Leading terms of Artin L-series at negative integers and annihilation of higher K-groups

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Abstract

Let L/K be a finite Galois extension of number fields with Galois group G. We use leading terms of Artin L-series at strictly negative integers to construct elements which we conjecture to lie in the annihilator ideal associated to the Galois action on the higher dimensional algebraic K-groups of the ring of integers in L. For abelian G our conjecture coincides with a conjecture of Snaith and thus generalizes also the well known Coates-Sinnott conjecture. We show that our conjecture is implied by the appropriate special case of the equivariant Tamagawa number conjecture (ETNC) provided that the Quillen-Lichtenbaum conjecture holds. Moreover, we prove induction results for the ETNC in the case of Tate motives $h^0(\mathrm{Spec}(L))(r)$, where r is a strictly negative integer. In particular, this implies the ETNC for the pair $(h^0(\mathrm{Spec}(L))(r), \mathfrak{M})$, where L is totally real, $r < 0$ is odd and $\mathfrak{M}$ is a maximal order containing $\mathbb{Z}[\frac{1}{2}]G$, and will also provide some evidence for our conjecture.

1. Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. To each finite set S of places of K which contains all the infinite places, one can associate a so-called "Stickelberger element" θ_S in the center of the group ring algebra $\mathbb{C}G$. This Stickelberger element is defined via L-values at zero of S-truncated Artin L-functions attached to the (complex) characters of G. Let us denote the roots of unity of L by μ_L and the class group of L by cl_L. Assume that S contains the set S_{ram} of all finite primes of K which ramify in L/K. Then it was independently shown in [Ca79], [DR80] and [Ba77] that for abelian G one has

$$\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)\theta_S \subset \mathbb{Z}G, \tag{1}$$

where we denote by $\mathrm{Ann}_\Lambda(M)$ the annihilator ideal of M regarded as a module over the ring Λ. Now Brumer's conjecture asserts that $\mathrm{Ann}_{\mathbb{Z}G}(\mu_L)\theta_S$ annihilates cl_L. Using L-values at strictly negative integers r, one can define higher Stickelberger elements $\theta_S(r)$. Coates and Sinnott [CS74] conjectured that these elements can be used to construct annihilators of the higher K-groups $K_{-2r}(\mathfrak{o}_{L,S})$, where we denote by $\mathfrak{o}_{L,S}$ the ring of $S(L)$-integers in L for any finite set S of places of K; here, we write $S(L)$ for the set of places of L which lie above those in S. But if, for example, L is totally real and r is even, this conjecture merely predicts that zero annihilates $K_{-2r}(\mathfrak{o}_{L,S})$. Assuming the validity of a conjecture of Gross [Gr05] which is a higher analogue of Stark's conjecture, Snaith [Sn06] constructed a fractional ideal $\mathcal{J}_r^S$ in the rational group ring $\mathbb{Q}G$, the so-called canonical fractional Galois ideal. The construction involves leading terms rather than values of Artin L-functions at negative integers.

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Snaith conjectured that for any odd prime p one has

$$\mathrm{Ann}_{\mathbb{Z}_pG}(K_{1-2r}(\mathfrak{o}_{L,S})_{\mathrm{tor}} \otimes \mathbb{Z}_p) \cdot \mathcal{J}_r^S \cap \mathbb{Z}_pG \subset \mathrm{Ann}_{\mathbb{Z}_pG}(K_{-2r}(\mathfrak{o}_{L,S}) \otimes \mathbb{Z}_p). \tag{2}$$

Here, we write M_{tor} for the R-torsion submodule of an RG-module M if R is a commutative ring and G is a finite group.
The Quillen-Lichtenbaum conjecture relates K-groups to étale cohomology, predicting that for all odd primes p, $r \in \mathbb{Z}_{<0}$ and $i = 0, 1$ the canonical p-adic Chern class maps

$$\mathrm{ch}_{1-r,2-i}^{(p)} : K_{i-2r}(\mathfrak{o}_L) \otimes \mathbb{Z}_p \to H_{\text{ét}}^{2-i}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r)) \tag{3}$$

constructed by Soulé [So79] are isomorphisms. Soulé proved surjectivity and recent unpublished work of Rost and Weibel seems to have led to a proof of the Quillen-Lichtenbaum conjecture such that one may replace the K-groups in the above conjecture by the corresponding étale cohomology groups.
In [BS10] the authors generalize the notion of the canonical fractional Galois ideal to arbitrary Galois extensions of number fields. We will introduce a different generalization of $\mathcal{J}_r^S$ to non-abelian Galois extensions and, as in the abelian case, we conjecture that $\mathcal{J}_r^S$ can be used to construct annihilators of $K_{-2r}(\mathfrak{o}_{L,S}) \otimes \mathbb{Z}_p$ for all odd primes p. We show that $\mathcal{J}_r^S$ can be expressed in terms of (non-commutative) Fitting invariants. Assuming the validity of the Quillen-Lichtenbaum conjecture, this gives the connection to the appropriate special case of the equivariant Tamagawa number conjecture (ETNC) as formulated by Burns and Flach [BF01]. More precisely, the ETNC for the Tate motive $\mathbb{Q}(r)_L := h^0(\mathrm{Spec}(L))(r)$ with coefficients in $\mathbb{Z}[\frac{1}{2}]G$ implies the cohomological version of our conjecture (and hence the cohomological version of Snaith's conjecture). This fits well into the picture that the ETNC should be the central conjecture in the field. Together with a recent result of Burns [Bu] this will provide some evidence for our conjecture.
We use a new formulation of the ETNC for Tate motives $\mathbb{Q}(r)_L$, $r \in \mathbb{Z}_{<0}$ due to Burns [Bu10] to prove induction results which are well known in the case $r = 0$. This will provide some further evidence for our conjecture and in fact will lead to a proof of the ETNC for the pair $(\mathbb{Q}(r)_L, \mathfrak{M})$, where L is totally real, $r < 0$ is odd and $\mathfrak{M}$ is a maximal order containing $\mathbb{Z}[\frac{1}{2}]G$. As a byproduct we obtain spectral sequences for χ-twists which have been introduced by Burns [Bu08].
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2. The conjectures

Let L/K be a finite Galois extension of number fields with Galois group G. For any place v of K we fix a place w of L above v and write G_w resp. I_w for the decomposition group resp. inertia subgroup of L/K at w. Moreover, we choose a lift $\phi_w \in G_w$ of the Frobenius automorphism at w. Let S be a finite set of places of K containing the set S_∞ of all infinite primes and the set S_{ram} of all primes which ramify in L/K. We denote the S-truncated Artin L-function attached to a (complex) character χ of G by $L_S(s, \chi)$, and the leading coefficient of the Taylor expansion of $L_S(s, \chi)$ at $s = r$, $r \in \mathbb{Z}$ by $L_S^*(r, \chi)$. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}G) = \prod_{\chi \in \mathrm{Irr}\,(G)} \mathbb{C}$, where $\mathrm{Irr}\,(G)$ denotes the set of irreducible complex characters of G, and where we write $\zeta(\Lambda)$ for the center of any ring Λ. We define the equivariant Artin L-function to be the meromorphic $\zeta(\mathbb{C}G)$-valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \mathrm{Irr}\,(G)}.$$

Let us denote by $^\sharp : \mathbb{C}G \to \mathbb{C}G$ the involution which is induced by mapping each $g \in G$ to its inverse, and let $\mathrm{nr} : \mathbb{C}G \to \zeta(\mathbb{C}G)$ be the reduced norm map. If T is a second finite set of places of K sucht that $S \cap T = \emptyset$, we define $\delta_T(s) := (\delta_T(s, \chi))_{\chi \in \mathrm{Irr}\,(G)}$, where $\delta_T(s, \chi) = \prod_{v \in T} \mathrm{nr}(1 - N(v)^{1-s} \phi_w^{-1})$, and put

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\sharp.$$

These functions are the so-called (S,T)-modified G-equivariant L-functions and for $r \in \mathbb{Z}_{\leqslant 0}$ we define Stickelberger elements

$$\theta_S^T(r) := \Theta_{S,T}(r) \in \zeta(\mathbb{C}G).$$

Note that these elements actually lie in the center of the rational group ring $\mathbb{Q}G$ by a classical result of Siegel [Si70]. If T is empty, we abbreviate $\theta_S^T(r)$ by $\theta_S(r)$. Now the higher K-theoretic analogue of Brumer's conjecture asserts the following:

Conjecture 2.1 Coates-Sinnott. For all abelian extensions L/K, S and $p \neq 2$ as above, and all $r \in \mathbb{Z}_{<0}$ we have

$$\mathrm{Ann}_{\mathbb{Z}_pG}(K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tor}} \otimes \mathbb{Z}_p) \cdot \theta_S(r) \subset \mathrm{Ann}_{\mathbb{Z}_pG}(K_{-2r}(\mathfrak{o}_L) \otimes \mathbb{Z}_p).$$

Since we would like to use étale cohomology rather than K-theory, we state the following conjecture:

Conjecture 2.2 Quillen-Lichtenbaum. The p-adic Chern class maps $\mathrm{ch}^{(p)}_{1-r,2-i}$ are isomorphisms for all primes $p \neq 2$, $r \in \mathbb{Z}_{<0}$ and $i = 0, 1$.

Note that Dwyer and Friedlander [DF85] have constructed Chern class maps also for $p = 2$, but these maps are in general neither surjective nor injective. Since the Chern class maps are G-equivariant, the following conjecture should be equivalent to Conjecture 2.1.

