak{P}$, and $\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1})$ is the refined Euler characteristic associated

to the perfect 2-extension $A_S \to B_S$ whose extension class is τ_S, metrised by λ_S^{-1}. For more precise definitions we refer the reader to [Bu01]. The ETNC for the motive $h^0(L)$ with coefficients in $\mathbb{Z}G$ in this context asserts that the element $T\Omega(L/K,0)$ is zero. Note that this statement is also equivalent to the Lifted Root Number Conjecture formulated by Gruenberg, Ritter and Weiss [GRW99].
It is also proven in [Bu01] that $T\Omega(L/K,0)$ lies in $K_0(\mathbb{Z}G,\mathbb{Q})$ if and only if Stark's conjecture holds. In this case the ETNC decomposes into local conjectures at each prime p by means of the isomorphism

$$K_0(\mathbb{Z}G,\mathbb{Q}) \simeq \bigoplus_{p\nmid\infty} K_0(\mathbb{Z}_pG,\mathbb{Q}_p).$$

Since Stark's conjecture is known for odd characters (cf. [Ta84], Th. 1.2, p. 70), $T\Omega(L/K,0)$ has a well defined image $T\Omega(L/K,0)_p^-$ in $K_0(\Lambda,A)$. Let us fix an embedding $\iota:\mathbb{C}\rightarrowtail\mathbb{C}_p$; then the image of $L(0)$ (which actually lies in $\zeta(\mathbb{Q}G)$) in $\zeta(\mathbb{Q}_pG)$ via the canonical embedding

$$\zeta(\mathbb{Q}G)\rightarrowtail\zeta(\mathbb{Q}_pG)=\bigoplus_{\chi\in\mathrm{Irr}_p(G)/\sim}\mathbb{Q}_p(\chi),$$

is given by $\sum_{\chi\in\mathrm{Irr}_p(G)/\sim}L(0,\chi^{\iota^{-1}})^{\iota}$. Here the sum runs over all $\mathbb{C}_p$-valued irreducible characters of G modulo Galois action.
We denote the class group of L by cl_L and the roots of unity in L by μ_L. We are ready to state a non-abelian generalization of [Gr07], Theorem 8.8.

Theorem 7.1. *Let L/K be a finite Galois CM-extension of number fields and p an odd prime. If $\mu_L\otimes\mathbb{Z}_p$ is G-c.t. and $T\Omega(L/K,0)_p^-=0$, then*

$$L(0)^\sharp\mathrm{nr}(a)\prod_{\mathfrak{p}\in S_{\mathrm{ram}}}\mathrm{nr}(U_{\mathfrak{P}})\subset\mathrm{Fitt}_\Lambda^{\max}((\mathrm{cl}_L\otimes\mathbb{Z}_p)^{\vee-})^\sharp, \tag{9}$$

where S_{ram} denotes the set of finite places of K which ramify in L/K and a is a generator of $\mathrm{Ann}_\Lambda(\mu_L\otimes\mathbb{Z}_p)$.

Remark 8. It follows from the results in [BJ] that the inclusion in (9) becomes an equality over Λ' if Λ' is a maximal order containing Λ, and indeed

$$\zeta(\Lambda')\otimes_{\zeta(\Lambda)}\mathrm{Fitt}_\Lambda^{\max}((\mathrm{cl}_L\otimes\mathbb{Z}_p)^{\vee-})^\sharp=\mathrm{Fitt}_{\Lambda'}(\Lambda'\otimes_\Lambda(\mathrm{cl}_L\otimes\mathbb{Z}_p)^-).$$

Note that it suffices to assume the Strong Stark Conjecture rather than the ETNC to obtain results over Λ'. This conjecture is known to be true in many cases (cf. [Ni] Corollary 2).