Conjecture 2.3 Coates-Sinnott, cohomological version. For all abelian extensions L/K, S and $p \neq 2$ as above, and all $r \in \mathbb{Z}_{<0}$ we have

$$\mathrm{Ann}_{\mathbb{Z}_pG}(H^1_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r))_{\mathrm{tor}}) \cdot \theta_S(r) \subset \mathrm{Ann}_{\mathbb{Z}_pG}(H^2_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r))).$$

Let us denote the cyclotomic $\mathbb{Z}_p$-extension of a number field K by K^p_∞, and the maximal real subfield of K by K^+. Then the strongest piece of evidence in support of Conjecture 2.3 is the following:

Theorem 2.4 Burns-Greither, [BG03b], Cor. 5.3. *Conjecture 2.3 is true if K is totally real, L is either totally real or CM, p is a prime such that $L \cap K^p_\infty = K$, and the Iwasawa μ-invariant attached to L^+ and p vanishes.*

The vanishing of the μ-invariant is a long-standing conjecture of Iwasawa theory. The most general result is still due to Ferrero and Washington [FW79] and says that $\mu = 0$ for absolutely abelian extensions. The condition $L \cap K^p_\infty = K$ can be relaxed (cf. [Ng05]).
Let $\Sigma(L)$ denote the set of embeddings of L into the complex numbers; we have $|\Sigma(L)| = r_1 + 2r_2$, where r_1 and r_2 are the number of real embeddings and the number of pairs of complex embeddings, respectively. For $r \in \mathbb{Z}_{<0}$ we define

$$H_r(L) := \bigoplus_{\Sigma(L)} (2\pi i)^{-r}\mathbb{Z}$$

which is endowed with a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action, diagonally on $\Sigma(L)$ and on $(2\pi i)^{-r}$. We denote the fixed points of $H_r(L)$ under this action by $H_r^+(L)$, and it is easily seen that

$$\mathrm{rk}_{\mathbb{Z}}(H_r^+(L)) = d_{1-r} := \begin{cases} r_2 & \text{if} \quad 2 \nmid r \\ r_1 + r_2 & \text{if} \quad 2 \mid r. \end{cases}$$

Borel [Bo74] has determined the group structure of the higher K-groups: For $r < 0$ the groups $K_{-2r}(\mathfrak{o}_L)$ are finite, $K_{1-2r}(\mathfrak{o}_L)$ has a finite torsion submodule and $K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tf}} \simeq \mathbb{Z}^{d_{1-r}}$, where we use the notation $M_{\mathrm{tf}} := M/M_{\mathrm{tor}}$ for any G-module M. Indeed, Borel defined higher regulator maps

$$\rho_r : K_{1-2r}(\mathfrak{o}_L) \longrightarrow H_r^+(L) \otimes \mathbb{R}$$

and showed that the kernel is finite and the image is a full lattice in $H_r^+(L) \otimes \mathbb{R}$. The covolume of this lattice is called the Borel regulator and will be denoted by $R_r^B(L)$. Moreover, Borel obtained that there is a nonzero rational number q_r such that

$$\zeta_L^*(r) = q_r \cdot R_r^B(L),$$

where $\zeta_L(s)$ denotes the Dedekind zeta function attached to L. Lichtenbaum conjectured the following generalization of the classical class number formula:

Conjecture 2.5 Lichtenbaum. For all $r < 0$ we have

$$\zeta_L^*(r) = \pm \frac{|K_{-2r}(\mathfrak{o}_L)|}{|K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tor}}|} \cdot R_r^B(L)$$

up to powers of 2.

In view of Conjecture 2.2 we may also state the following cohomological version of this conjecture:

Conjecture 2.6 Lichtenbaum, cohomological version. For all $r < 0$ and all primes $p \neq 2$ we have

$$\zeta_L^*(r) \sim_p \frac{|H^2_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r))|}{|H^1_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r))_{\mathrm{tor}}|} \cdot R_r^B(L),$$

where $\sim_p$ means "equal up to a p-adic unit".

As a consequence of his proof of the Iwasawa main conjecture for totally real fields, Wiles (cf. [Wi90]) obtained the following result:

Theorem 2.7 Wiles. *If L is totally real and $r < 0$ is odd, then Conjecture 2.6 holds.*

Note that in this case $\zeta_L^*(r) = \zeta_L(r)$ and the Borel regulator is trivial, since $d_{1-r} = 0$. In the case of absolutely abelian extensions Conjecture 2.6 is known in full generality (cf. [KNF96, BN02, HK03, BG03a]).
Since the Borel regulator map induces an isomorphism of $\mathbb{R}G$-modules, the Noether-Deuring theorem implies the existence of $\mathbb{Q}G$-isomorphisms

$$\phi_r : H_r^+(L) \otimes \mathbb{Q} \xrightarrow{\simeq} K_{1-2r}(\mathfrak{o}_L) \otimes \mathbb{Q}. \tag{4}$$

Let $R(G)$ denote the ring of virtual characters with values in the algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$. Let F be a finite Galois extension of $\mathbb{Q}$ such that each representation of G can be realized over F, and let V_χ be a FG-module with character χ. We form regulator maps

$$\begin{aligned} R_{\phi_r} : R(G) &\longrightarrow \mathbb{C}^\times \\ \chi &\mapsto \det(\rho_r \circ \phi_r | \mathrm{Hom}_G(V_{\check{\chi}}, H_r^+(L) \otimes \mathbb{C})), \end{aligned}$$

where $\check{\chi}$ is the character contragradient to χ. Moreover, we define a function

$$\begin{aligned} A^S_{\phi_r} : R(G) &\longrightarrow \mathbb{C}^\times \\ \chi &\mapsto R_{\phi_r}(\chi)/L_S^*(r, \chi). \end{aligned}$$

The higher analogue of Stark's conjecture is the following (cf. [Gr05], Conj. 3.11):

Conjecture 2.8 Gross. We have $A^S_{\phi_r}(\chi^\sigma) = A^S_{\phi_r}(\chi)^\sigma$ for all $r < 0$ and all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Mimicking the construction of Snaith [Sn06], §4.3, we now define the canonical fractional Galois ideal for an arbitrary finite extension of number fields. Assume for the moment that K is any field, A is a finite dimensional semi-simple K-algebra and Λ is an R-order in A, where R is a commutative noetherian domain with field of quotients K. If P is a finitely generated projective Λ-module, and α is an endomorphism of $P \otimes K$, then we may choose a finitely generated (projective) Λ-module Q such that $P \oplus Q$ is free, and taking the reduced norm of $\alpha \oplus 1$ with respect to a chosen Λ-basis yields a well defined element

$$\mathrm{nr}_P(\alpha) \in \zeta(A).$$

Since $H_r(L)$ is a $\mathbb{Z}[G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module which is free over $\mathbb{Z}G$, we may apply the above construction to the projective $\mathbb{Z}[\frac{1}{2}]G$-module $H_r^+(L) \otimes \mathbb{Z}[\frac{1}{2}]$: Choose an isomorphism ϕ_r as in (4) and define $\mathcal{I}_{\phi_r}$ to be the (finitely generated) $\zeta(\mathbb{Z}[\frac{1}{2}]G)$-submodule of $\zeta(\mathbb{Q}G)$ generated by all the elements $\mathrm{nr}_{H_r^+(L)\otimes\mathbb{Z}[\frac{1}{2}]}(\alpha)$, where α runs through the $\mathbb{Q}G$-endomorphisms of $H_r^+(L) \otimes \mathbb{Q}$ satisfying the integrality condition

$$\alpha f_r(K_{1-2r}(\mathfrak{o}_L)) \subset H_r^+(L),$$

where $f_r := \phi_r^{-1}$. We may regard $A_{\phi_r}^S$ as an element of $\zeta(\mathbb{C}G)^\times$; its χ-component via the isomorphism $\zeta(\mathbb{C}G) \simeq \bigoplus_{\chi \in \mathrm{Irr}\,(G)} \mathbb{C}$ is just $A_{\phi_r}^S(\chi)$. We put

$$\mathcal{J}_r^S = \mathcal{J}_r^S(L/K) := \mathcal{I}_{\phi_r} \cdot ((A_{\phi_r}^S)^{-1})^\sharp.$$

Definition 2.9. Assume that Gross' conjecture (Conjecture 2.8) holds. Then $\mathcal{J}_r^S$ is a finitely generated $\zeta(\mathbb{Z}[\frac{1}{2}]G)$-submodule of $\zeta(\mathbb{Q}G)$ which is called the **canonical fractional Galois ideal**.

It is clear that our definition coincides with the definition of Snaith for abelian G; for non-abelian G, however, our definition differs from that in [BS10]. Even if the definition involves the choice of an isomorphism ϕ_r, we have the following proposition which is proved along the lines of [Sn06], Prop. 4.5.

Proposition 2.10. *The definition of $\mathcal{J}_r^S$ does not depend on the choice of ϕ_r.*

Before we can state our new conjecture, we have to introduce the following construction in order to get rid of possible denominators. Assume that Λ is an R-order in a semi-simple finite dimensional K-algebra A, where R is an integrally closed commutative noetherian ring with field of quotients K. We choose a maximal order Λ' containing Λ. For any matrix $H \in M_{b\times b}(\Lambda)$ there is a matrix $H^* \in M_{b\times b}(\Lambda')$ such that $H^*H = HH^* = \mathrm{nr}(H) \cdot 1_{b\times b}$ (cf. [Ni10], Lemma 4.1; the additional assumption on R to be complete local is not necessary), where for any ring Λ we denote by $M_{a\times b}(\Lambda)$ the set of all $a \times b$ matrices with entries in Λ. If $\tilde{H} \in M_{b\times b}(\Lambda)$ is a second matrix, then $(H\tilde{H})^* = \tilde{H}^*H^*$. We define

$$\mathcal{H}(\Lambda) := \{x \in \zeta(\Lambda) | xH^* \in M_{b\times b}(\Lambda) \forall b \in \mathbb{N}\ \forall H \in M_{b\times b}(\Lambda)\}.$$

Note that in particular $x \cdot \mathrm{nr}(H) \in \Lambda$ for all $x \in \mathcal{H}(\Lambda)$ and all matrices H with entries in Λ. If p is a prime, we will abbreviate $\mathcal{H}(\mathbb{Z}_pG)$ by $\mathcal{H}_p(G)$. We now state:

Conjecture 2.11. Let L/K be a finite Galois extension of number fields with Galois group G, $r \in \mathbb{Z}_{<0}$ and S a finite set of primes of K containing all the infinite primes and those which ramify in L/K. Then for all odd primes p and any $x \in \mathrm{Ann}_{\mathbb{Z}_pG}(K_{1-2r}(\mathfrak{o}_{L,S})_{\mathrm{tor}} \otimes \mathbb{Z}_p)$ we have

$$\mathrm{nr}(x) \cdot \mathcal{H}_p(G) \cdot \mathcal{J}_r^S \subset \mathrm{Ann}_{\mathbb{Z}_pG}(K_{-2r}(\mathfrak{o}_{L,S}) \otimes \mathbb{Z}_p).$$

As before, we may also formulate a cohomological version of this conjecture:

Conjecture 2.12. Let r and S as above. Then for all odd primes p and any $x \in \mathrm{Ann}_{\mathbb{Z}_pG}(H^1_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))_{\text{tor}})$ we have

$$\mathrm{nr}(x) \cdot \mathcal{H}_p(G) \cdot \mathcal{J}_r^S \subset \mathrm{Ann}_{\mathbb{Z}_pG}(H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))).$$

Remark 1. i) Since $\mathcal{H}_p(G) = \mathbb{Z}_pG$ for abelian G, Conjecture 2.11 recovers Snaith's conjecture ([Sn06], Conj. 5.1, see (2) above), but with an additional integrality statement which was posed as Question 5.3 in loc.cit.

ii) The localization sequence in p-adic étale cohomology leads to an isomorphism (cf. [So79])

$$H^1_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r)) \simeq H^1_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))$$

and to an exact sequence

$$H^2_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r)) \rightarrowtail H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r)) \twoheadrightarrow \bigoplus_{\substack{w \in S(L) \\ w \nmid p\infty}} H^1_{\text{ét}}(L(w), \mathbb{Z}_p(-r)), \qquad (5)$$

where $L(w)$ denotes the residue field at w. Let $G_L := \mathrm{Gal}(\mathbb{Q}^c/L)$; the above isomorphism implies that

$$H^1_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))_{\text{tor}} \simeq H^0_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Q}_p/\mathbb{Z}_p(1-r)) = \mathbb{Q}_p/\mathbb{Z}_p(1-r)^{G_L} =: \mu_{1-r}(L)$$

whose annihilator is generated by the elements $1 - \phi_w \cdot N(v)^{r-1}$, where v runs through the finite places of K not above p, $v \notin S_{\text{ram}}$ (cf. [Co77]). Similarly, the annihilator of $H^1_{\text{ét}}(L(w), \mathbb{Z}_p(-r))$ is generated by $1 - \phi_w \cdot N(v)^r$. Moreover, if S' is a second finite set of places of K containing S, we have

$$\mathcal{J}_r^{S'} = \mathcal{J}_r^S \cdot \delta_{S' \setminus S}(1-r)^\sharp = \mathcal{J}_r^S \cdot \prod_{v \in S' \setminus S} \mathrm{nr}(1 - \phi_w \cdot N(v)^r).$$

Assume that $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_pG)$ which is fulfilled, for instance, if G is abelian or if $p \nmid |G|$. Then sequence (5) implies that the validity of Conjecture 2.12 for the set S implies Conjecture 2.12 for the set S'. For the required fact that $\mathrm{nr}(1 - \phi_w \cdot N(v)^r)$ annihilates $H^1_{\text{ét}}(L(w), \mathbb{Z}_p(-r))$ see Theorem 4.3 below.

iii) Assume that L is totally real and $r < 0$ is odd. Then

$$\mathcal{J}_r^S = \theta_S(r) \cdot \zeta(\mathbb{Z}[\tfrac{1}{2}]G)$$

and Conjecture 2.12 predicts that in particular

$$\mathcal{H}_p(G) \cdot \theta_S^T(r) \subset \mathrm{Ann}_{\mathbb{Z}_pG}(H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r)))$$

for all finite sets T of places of K such that $S \cap T = \emptyset$. By virtue of sequence (5) we see that Conjecture 2.12 generalizes Conjecture 2.3. Similar observations hold if L/K is a CM-extension and $r < 0$ is even.

iv) In [BdJG] the authors conjecture the existence of elements in odd degree higher algebraic K-groups of number fields that are related in an explicit way to the values at strictly negative integers of the first derivatives of Artin L-functions. These elements can be used to construct conjectural annihilators of even degree higher algebraic K-groups (cf. loc.cit., Th. 3.1). Their approach is different from ours (as they use values of first derivatives instead of leading terms), and probably more convenient for providing numerical evidence.

At last, we have to deal with the ETNC for the Tate motive $\mathbb{Q}(r)_L$. We will give a reformulation due to Burns [Bu10]. Let G be a finite group and let $\mathfrak{A}$ be an R-order in $\mathbb{Q}G$ containing the group ring RG, where R is a finitely generated subring of $\mathbb{Q}$. The reduced norm induces an injective homomorphism $\mathrm{nr} : K_1(\mathbb{R}G) \to \zeta(\mathbb{R}G)^\times$, and there exists an extended boundary homomorphism (due to Burns and Flach [BF01])

$$\hat{\partial}_{\mathfrak{A}} : \zeta(\mathbb{R}G)^\times \longrightarrow K_0(\mathfrak{A}, \mathbb{R}G)$$

such that $\hat{\partial}_{\mathfrak{A}} \circ \mathrm{nr}$ is the usual boundary homomorphism of the localization sequence of relative K-theory (cf. [CR87], p. 72).

DEFINITION 2.13. Let H^0 and H^1 be finitely generated $\mathfrak{A}$-modules.

i) An element $\epsilon \in \mathrm{Ext}^2_{\mathfrak{A}}(H^1, H^0)$ is called **perfect** if it can be represented as a Yoneda extension by an exact sequence

$$H^0 \rightarrowtail A \to B \twoheadrightarrow H^1$$

in which A and B are both finitely generated $\mathfrak{A}$-modules of finite projective dimension.

ii) An **augmented trivialized extension** (a.t.e. for short) of $\mathfrak{A}$-modules is a triple $\tau = (\epsilon_\tau, \lambda_\tau, \mathfrak{L}^*_\tau)$ comprising a perfect 2-extension $\epsilon_\tau \in \mathrm{Ext}^2_{\mathfrak{A}}(H^1_\tau, H^0_\tau)$ of finitely generated $\mathfrak{A}$-modules, an isomorphism $\lambda_\tau : H^0_\tau \otimes_R \mathbb{R} \simeq H^1_\tau \otimes_R \mathbb{R}$ of $\mathbb{R}G$-modules and an element $\mathfrak{L}^*_\tau \in \zeta(\mathbb{R}G)^\times$.

iii) The **Euler characteristic** $\chi(\tau)$ of an a.t.e. τ is defined to be

$$\chi(\tau) := \chi_{\mathfrak{A},\mathbb{R}G}(\epsilon_\tau, \lambda_\tau) - \hat{\partial}_{\mathfrak{A}}(\mathfrak{L}^*_\tau) \in K_0(\mathfrak{A}, \mathbb{R}G),$$

where the first term on the right hand side denotes the refined Euler characteristic of the perfect complex $A \to B$ whose extension class is ϵ_τ, trivialized by λ_τ (cf. [Bu03]).

We now state the following proposition due to Burns ([Bu10], Prop. 4.2.6) which provides a reformulation of the ETNC for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}G)$. As before, let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty$.

PROPOSITION 2.14. *Let $r \in \mathbb{Z}_{<0}$ and assume that the Quillen-Lichtenbaum Conjecture (Conjecture 2.2) holds for all odd primes p. Then there exists an a.t.e. τ_r of $\mathbb{Z}G$-modules with the following properties:*

i) $H^0_{\tau_r} \otimes \mathbb{Z}[\frac{1}{2}] = K_{1-2r}(\mathfrak{o}_L) \otimes \mathbb{Z}[\frac{1}{2}]$ *and* $(H^1_{\tau_r})_{\mathrm{tor}} \otimes \mathbb{Z}[\frac{1}{2}] = K_{-2r}(\mathfrak{o}_{L,S}) \otimes \mathbb{Z}[\frac{1}{2}]$ *and* $(H^1_{\tau_r})_{\mathrm{tf}} \otimes \mathbb{Z}[\frac{1}{2}] = H^+_r \otimes \mathbb{Z}[\frac{1}{2}]$;

ii) λ_{τ_r} *is induced by* -1 *times the Beilinson regulator map (as described in [BG02]);*

iii) $\mathfrak{L}^*_{\tau_r} = L^*_S(r)^\sharp$;

iv) *The Euler characteristic* $\chi(\tau_r)$ *vanishes if and only if the ETNC holds for the pair* $(\mathbb{Q}(r)_L, \mathbb{Z}G)$.

Since the Borel regulator is twice the Beilinson regulator by a result of Burgos Gil [BG02], there will be no essential difference if we replace Borel's regulator by Beilinson's throughout the above mentioned conjectures.