Corollary 7.2. *Let L/K be a finite Galois CM-extension of number fields and p an odd prime such that $\mu_L\otimes\mathbb{Z}_p$ is G-c.t. and $T\Omega(L/K,0)_p^-=0$. Let $x\in\zeta(\Lambda')$ such that $x\cdot H^*\in M_{b\times b}(\Lambda)$ for any $H\in M_{b\times b}(\Lambda)$ and any $b\in\mathbb{N}$. Then for any $y\in L(0)^\sharp\mathrm{nr}(a)\prod_{\mathfrak{p}\in S_{\mathrm{ram}}}\mathrm{nr}(U_{\mathfrak{P}})$, the product $x\cdot y$ belongs to $\zeta(\mathbb{Z}_pG)$ and annihilates $\mathrm{cl}_L\otimes\mathbb{Z}_p$. In particular, if $x=(x_\chi)_\chi\in\bigoplus_{\chi\in\mathrm{Irr}_p(G)/\sim}\mathcal{D}^{-1}(\mathbb{Z}_p[\chi]/\mathbb{Z}_p)$ and S is a set of places of K containing $S_{\mathrm{ram}}\cup S_\infty$, then*

$$\mathrm{nr}(a)\cdot\sum_{\chi\in\mathrm{Irr}_p(G)/\sim}x_\chi L_S(0,\overline{\chi}^{\iota^{-1}})^\iota\mathrm{pr}_\chi\in\zeta(\mathbb{Z}_pG) \tag{10}$$

annihilates $\mathrm{cl}_L\otimes\mathbb{Z}_p$. Moreover, if G is abelian, then Brumer's conjecture is true outside the 2-part.

The last statement is, of course, still contained in [Gr07] (see Corollary 8.11). In the non-abelian case, the above corollary predicts more annihilators than [BJ], Theorem 1.2. But note that the explicit annilators (10) are the same as in loc.cit. We conclude with the

of Theorem 7.1. We briefly review the parts of the construction in [Gr07] which are of interest for us. For the set S_∞ of all infinite primes of K, there is a Tate sequence (cf. [RW96])

$$E_{S_\infty(L)} \rightarrowtail A_\infty \to B_\infty \twoheadrightarrow \nabla, \tag{11}$$

where A_∞ is G-c.t., B is $\mathbb{Z}G$-projective and ∇ fits into an exact sequence

$$\mathrm{cl}_L \rightarrowtail \nabla \twoheadrightarrow \overline{\nabla},$$

where $\overline{\nabla}$ is a $\mathbb{Z}G$-lattice. On minus parts, there is an isomorphism $\overline{\nabla}^- \simeq \bigoplus_{\mathfrak{p}\in S_{\mathrm{ram}}}(\mathrm{ind}_{G_\mathfrak{P}}^G(W_\mathfrak{P}^0))^-$, where $W_\mathfrak{P}^0$ can be described as the cokernel of the map (cf. [Gr07], §5)

$$\begin{array}{rcl} \mathbb{Z}\overline{G_\mathfrak{P}} & \longrightarrow & \mathbb{Z}G_\mathfrak{P}/(N_{G_\mathfrak{P}}) \times \mathbb{Z}\overline{G_\mathfrak{P}} \\ 1 & \mapsto & (N_{I_\mathfrak{P}}, 1-\phi_\mathfrak{P}^{-1}). \end{array}$$

Let κ be the canonical epimorphism $\mathbb{Z}G_\mathfrak{P} \oplus \mathbb{Z}G_\mathfrak{P} \twoheadrightarrow W_\mathfrak{P}^0$ and define a map $\delta_\mathfrak{p} : \mathbb{Z}G_\mathfrak{P} \cdot x_\mathfrak{p} \to W_\mathfrak{P}^0$ by $\delta(x_\mathfrak{p}) := \kappa((-1,1))$. We induce this map to $\mathbb{Z}G$ and sum over all ramified primes $\mathfrak{p}$ such that we obtain a map $\delta_0 : C^- \rightarrowtail \overline{\nabla}^-$, where C is $\mathbb{Z}G$-free with basis $x_\mathfrak{p}$, $\mathfrak{p} \in S_{\mathrm{ram}}$. Finally, let $\delta : C^- \rightarrowtail \nabla^-$ be any lift of δ_0 and choose a natural number x such that $x\nabla^- \subset \delta(C)^-$. Then there is a four-term exact sequence (cf. [Gr07], proof of Lemma 8.2 or [BJ], proof of Proposition 9.1)

$$\mathrm{cl}_L^- \rightarrowtail \nabla^-/\delta(C^-) \to x^{-1}\delta(C^-)/\delta(C^-) \twoheadrightarrow x^{-1}\delta(C^-)/\overline{\nabla}^-. \tag{12}$$