In fact, the proof of Proposition 2.14 shows more: For each prime p we set

$$C^\cdot_{p,r} := R\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(r)), \mathbb{Z}_p[-2]),$$

where $R\Gamma_c(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(r))$ is the complex of $\mathbb{Z}_pG$-modules given by the cohomology with compact support as defined in [BF01], p. 522. For any ring Λ we write $\mathcal{D}(\Lambda)$ for the derived category of Λ-modules and denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$ consisting of perfect complexes by $\mathcal{D}^{\mathrm{perf}}(\Lambda)$. Then the complex $C^\cdot_{p,r}$ belongs to $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_pG)$ and is acyclic outside degrees 0 and 1. Then for any prime p there are isomorphisms

$$H^i_{\tau_r} \otimes \mathbb{Z}_p \simeq H^i(C^\cdot_{p,r}),\ i = 0, 1$$

and the extension class ϵ_{τ_r} corresponds to $\oplus C^\cdot_{p,r}$ under the homomorphism

$$\mathrm{Ext}^2_{\mathbb{Z}G}(H^1_{\tau_r}, H^0_{\tau_r}) \longrightarrow \bigoplus_p \mathrm{Ext}^2_{\mathbb{Z}_pG}(H^1(C^\cdot_{p,r}), H^0(C^\cdot_{p,r})).$$

Moreover, $C^{\cdot}_{p,r}$ fits into an exact triangle in $\mathcal{D}(\mathbb{Z}_pG)$ (cf. [BF98], Prop. 4.1).:

$$\bigoplus_{w\in S_\infty(L)} R\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_\Delta(L(w),\mathbb{Z}_p(r)),\mathbb{Z}_p)[-3] \longrightarrow R\Gamma(\mathfrak{o}_{L,S}[1/p],\mathbb{Z}_p(1-r)) \longrightarrow C^{\cdot}_{p,r}[-1] \longrightarrow \qquad (6)$$

where $R\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_\Delta(L(w),\mathbb{Z}_p(r)),\mathbb{Z}_p)$ is given by $\mathbb{Z}_p(-r)$ (placed in degree zero) if w is complex, and by

$$\mathbb{Z}_p \xrightarrow{\delta_1} \mathbb{Z}_p \xrightarrow{\delta_0} \mathbb{Z}_p \xrightarrow{\delta_1} \dots$$

if w is real and δ_i is multiplication with $1-(-1)^{r+i+1}$ for $i=0,1$; here, the first $\mathbb{Z}_p$ is placed in degree 0. Let us denote the sets of real and complex places of K by $S_\mathbb{R}$ and $S_\mathbb{C}$, respectively. Then the triangle (6) gives the following description of $H^i_{\tau_r}$, $i=0,1$: For each prime p there are isomorphisms

$$H^0_{\tau_r}\otimes\mathbb{Z}_p \simeq H^1_{\text{ét}}(\mathfrak{o}_L[1/p],\mathbb{Z}_p(1-r)),\ \ (H^1_{\tau_r})_{\text{tor}}\otimes\mathbb{Z}_p \simeq H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p],\mathbb{Z}_p(1-r)).$$

Moreover, we have an isomorphism

$$(H^1_{\tau_r})_{\text{tf}}\otimes\mathbb{Z}_p \simeq H^+_r(L)\otimes\mathbb{Z}_p$$

unless $p=2$ and r is even; in this case we have an exact sequence

$$(H^1_{\tau_r})_{\text{tf}}\otimes\mathbb{Z}_2 \rightarrowtail H^+_r(L)\otimes\mathbb{Z}_2 \twoheadrightarrow \bigoplus_{w\in S_\mathbb{R}(L)} \mathbb{Z}/2\mathbb{Z}.$$

Globally, we thus obtain

$$H^0_{\tau_r} \simeq H^1(\mathfrak{o}_L,\mathbb{Z}(1-r)),$$

$$H^2(\mathfrak{o}_{L,S},\mathbb{Z}(1-r)) \rightarrowtail H^1_{\tau_r} \to H^+_r(L) \twoheadrightarrow \bigoplus_{w\in S_\mathbb{R}(L)} \mathbb{Z}/2\mathbb{Z} \qquad (7)$$

if r is even, and without the rightmost term if r is odd. Here we use the global "models" for the étale cohomology groups as defined in [CKPS98], §3. Hence $H^2(\mathfrak{o}_{L,S},\mathbb{Z}(1-r))$ is simply the direct product $\prod_p H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p],\mathbb{Z}_p(1-r))$, whereas the construction of $H^1(\mathfrak{o}_L,\mathbb{Z}(1-r))$ is more involved: the Chern class maps $\mathrm{ch}^{(p)}_{1-r,1}$ for all primes p yield a homomorphism

$$\mathrm{ch}_{1-r,1}: K_{1-2r}(\mathfrak{o}_L)\otimes\hat{\mathbb{Z}} \longrightarrow H^1_\infty(\mathfrak{o}_L,\mathbb{Z}(1-r)) := \prod_p H^1_{\text{ét}}(\mathfrak{o}_L[1/p],\mathbb{Z}_p(1-r))$$

with finite cokernel T of 2-power order. Then there is an exact sequence (cf. [CKPS98], Lemma 3.8)

$$\mathrm{ch}_{1-r,1}(K_{1-2r}(\mathfrak{o}_L)) \rightarrowtail H^1(\mathfrak{o}_L,\mathbb{Z}(1-r)) \twoheadrightarrow T$$

and the isomorphism λ_{τ_r} of Proposition 2.14 is $-\lambda_r\circ\mathrm{ch}^{-1}_{1-r,1}$, where $\lambda_r=\frac{1}{2}\cdot\rho_r$ is the Beilinson regulator. Since the proof of Proposition 2.14 makes use of the Quillen-Lichtenbaum Conjecture only to derive assertion (i), we obtain the following result (for the last two assertions compare [Bu10], Lemma 6.1.1, Th. 5.1.1 and Remark 5.1.3).

Proposition 2.15. *Let L/K be a Galois extension of number fields with Galois group G and let $r\in\mathbb{Z}_{<0}$. Then there exists an a.t.e. τ_r of $\mathbb{Z}G$-modules with the following properties:*

i) *$H^i_{\tau_r}$ are as described in (7), $i=0,1$;*
ii) *$\lambda_{\tau_r}=-\lambda_r\circ\mathrm{ch}^{-1}_{1-r,1}$;*
iii) *$\mathfrak{L}^*_{\tau_r}=L^*_S(r)^\sharp$;*
iv) *The Euler characteristic $\chi(\tau_r)$ vanishes if and only if the ETNC holds for the pair $(\mathbb{Q}(r)_L,\mathbb{Z}G)$;*
v) *$\chi(\tau_r)\in K_0(\mathbb{Z}G,\mathbb{Q}G)$ if and only if Gross' Conjecture (Conjecture 2.8) holds for r;*

vi) *Assume that Gross' Conjecture holds, and let $\mathfrak{A}$ be an R-order in $\mathbb{Q}G$ containing $\mathbb{Z}G$, where R is a finitely generated subring of $\mathbb{Q}$. Then $\chi(\tau_r)$ lies in the kernel of the natural map $K_0(\mathbb{Z}G, \mathbb{Q}G) \to K_0(\mathfrak{A}, \mathbb{Q}G)$ if and only if the ETNC holds for the pair $(\mathbb{Q}(r)_L, \mathfrak{A})$.*

3. An alternative definition of the canonical fractional Galois ideal

In this section we fix an integer $r < 0$, an *abelian* extension of number fields L/K with Galois group G and a finite set S of places of K containing $S_{\mathrm{ram}} \cup S_\infty$. For any $\mathbb{Z}G$-module (resp. $\mathbb{Z}_pG$-module) M we denote the Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ (resp. $\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$) of M by $M^\vee$ which is equipped with the natural G-action $(gf)(m) = f(g^{-1}m)$ for $f \in M^\vee$, $g \in G$ and $m \in M$. Assuming Conjecture 2.8, we choose an equivariant injection

$$\phi_r : H_r^+(L) \otimes \mathbb{Z}[\tfrac{1}{2}] \rightarrowtail K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tf}} \otimes \mathbb{Z}[\tfrac{1}{2}]$$

and define a $\mathbb{Z}[\frac{1}{2}]G$-submodule $\widetilde{\mathcal{J}}_r^S = \widetilde{\mathcal{J}}_r^S(L/K)$ of $\mathbb{Q}G$ by declaring

$$(\widetilde{\mathcal{J}}_r^S)^\sharp = \mathrm{Fitt}_{\mathbb{Z}[\frac{1}{2}]G}((\mathrm{cok}\,\phi_r)^\vee) \cdot (A_{\phi_r}^S)^{-1}.$$

We will need the following lemma.

LEMMA 3.1. *Let R be a commutative ring with identity and let $\pi : P \twoheadrightarrow M$ be an epimorphism of finitely generated R-modules with projective P. Then $\mathrm{Fitt}_R(M)$ is generated by the elements $\mathrm{nr}_P(\alpha)$, where α runs through the endomorphisms of P such that $\mathrm{im}\,(\alpha)$ is contained in the kernel of π.*

Proof. Two R-ideals are equal if they become equal after localization at each prime ideal. But a projective module over a commutative local ring is free, and in this case the above assertion is clear by the definition of Fitting ideals and the fact that they are well defined. $\square$

PROPOSITION 3.2. *Let L/K be an abelian extension of number fields, r and S as above. Then we have an equality*

$$\widetilde{\mathcal{J}}_r^S = \mathcal{J}_r^S.$$

In particular, the definition of $\widetilde{\mathcal{J}}_r^S$ does not depend on ϕ_r.

Proof. As before, we will drop $\_ \otimes \mathbb{Z}[\frac{1}{2}]$ from the notation. Choose a G-equivariant embedding $f_r : K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tf}} \rightarrowtail H_r^+(L)$ and put $\phi_r := (\mathbb{Q}f_r)^{-1}$. Moreover, choose an integer n such that $n \cdot \phi_r$ maps $H_r^+(L)$ into $K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tf}}$. Since $f_r \circ (n\phi_r)$ is multiplication by n on $H_r^+(L)$, we have an exact sequence

$$(\mathrm{cok}\, f_r)^\vee \rightarrowtail H_r^+(L)/n \overset{\tilde{\pi}}{\twoheadrightarrow} (\mathrm{cok}\,(n\phi_r))^\vee. \tag{8}$$

where the epimorphism $\tilde{\pi}$ is induced by f_r. We now show that $\mathcal{J}_r^S$ contains $\widetilde{\mathcal{J}}_r^S$. Let $\pi : H_r^+(L) \twoheadrightarrow (\mathrm{cok}\,(n\phi_r))^\vee$ be the epimorphism which is obtained by composing the natural projection $H_r^+(L) \twoheadrightarrow H_r^+(L)/n$ and $\tilde{\pi}$. Let α be an endomorphism of $H_r^+(L)$ such that $\mathrm{im}\,(\alpha)$ lies in the kernel of π. By Lemma 3.1 it suffices to show that $\mathrm{nr}_{H_r^+(L)}(\alpha)^\sharp$ belongs to $\mathcal{I}_{n\phi_r}$. Let P be a finitely generated projective $\mathbb{Z}[\frac{1}{2}]G$-module such that $H_r^+(L) \oplus P$ is free of rank m, say. Moreover, let $\alpha' = \alpha \oplus \mathrm{id}_P$ and let $\overline{\alpha}'$ be the endomorphism of $(H_r^+(L) \oplus P)/n$ induced by α'. Then the cokernel of $\overline{\alpha}'$ projects onto $(\mathrm{cok}\,(n\phi_r))^\vee$ and hence there exists an injection

$$\mathrm{cok}\,(n\phi_r) \rightarrowtail (\mathrm{cok}\,(\overline{\alpha}'))^\vee \simeq \ker((\overline{\alpha}')^{T,\sharp}),$$

where we identify α' with the corresponding $m \times m$ matrix, $(\alpha')^T$ denotes the transpose of this matrix, and the isomorphism on the right hand side follows from [Ni10], Lemma 5.2 and the proof of loc.cit., Prop. 5.3, where it is shown that $\ker((\overline{\alpha}')^{T,\sharp}) \simeq \mathrm{cok}\,((\overline{\alpha}')^{T,\sharp})$ (see four lines after equation

(6) of the proof). We may write $(\alpha')^{T,\sharp} = \alpha^{T,\sharp} \oplus \mathrm{id}_P$, where $\alpha^{T,\sharp}$ is an endomorphism of $H_r^+(L)$ such that its reduced norm equals $\mathrm{nr}_{H_r^+(L)}(\alpha)^\sharp$. But since $\overline{\alpha}^{T,\sharp}$ induces the zero map on $\mathrm{cok}\,(n\phi_r)$, the dual of sequence (8) implies that

$$K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tf}} \xrightarrow{f_r} H_r^+(L) \twoheadrightarrow H_r^+(L)/n \xrightarrow{\overline{\alpha}^{T,\sharp}} H_r^+(L)/n$$

is the zero map and hence $\alpha^{T,\sharp} n^{-1} f_r(K_{1-2r}(\mathfrak{o}_L)) \subset H_r^+(L)$ as desired. Conversely, let α be an endomorphism of $H_r^+(L) \otimes \mathbb{Q}$ such that $\alpha n^{-1} f_r(K_{1-2r}(\mathfrak{o}_L)) \subset H_r^+(L)$. Since $n^{-1} f_r(K_{1-2r}(\mathfrak{o}_L))$ contains $H_r^+(L)$, we see that α is actually an endomorphism of $H_r^+(L)$ such that $\overline{\alpha} : H_r^+(L)/n \to H_r^+(L)/n$ induces the zero map on $\mathrm{cok}\,(n\phi_r)$ (via the dual of sequence (8)). Taking duals, we find that $\overline{\alpha}^{T,\sharp}$ induces the zero map on $(\mathrm{cok}\,(n\phi_r))^\vee$ and we obtain epimorphisms

$$\mathrm{cok}\,(\alpha^{T,\sharp}) \twoheadrightarrow \mathrm{cok}\,(\overline{\alpha}^{T,\sharp}) \twoheadrightarrow (\mathrm{cok}\,(n\phi_r))^\vee$$

such that $\mathrm{Fitt}_{\mathbb{Z}[\frac{1}{2}]G}((\mathrm{cok}\,(n\phi_r))^\vee)$ contains $\mathrm{nr}(\alpha^{T,\sharp}) = \mathrm{nr}(\alpha)^\sharp$. As any equivariant injection $\psi_r : H_r^+(L) \otimes \mathbb{Z}[\frac{1}{2}] \rightarrowtail K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tf}} \otimes \mathbb{Z}[\frac{1}{2}]$ arises as $n \cdot (\mathbb{Q}f_r)^{-1}$ for appropriate n and f_r as above (put $f_r = n \cdot (\mathbb{Q}\psi_r)^{-1}$ for suitable n), we are done. $\square$

4. The relation to the equivariant Tamagawa number conjecture

The aim of this section is to prove the following result.

Theorem 4.1. *Let L/K be a Galois extension of number fields with Galois group G and $r \in \mathbb{Z}_{<0}$. Assume that the ETNC holds for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[\frac{1}{2}]G)$. Then Conjecture 2.12 is true for r and all admissible sets S. In particular, Conjecture 2.11 holds for r and S if the Quillen-Lichtenbaum conjecture (Conjecture 2.2) holds for r.*

If L/K is a CM-extension, we denote by $j \in G$ the unique automorphism induced by complex conjugation and put $e_- := \frac{1-j}{2} \in \mathbb{Z}[\frac{1}{2}]G$. Theorem 4.1 together with [Bu], Cor. 2.10 immediately implies the following result.

Corollary 4.2. i) *Let L/K be a Galois extension of totally real fields with Galois group G and let $r < 0$ be odd. Assume that the Iwasawa μ-invariant vanishes for all odd primes p. Then conjecture 2.12 is true for r and all admissible sets S.*

ii) *Let L/K be a Galois CM-extension with Galois group G and let $r < 0$ be even. Assume that the Iwasawa μ-invariant vanishes for all odd primes p. Then conjecture 2.12 is true for r and all admissible sets S if we replace $\mathcal{J}_r^S$ by $\mathcal{J}_r^S e_-$.*

Since we will not assume that G is abelian, we give a short introduction to non-commutative Fitting invariants (cf. [Ni10]).
If R is a ring and $n \in \mathbb{N}$, we denote the group of all invertible elements of $M_{n\times n}(R)$ by $\mathrm{Gl}_n(R)$. Let A be a separable K-algebra and Λ be an R-order in A, finitely generated as R-module, where R is a complete commutative noetherian local ring with field of quotients K. Moreover, we will assume that the integral closure of R in K is finitely generated as R-module. The group ring $\mathbb{Z}_pG$ of a finite group G will serve as a standard example. Let N and M be two $\zeta(\Lambda)$-submodules of an R-torsionfree $\zeta(\Lambda)$-module. Then N and M are called $\mathrm{nr}(\Lambda)$-**equivalent** if there exists an integer n and a matrix $U \in \mathrm{Gl}_n(\Lambda)$ such that $N = \mathrm{nr}(U) \cdot M$. We denote the corresponding equivalence class by $[N]_{\mathrm{nr}(\Lambda)}$. We say that N is $\mathrm{nr}(\Lambda)$-contained in M (and write $[N]_{\mathrm{nr}(\Lambda)} \subset [M]_{\mathrm{nr}(\Lambda)}$) if for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$ there exists $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N' \subset M'$. Note that it suffices to check this property for one $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$. We will say that x is contained in $[N]_{\mathrm{nr}(\Lambda)}$ (and write $x \in [N]_{\mathrm{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$ such that $x \in N_0$.

Now let M be a finitely presented (left) Λ-module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M \tag{9}$$

be a finite presentation of M. We identify the homomorphism h with the corresponding matrix in $M_{a\times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geqslant b$. In the case $a = b$ we call (9) a **quadratic presentation**. The Fitting invariant of h over Λ is defined to be

$$\mathrm{Fitt}_\Lambda(h) = \begin{cases} [0]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a < b \\ \left[\langle \mathrm{nr}(H) | H \in S(h)\rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a \geqslant b. \end{cases}$$

We call $\mathrm{Fitt}_\Lambda(h)$ a **(non-commutative) Fitting invariant** of M over Λ. One defines $\mathrm{Fitt}_\Lambda^{\max}(M)$ to be the unique Fitting invariant of M over Λ which is maximal among all Fitting invariants of M with respect to the partial order "$\subset$". If M admits a quadratic presentation h, one also puts $\mathrm{Fitt}_\Lambda(M) := \mathrm{Fitt}_\Lambda(h)$ which is independent of the chosen quadratic presentation. The following result is [Ni10], Th. 4.2.

THEOREM 4.3. *If R is an integrally closed complete commutative noetherian local ring and M is a finitely presented Λ-module, then*

$$\mathcal{H}(\Lambda) \cdot \mathrm{Fitt}_\Lambda^{\max}(M) \subset \mathrm{Ann}_\Lambda(M).$$

LEMMA 4.4. *Let $\epsilon : H^0 \rightarrowtail A \to B \twoheadrightarrow H^1$ be a perfect 2-extension of $\mathbb{Z}_pG$-modules, where H^i is finite, $i = 0, 1$. If $\chi_{\mathbb{Z}_pG,\mathbb{Q}_pG}(\epsilon, 0) = \hat{\partial}_{\mathbb{Z}_pG}(\mathcal{L})$, where $\mathcal{L} \in \zeta(\mathbb{Q}_pG)^\times$, then we have an equality*

$$\mathrm{Fitt}_{\mathbb{Z}_pG}^{\max}((H^0)^\vee)^\sharp = \mathrm{Fitt}_{\mathbb{Z}_pG}^{\max}(H^1) \cdot \mathcal{L}.$$

Proof. Since the refined Euler characteristic $\chi_{\mathbb{Z}_pG,\mathbb{Q}_pG}(\epsilon, 0)$ only depends upon the isomorphism class of the perfect complex $A \to B$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_pG)$, we may assume that A and B are finite G-modules. But then [Ni10], Prop. 5.3 gives the first equality in

$$\begin{aligned}\mathrm{Fitt}_{\mathbb{Z}_pG}^{\max}((H^0)^\vee)^\sharp &= \mathrm{Fitt}_{\mathbb{Z}_pG}^{\max}(H^1)\mathrm{Fitt}_{\mathbb{Z}_pG}(A)\mathrm{Fitt}_{\mathbb{Z}_pG}(B)^{-1} \\ &= \mathrm{Fitt}_{\mathbb{Z}_pG}^{\max}(H^1) \cdot \mathcal{L},\end{aligned}$$

whereas the second equality follows from

$$\hat{\partial}_{\mathbb{Z}_pG}(\mathcal{L}) = \chi_{\mathbb{Z}_pG,\mathbb{Q}_pG}(\epsilon, 0) = (A) - (B)$$