Since the minus part of the global units consists of the roots of unity, sequence (11) and the hypothesis on μ_L imply that the Λ-module $\nabla^- \otimes \mathbb{Z}_p$ is c.t. But C is $\mathbb{Z}G$-free and hence $\nabla^-/\delta(C^-) \otimes \mathbb{Z}_p$ is also c.t. It follows that we can apply Proposition 5.3 to sequence (12) tensored with $\mathbb{Z}_p$.
Let s denote the number of finite primes of K which ramify in L/K. Since $C^- \otimes \mathbb{Z}_p \simeq \Lambda^s$, we have a quadratic presentation

$$\Lambda^s \stackrel{\cdot x}{\longrightarrow} \Lambda^s \stackrel{x^{-1}\delta}{\twoheadrightarrow} (x^{-1}\delta(C^-)/\delta(C^-)) \otimes \mathbb{Z}_p \tag{13}$$

and thus

$$\mathrm{Fitt}_\Lambda((x^{-1}\delta(C^-)/\delta(C^-)) \otimes \mathbb{Z}_p) = [\langle \mathrm{nr}(x)^s \rangle]_{\mathrm{nr}(\Lambda)}. \tag{14}$$

Following the notation of [Gr07] and [BJ] we put $g_\mathfrak{p} := |I_\mathfrak{P}| + 1 - \phi_\mathfrak{P}^{-1}$ and $h_\mathfrak{p} = g_\mathfrak{p} e'_\mathfrak{p} + e''_\mathfrak{p}$ for $\mathfrak{p} \in S_{\mathrm{ram}}$. Since C is projective, sequence (11) gives rise to an exact sequence of finite c.t. Λ-modules

$$\mu_L \otimes \mathbb{Z}_p \rightarrowtail A_\infty^- \otimes \mathbb{Z}_p \to (B_\infty^-/\delta(C^-)) \otimes \mathbb{Z}_p \twoheadrightarrow (\nabla^-/\delta(C^-)) \otimes \mathbb{Z}_p.$$

Now we reinterpret [BJ], Proposition 8.7 in terms of Fitting invariants: If $T\Omega(L/K,0)_p^- = 0$, then

$$\mathrm{Fitt}_\Lambda((\nabla^-/\delta(C^-)) \otimes \mathbb{Z}_p) = \mathrm{Fitt}_\Lambda(\mu_L \otimes \mathbb{Z}_p) \cdot \left[\langle L^\sharp(0)\mathrm{nr}(h_{glob})\rangle\right]_{\mathrm{nr}(\Lambda)}, \tag{15}$$

where $h_{glob} = \prod_{\mathfrak{p}\in S_{\mathrm{ram}}} h_\mathfrak{p}$. Since μ_L is cyclic, there is an exact sequence

$$\Lambda \stackrel{\cdot a}{\rightarrowtail} \Lambda \twoheadrightarrow \mu_L \otimes \mathbb{Z}_p.$$

Then a clearly generates the Λ-annihilator of $\mu_L \otimes \mathbb{Z}_p$ and $\mathrm{Fitt}_\Lambda(\mu_L \otimes \mathbb{Z}_p)$ is generated by $\mathrm{nr}(a)$. Since there is an isomorphism (cf. [BJ], proof of Prop. 9.1)

$$(x^{-1}\delta(C^-)/\overline{\nabla}^-) \otimes \mathbb{Z}_p \simeq \bigoplus_{\mathfrak{p}\in S_{\mathrm{ram}}} \Lambda/x\Lambda(h_\mathfrak{p}^{-1}U_\mathfrak{P}),$$

the maximal Fitting invariant of this module contains $\prod_{\mathfrak{p}\in S_{\mathrm{ram}}} \mathrm{nr}(xh_\mathfrak{p}^{-1}U_\mathfrak{P})$. Now Proposition 5.3 together with (15) and (14) implies that

$$\mathrm{nr}(a)L^\sharp(0)\mathrm{nr}(h_{glob})\mathrm{nr}(x)^{-s} \prod_{\mathfrak{p}\in S_{\mathrm{ram}}} \mathrm{nr}(xh_\mathfrak{p}^{-1}U_\mathfrak{P}) = \mathrm{nr}(a)L^\sharp(0) \prod_{\mathfrak{p}\in S_{\mathrm{ram}}} \mathrm{nr}(U_\mathfrak{P}).$$

is contained in $\mathrm{Fitt}_\Lambda^{\max}((\mathrm{cl}_L \otimes \mathbb{Z}_p)^{\vee-})^\sharp$. □

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