(here we regard (A) and (B) as elements of $K_0(\mathbb{Z}_pG, \mathbb{Q}_p)$ under the canonical isomorphism $K_0(\mathbb{Z}_pG, \mathbb{Q}_p) \simeq K_0T(\mathbb{Z}_pG)$, where $K_0T(\mathbb{Z}_pG)$ denotes the Grothendieck group of the category of all $\mathbb{Z}_p$-torsion $\mathbb{Z}_pG$-modules of finite projective dimension) and the following observation: Let A be a finite c.t. (short for cohomologically trivial) $\mathbb{Z}_pG$-module; we may choose a quadratic presentation

$$(\mathbb{Z}_pG)^n \overset{h}{\rightarrowtail} (\mathbb{Z}_pG)^n \twoheadrightarrow A$$

and, by definition, $\mathrm{nr}(h)$ is a generator of $\mathrm{Fitt}_{\mathbb{Z}_pG}(A)$. Since the class of $\mathbb{Q}_p h$ in $K_1(\mathbb{Q}_pG)$ is a preimage of (A), we have $\hat{\partial}_{\mathbb{Z}_pG}(\mathrm{nr}(h)) = (A)$. Now assume that $\mathcal{L}' \in \zeta(\mathbb{Q}_pG)^\times$ is such that $\hat{\partial}_{\mathbb{Z}_pG}(\mathcal{L}') = (A)$, then $\mathcal{L}'$ is also a generator of $\mathrm{Fitt}_{\mathbb{Z}_pG}(A)$, since $\mathrm{nr}(h)(\mathcal{L}')^{-1} \in \mathrm{nr}(K_1(\mathbb{Z}_pG))$. Now apply this to $\mathcal{L}' = \mathcal{L} \cdot \mathrm{nr}(g)$, where g is a quadratic presentation of B. $\square$

Proof of Theorem 4.1. Let τ_r be the a.t.e. described in Proposition 2.15 and let

$$\epsilon : H^0 \rightarrowtail A \to B \twoheadrightarrow H^1$$

be the corresponding perfect 2-extension. As before, we will implicitly tensor with $\mathbb{Z}[\frac{1}{2}]$ such that we have isomorphisms

$$H^0 \simeq H^1(\mathfrak{o}_L, \mathbb{Z}(1-r)), \ H^1 \simeq H_r^+(L) \oplus H^2(\mathfrak{o}_{L,S}, \mathbb{Z}(1-r))$$

by projectivity of $H_r^+(L)$. By the same reason we have an isomorphism $B \simeq H_r^+(L) \oplus B'$. We now choose a G-equivariant embedding

$$\phi : H^1_{\mathrm{tf}} \rightarrowtail H^0$$

which induces an embedding $\overline{\phi} : H^1_{\mathrm{tf}} \rightarrowtail H^0_{\mathrm{tf}}$. For each prime $p \neq 2$ we define

$$\mathcal{K}_r^S(p) := \left(\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}((\mathrm{cok}\,\phi \otimes \mathbb{Z}_p)^\vee) \cdot (A_\phi^S)^{-1}\right)^\sharp, \tag{10}$$

$$\widetilde{\mathcal{J}}_r^S(p) := \left(\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}((\mathrm{cok}\,\overline{\phi} \otimes \mathbb{Z}_p)^\vee) \cdot (A_\phi^S)^{-1}\right)^\sharp.$$

As before, these definitions do not depend on ϕ (but we will make no use of this fact). We observe that $H^1_{\mathrm{tf}} = H_r^+(L)$ and $\mathrm{ch}^{-1}_{1-r,1}$ induces an isomorphism $H^0_{\mathrm{tf}} \simeq K_{1-2r}(\mathfrak{o}_L)_{\mathrm{tf}}$, since the kernel of the Chern class maps are finite. A similar proof as of Proposition 3.2 shows that for any prime $p \neq 2$ we have an inclusion

$$\widetilde{\mathcal{J}}_r^S(p) \supseteq \mathcal{J}_r^S \otimes \mathbb{Z}_p. \tag{11}$$

Obviously, this inclusion is an equality for abelian G by Proposition 3.2. For arbitrary G, the proof only shows that ($^\sharp$ applied to) the right hand side generates a Fitting invariant of $(\mathrm{cok}\,\overline{\phi})^\vee$ times $(A_\phi^S)^{-1}$, but maybe not the maximal one. Moreover, the dual of the exact sequence

$$H^0_{\mathrm{tor}} \rightarrowtail \mathrm{cok}\,\phi \twoheadrightarrow \mathrm{cok}\,\overline{\phi}$$

implies that

$$\mathcal{K}_r^S(p) \supseteq \widetilde{\mathcal{J}}_r^S(p)\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}((H^0_{\mathrm{tor}} \otimes \mathbb{Z}_p)^\vee)^\sharp. \tag{12}$$

Consider the following commutative diagram whose middle row is the 2-extension ϵ:

$$\begin{tikzcd}
H_r^+(L) \arrow[r, equal] \arrow[d, hook, "\phi"] & H_r^+(L) \arrow[r, "0"] \arrow[d, hook] & H_r^+(L) \arrow[r, equal] \arrow[d, hook] & H_r^+(L) \arrow[d, hook] \\
H^0 \arrow[r, hook] \arrow[d, two heads] & A \arrow[r] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & H^1 \arrow[d, two heads] \\
\mathrm{cok}\,\phi \arrow[r, hook] & A' \arrow[r] & B' \arrow[r, two heads] & H^1_{\mathrm{tor}}
\end{tikzcd}$$

If we denote the extension class of the bottom and top row by ϵ' and ϵ'', respectively, we have equalities

$$\begin{aligned}
\chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon, \phi^{-1}) &= \chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon', 0) + \chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon'', \mathrm{id}_{H_r^+(L)}) \\
&= \chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon', 0) + (H_r^+(L), \mathrm{id}_{H_r^+(L)}, H_r^+(L)) \\
&= \chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon', 0).
\end{aligned} \tag{13}$$

Moreover, we have

$$\begin{aligned}
\chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon, \phi^{-1}) &= -\chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon, \lambda_{\tau_r}) + \hat{\partial}_{\mathbb{Z}[\frac{1}{2}]G}((R_\phi^\sharp)) \\
&= -\chi(\tau_r) + \hat{\partial}_{\mathbb{Z}[\frac{1}{2}]G}((A_\phi^S)^\sharp).
\end{aligned} \tag{14}$$

By Proposition 2.15 (iv) the ETNC for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[\frac{1}{2}]G)$ asserts that $\chi(\tau_r)$ vanishes; this together with (13) and (14) yields

$$\hat{\partial}_{\mathbb{Z}[\frac{1}{2}]G}((A_\phi^S)^\sharp) = \chi_{\mathbb{Z}[\frac{1}{2}]G,\mathbb{R}G}(\epsilon', 0).$$

Now Lemma 4.4 implies that for $p \neq 2$ we have

$$\begin{aligned}\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}(H^1_{\mathrm{tor}} \otimes \mathbb{Z}_p) &= \mathcal{K}^S_r(p) \\ &\supseteq (\mathcal{J}^S_r \otimes \mathbb{Z}_p) \cdot \mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}((H^0_{\mathrm{tor}} \otimes \mathbb{Z}_p)^\vee)^\sharp\end{aligned}$$

by (11) and (12). But since $H^1_{\mathrm{tor}} \otimes \mathbb{Z}_p \simeq H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))$ and $H^0_{\mathrm{tor}} \otimes \mathbb{Z}_p \simeq H^1_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))$ is cyclic, Theorem 4.3 implies the desired result. □

Remark 2. The proof shows that the ETNC implies

$$\mathcal{H}_p(G) \cdot \mathcal{K}^S_r(p) \subset \mathrm{Ann}_{\mathbb{Z}_pG}(H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))) \tag{15}$$

and in fact $\mathcal{K}^S_r(p)$ has better base change properties than $\mathcal{J}^S_r \otimes \mathbb{Z}_p$ (cf. Corollary 6.7 below and [BS10], Prop. 3.3). But note that the inclusion (12) happens to be an equality in many cases. For example, this is true if K is totally real and L is either totally real or CM, since in this case H^0 decomposes into $H^0_{\mathrm{tor}} \oplus H^0_{\mathrm{tf}}$ as $\mathbb{Z}[\frac{1}{2}]G$-module: If L is totally real, either H^0_{tor} or H^0_{tf} is trivial; if L is CM, complex conjugation acts trivially on one of these modules and as -1 on the other (depending on the parity of r). Likewise, the inclusion (11) is an equality for abelian G and if $p \nmid |G|$.

5. χ-twists and q-indices

For the following, we largely adopt the treatment of [Bu08], §1. Let G be a finite group and let F be a finite Galois extension of $\mathbb{Q}$ such that each representation of G can be realized over F. For any irreducible character $\chi \in \mathrm{Irr}\,(G)$ we define a central primitive idempotent of FG by $e_\chi := \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g$. We write $\mathfrak{o}_F$ for the ring of integers of F and choose a maximal $\mathfrak{o}_F$-order $\mathfrak{M}$ in FG which contains $\mathfrak{o}_FG$. We fix an indecomposable idempotent f_χ of $e_\chi\mathfrak{M}$ and define an $\mathfrak{o}_F$-torsionfree right $\mathfrak{o}_FG$-module by setting $T_\chi := f_\chi\mathfrak{M}$. Note that this slightly differs from the definition in [Bu08]. T_χ is locally free of rank $\chi(1)$ over $\mathfrak{o}_F$ and the associated right FG-module $V_\chi = T_\chi \otimes_{\mathfrak{o}_F} F$ has character χ. For any left G-module M we set $M[\chi] := T_\chi \otimes_{\mathbb{Z}} M$, upon which G acts on the left by $t \otimes m \mapsto tg^{-1} \otimes g(m)$ for $t \in T_\chi$, $m \in M$ and $g \in G$. For any integer i we write $H^i(G, M)$ for the Tate cohomology in degree i of M with respect to G. Moreover, we write M^G resp. M_G for the maximal submodule resp. the maximal quotient module of M upon which G acts trivially. We obtain a left exact functor $M \mapsto M^\chi$ and a right exact functor $M \mapsto M_\chi$ from the category of left G-modules to the category of $\mathfrak{o}_F$-modules by setting $M^\chi := M[\chi]^G$ and $M_\chi := M[\chi]_G$. We denote the right and left derived functors by $H^i(\chi, \_)$ and $H_i(\chi, \_)$, respectively. Obviously, we have isomorphisms

$$H^i(\chi, M) \simeq H^i(G, M[\chi]), \ H_i(\chi, M) \simeq H^{-i-1}(G, M[\chi])$$

for all $i > 0$. The action of $N_G := \sum_{g\in G} g$ induces a homomorphism $t(M, \chi) : M_\chi \to M^\chi$ with kernel $H^{-1}(G, M[\chi])$ and cokernel $H^0(G, M[\chi])$. Thus $M_\chi \simeq M^\chi$ whenever M and hence also $M[\chi]$ is a c.t. G-module. We may extend the above constructions to arbitrary characters $\chi' = \sum_{\chi\in\mathrm{Irr}\,(G)} n_\chi\chi$, as we may replace T_χ by $T_{\chi'} := \oplus_\chi n_\chi T_\chi$ throughout.

If H is a subgroup of G and ψ is a character of H, we write $\mathrm{ind}^G_H\psi$ for the induced character; if H is normal and ψ is a character of $\overline{G} := G/H$, we write $\mathrm{infl}^G_{\overline{G}}\,\psi$ for the character which maps $g \in G$ to $\psi(g \bmod H)$.

Lemma 5.1. *Let G be a finite group, H a normal subgroup of G and M a finitely generated left G-module. If ψ is a character of $\overline{G} = G/H$ and $\chi = \mathrm{infl}^G_{\overline{G}}\,\psi$, we have isomorphisms*

$$M_\chi \simeq (M_H)_\psi, \ M^\chi \simeq (M^H)^\psi.$$

Proof. We have $T_\chi = T_\psi$ such that

$$M_\chi = T_\chi \otimes_{\mathbb{Z}G} M = T_\chi \otimes_{\mathbb{Z}\overline{G}} M_H = (M_H)_\psi,$$

$$M^{\chi} \simeq \mathrm{Hom}_{\mathbb{Z}G}(T_{\check{\chi}}, M) = \mathrm{Hom}_{\mathbb{Z}\overline{G}}(T_{\check{\chi}}, M^H) = (M^H)^{\psi}.$$

$\square$

COROLLARY 5.2. *Let H be a normal subgroup of a finite group G, ψ a character of $\overline{G} = G/H$ and $\chi = \mathrm{infl}_{\overline{G}}^{G}\, \psi$. Let M be a finitely generated G-module.*

i) *We have a cohomological spectral sequence*

$$H^i(\psi, H^j(H, M)) \Longrightarrow H^{i+j}(\chi, M).$$

In particular, we have a five-term exact sequence

$$H^1(\psi, M^H) \rightarrowtail H^1(\chi, M) \to H^1(H, M)^{\psi} \to H^2(\psi, M^H) \to H^2(\chi, M).$$

ii) *We have a homological spectral sequence*

$$H_i(\psi, H^{-j-1}(H, M)) \Longrightarrow H_{i+j}(\chi, M).$$

In particular, we have a five-term exact sequence

$$H_2(\chi, M) \to H_2(\psi, M_H) \to H^{-2}(H, M)_{\psi} \to H_1(\chi, M) \twoheadrightarrow H_1(\psi, M_H).$$

Remark 3. In the case, where ψ is the trivial character, Corollary 5.2 (i) recovers the Hochschild-Serre spectral sequence (cf. [NSW00], Th. 2.4.1) in the case of finite groups.

DEFINITION 5.3. If $\mathfrak{o}$ is a Dedekind domain and f is an $\mathfrak{o}$-homomorphism with finite kernel and finite cokernel, then the q-**index** of f is defined to be

$$q(f) := \mathrm{Fitt}_{\mathfrak{o}}(\mathrm{cok}\, f) \cdot \mathrm{Fitt}_{\mathfrak{o}}(\ker f)^{-1}.$$

We now return to the case, where G is the Galois group of a finite extension of number fields L/K. As before, let r be a negative integer and let τ_r be the a.t.e. described in Proposition 2.15. Let

$$\epsilon : H^0 \rightarrowtail A \to B \twoheadrightarrow H^1$$

be the corresponding 2-extension and fix an equivariant homomorphism $\phi : H^1 \to H^0$ such that $\mathbb{Q}\phi$ is an isomorphism. Each character χ of G induces a map

$$\hat{\phi}_{\chi} : H^1_{\chi} \xrightarrow{t(H^1,\chi)} (H^1)^{\chi} \xrightarrow{\phi^{\chi}} (H^0)^{\chi}$$

and we define

$$q(\phi, \chi) := q(\hat{\phi}_{\check{\chi}}).$$

Note that $q(\phi, \chi)$ invisibly depends on a finite set S of places of K.

CONJECTURE 5.4 Burns. Assume that Gross' conjecture (Conjecture 2.8) holds for r and the character χ. Then $A_{\phi}^S(\chi)$ lies in the number field F and in F one has

$$A_{\phi}^S(\chi)\mathfrak{o}_F = q(\phi, \chi).$$

We will say, that the p-part of Conjecture 5.4 holds if the above equality is true after localization at $\mathfrak{p}$ for any prime $\mathfrak{p}$ in F above p. As in the case $r = 0$ one can prove (cf. [Bu10], Prop. 6.2.1):

PROPOSITION 5.5. *Let L/K be a Galois extension of number fields with Galois group G and $r \in \mathbb{Z}_{<0}$. Then Conjecture 5.4 holds for all characters of G if and only if the ETNC holds for the pair $(\mathbb{Q}(r)_L, \mathfrak{M})$, where $\mathfrak{M}$ is a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}G$.*

We now study the behavior of the q-index under induction and inflation. Let $\phi : H^1(L/K) \to H^0(L/K)$ be an equivariant homomorphism as above, where we write $H^i(L/K)$, $i = 0, 1$ to indicate that these modules belong to the extension L/K.

LEMMA 5.6. *Let H be a subgroup of G and $\overline{L} := L^H$.*

i) *If ψ is a character of H, then we have an equality*

$$q(\phi, \psi) = q(\phi, \mathrm{ind}\,_H^G \psi).$$

ii) *If H is normal and ψ is a character of $\overline{G} := G/H$, then there are equivariant homomorphisms ϕ and $\overline{\phi}$ such that*

$$\begin{tikzcd} H^1(L/K) \arrow[r, "\phi"] \arrow[d, two heads] & H^0(L/K) \\ H^1(\overline{L}/K) \arrow[r, "\overline{\phi}"] & H^0(\overline{L}/K) \arrow[u, hook] \end{tikzcd}$$

commutes and we have an equality

$$q(\overline{\phi}, \psi) = q(\phi, \mathrm{infl}\,_{\overline{G}}^G\, \psi).$$

Proof. The associated FG-module of the $\mathfrak{o}_F G$-module $\tilde{T}_\chi := T_\psi \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is $V_\chi = T_\chi \otimes F$. But the q-index does not change if we replace T_χ by $\tilde{T}_\chi$ throughout (cf. [Ta84], Th. II.6.4); this proves (i). For (ii) we observe that taking H-invariants of the complexes $C^{\cdot}_{p,r}(L/K)$ yields $C^{\cdot}_{p,r}(\overline{L}/K)$, since the same is true for the complexes $R\Gamma_c(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(r))$. This implies that

$$H^0(\overline{L}/K) = H^0(L/K)^H,\ H^1(\overline{L}/K) = H^1(L/K)_H$$

such that we may choose ϕ and $\overline{\phi}$ as desired. The equality of the q-indices now follows from Lemma 5.1. $\square$

EXAMPLE. We compute the q-index of the trivial character 1_G. Since $1_G = \mathrm{infl}\,_1^G\, 1$, we may assume that $L = K$. Moreover, we will choose ϕ as the composite of the natural projection $\pi : H^1 \twoheadrightarrow H^1_{\mathrm{tf}}$ and an embedding $\phi' : H^1_{\mathrm{tf}} \rightarrowtail H^0$. Then

$$\begin{aligned} q(\phi, 1) &= \frac{[H^0 : \phi'(H^1_{\mathrm{tf}})]}{|H^2(\mathfrak{o}_K, \mathbb{Z}(1-r))|} \\ &= \frac{|H^0_{\mathrm{tor}}| \cdot \det(\lambda_{\tau_r} \circ \phi)}{R^{\mathcal{M}}_{1-r}(K) \cdot |H^2(\mathfrak{o}_K, \mathbb{Z}(1-r))|} \\ &= \frac{|H^1_{\mathcal{M}}(K, \mathbb{Z}(1-r))_{\mathrm{tor}}| \cdot \det(\lambda_{\tau_r} \circ \phi)}{R^{\mathcal{M}}_{1-r}(K) \cdot |H^2_{\mathcal{M}}(K, \mathbb{Z}(1-r))|}, \end{aligned}$$

where $H^i_{\mathcal{M}}(K, \mathbb{Z}(1-r))$ are suitable motivic cohomology groups and $R^{\mathcal{M}}_{1-r}$ is a suitable motivic regulator. Hence Conjecture 5.4 is true for the trivial character if and only if the motivic Lichtenbaum conjecture holds (cf. for instance, the survey article [Ko03], p. 208). In particular, Conjecture 5.4 is true for the trivial character up to primes above 2 if and only if the cohomological Lichtenbaum conjecture (Conjecture 2.6) holds.

More generally, let χ be any irreducible character of G. Then, ignoring the primes above 2, we have

$$q(\phi, \check{\chi}) = \mathrm{Fitt}_{\mathfrak{o}_F}((\mathrm{cok}\,\phi)^\chi)\mathrm{Fitt}_{\mathfrak{o}_F}(\ker \pi_\chi)^{-1}.$$

For any prime $p \neq 2$, the p-part of $\ker(\pi_\chi)$ is $H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r))_\chi$, since $H^1_{\mathrm{tf}} \otimes \mathbb{Z}_p \simeq H^+_r(L) \otimes \mathbb{Z}_p$ is projective. For any finite G-module M, we have $(M^\vee)^\chi = (M_{\check{\chi}})^\vee$ (cf. [Ni10], §5 (4)) and for any finite $\mathfrak{o}_F$-module M, we have $\mathrm{Fitt}_{\mathfrak{o}_F}(M) = \mathrm{Fitt}_{\mathfrak{o}_F}(M^\vee)$. This implies

$$\mathrm{Fitt}_{\mathfrak{o}_F}((\mathrm{cok}\,\phi)^\chi) = \mathrm{Fitt}_{\mathfrak{o}_F}(((\mathrm{cok}\,\phi)^\vee)_{\check{\chi}}).$$

Let $\mathfrak{M}_p$ be a maximal order in $\mathbb{Q}_pG$ containing $\mathbb{Z}_pG$ and let M be a $\mathbb{Z}_pG$-module. Then computing the (non-commutative) Fitting invariant of $M \otimes \mathfrak{M}_p$ over $\mathfrak{M}_p$ is equivalent to computing the Fitting invariants $\mathrm{Fitt}_{\mathfrak{o}_F}(M_\chi)$ for all χ by loc.cit., Remark 7. We obtain that the p-part of Conjecture 5.4 holds if and only if

$$\mathrm{Fitt}_{\mathfrak{M}_p}(H^2_{\text{ét}}(\mathfrak{o}_{L,S}[1/p], \mathbb{Z}_p(1-r)) \otimes \mathfrak{M}_p) = ((A^S_\phi)^{-1} \cdot \mathrm{Fitt}_{\mathfrak{M}_p}((\operatorname{cok}\phi)^\vee \otimes \mathfrak{M}_p))^\sharp.$$

As in the proof of Theorem 4.1, we can show that the right hand side contains

$$(\mathcal{J}_r^S \otimes \mathbb{Z}_p) \cdot \mathrm{nr}\left(\mathrm{Ann}_{\mathbb{Z}_pG}(H^1_{\text{ét}}(\mathfrak{o}_L[1/p], \mathbb{Z}_p(1-r))_{\mathrm{tor}})\right) \cdot \zeta(\mathfrak{M}_p).$$

Let $\mathcal{F}_p(G) = \{x \in \zeta(\mathbb{Z}_pG) | x \cdot \mathfrak{M}_p \subset \mathbb{Z}_pG\}$ be the central conductor. Then $\mathcal{F}_p(G)$ is contained in $\mathcal{H}_p(G)$ and loc.cit., Prop. 5.1 implies the following maximal order analogue of Theorem 4.1.

THEOREM 5.7. *Let L/K be a Galois extension of number fields with Galois group G and $r \in \mathbb{Z}_{<0}$. Assume that the ETNC holds for the pair $(\mathbb{Q}(r)_L, \mathfrak{M})$, where $\mathfrak{M}$ is a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}[\frac{1}{2}]G$. Then Conjecture 2.12 is true for r and all admissible sets S if we replace $\mathcal{H}_p(G)$ by $\mathcal{F}_p(G)$.*

Remark 4. Since the ETNC also splits into local conjectures at each prime p, it is clear that local analogues of Theorem 4.1 and Theorem 5.7 are valid, too.

6. An induction result

If L/K is a Galois CM-extension with Galois group G, recall that we denote by $j \in G$ the unique automorphism induced by complex conjugation and that $e_- = \frac{1-j}{2} \in \mathbb{Z}[\frac{1}{2}]G$. A character χ of G is called even (resp. odd) if $\chi(j) = \chi(1)$ (resp. $\chi(j) = -\chi(1)$). We will prove the following proposition.

PROPOSITION 6.1. *Let L/K be a Galois extension of number fields with Galois group G, $r \in \mathbb{Z}_{<0}$ and let p be a prime.*

i) *If the p-part of Conjecture 5.4 holds for all abelian intermediate extensions of L/K, then it holds also for L/K.*
ii) *Assume that Gross' conjecture (Conjecture 2.8) is valid for L/K. Then the p-part of Conjecture 5.4 holds for L/K if it is true for all cyclic intermediate extensions of degree prime to p.*
iii) *Assume that the extension L/K is CM, $p \neq 2$ and Gross' conjecture holds for odd characters. Then the p-part of Conjecture 5.4 holds for L/K and odd characters if it is true for all intermediate CM-extensions whose Galois group is either C or $C \times \langle j \rangle$, where C is a cyclic group of order prime to p.*

COROLLARY 6.2. *Let L/K be a Galois extension of totally real fields with Galois group G and let $r < 0$ be odd. Then the ETNC holds for the pair $(\mathbb{Q}(r)_L, \mathfrak{M})$, where $\mathfrak{M}$ is a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}[\frac{1}{2}]G$. In particular, Conjecture 2.12 holds if we replace $\mathcal{H}_p(G)$ by $\mathcal{F}_p(G)$ for all primes p.*

Proof. Since Gross' Conjecture holds in this case by the earlier mentioned result of Siegel, Proposition 6.1 implies that it suffices to verify the p-part of the ETNC for cyclic extensions of totally real fields whose degree is prime to p. But this is a special case of [BG03b], Th. 5.1 if we assume that the Iwasawa μ-invariant vanishes, and follows from [Po09], Th. 5.2 and Th. 5.4 if not. □

COROLLARY 6.3. *Let L/K be a Galois CM-extension and let $r < 0$ be even. Assume that the Iwasawa μ-invariant vanishes for all odd primes p. Then the ETNC holds for the pair $(\mathbb{Q}(r)_L, \mathfrak{M}e_-)$, where $\mathfrak{M}$ is a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}[\frac{1}{2}]G$. In particular, Conjecture 2.12 holds if we replace $\mathcal{J}_r^S$ by $\mathcal{J}_r^S e_-$ and $\mathcal{H}_p(G)$ by $\mathcal{F}_p(G)$ for all primes p.*

Proof. As above, Proposition 6.1 reduces the assertion to a special case of [BG03b], Th. 5.1. □

Remark 5. Obviously, Corollary 6.3 is weaker than Corollary 4.2 (ii), but the author expects that it is possible to remove the hypothesis on the μ-invariants as in Corollary 6.2 by mimicking the proofs of [Po09], Th. 5.2 and Th. 5.4 and then applying Proposition 6.1.

Let G be a finite group, M a finitely generated $\mathbb{Z}G$-module and θ a $\mathbb{C}G$-automorphism of $M \otimes \mathbb{C}$. For any character χ of G we put

$$\delta(\chi, \theta) := \det(\theta | \mathrm{Hom}_G(V_{\check{\chi}}, M \otimes \mathbb{C})) \in \mathbb{C}^\times.$$

In particular, we have $R_{\phi_r}(\chi) = \delta(\chi, \rho_r \circ \phi_r)$. We will need the following lemma which is proved as [Ta84], I.6.4., but in fact most of the assertions are obvious.

LEMMA 6.4. i) $\delta(\chi + \chi', \theta) = \delta(\chi, \theta)\delta(\chi', \theta)$

ii) $\delta(\chi, \theta\theta') = \delta(\chi, \theta)\delta(\chi, \theta')$

iii) $\delta(\mathrm{ind}_H^G \chi, \theta) = \delta(\chi, \theta)$ *for all subgroups H of G*

iv) $\delta(\mathrm{infl}_{\overline{G}}^G \chi, \theta) = \delta(\chi, \theta|_{M^H \otimes \mathbb{C}})$ *for all normal subgroups H of G, where $\overline{G} = G/H$.*

v) $\delta(\chi^\alpha, \theta^\alpha) = \delta(\chi, \theta)^\alpha$ *for all $\alpha \in \mathrm{Aut}(\mathbb{C})$.*

Remark 6. For example, Lemma 6.4 can be used to prove that Gross' conjecture does not depend on the choice of the $\mathbb{Q}G$-isomorphism ϕ_r in (4): If ϕ_r' is a second $\mathbb{Q}G$-isomorphism, then by (ii) we have

$$A_{\phi_r}^S(\chi) = A_{\phi_r'}^S(\chi) \cdot \delta(\chi, \phi_r^{-1}\phi_r')$$

and the rightmost term commutes with Galois action by (v).

By Lemma 6.4 and Brauer induction we obtain the following, probably well known result (for instance, assertion (iv) is a special case of [BF01], Prop. 4.2.a).

COROLLARY 6.5. i) $A_{\phi_r}^S(\chi + \chi') = A_{\phi_r}^S(\chi)A_{\phi_r}^S(\chi')$

ii) $A_{\phi_r}^S(\mathrm{ind}_H^G \chi) = A_{\phi_r}^S(\chi)$ *for all subgroups H of G*

iii) $A_{\phi_r}^S(\mathrm{infl}_{\overline{G}}^G \chi) = A_{\overline{\phi}_r}^S(\chi)$ *for all normal subgroups H of G, where $\overline{G} = G/H$ and $\overline{\phi}_r$ is obtained as in Lemma 5.6 (ii).*

iv) *Gross' conjecture (Conjecture 2.8) holds for L/K if and only if it holds for all abelian intermediate extensions.*

COROLLARY 6.6. *Assume that Lichtenbaum's conjecture (Conjecture 2.6) holds for all intermediate fields of L/K. Then Conjecture 5.4 holds for all rational valued characters of G outside the 2-primary part.*

Proof. Since Lichtenbaum's conjecture is equivalent to Conjecture 5.4 for the trivial character, and both sides of Conjecture 5.4 behave well under induction by Lemma 5.6 and Corollary 6.5, we obtain that Conjecture 5.4 holds for permutation characters. But if χ is rational valued, there is an integer m such that $m\chi$ is a permutation character by a theorem of Artin (cf. [Ta84], Th. II.1.2). This implies $A_{\phi_r}^S(\chi) = q(\phi_r, \chi)$, since both sides agree if we raise to the m^{th} power. □

Let H be a normal subgroup of G and put $\overline{G} = G/H$ and $\overline{L} = L^H$. We denote the canonical projection map $\mathbb{Q}_p G \twoheadrightarrow \mathbb{Q}_p \overline{G}$ by $\pi_{\overline{G}}^G$. Recall from (10) the definition of $\mathcal{K}_r^S(p) = \mathcal{K}_r^S(L/K, p)$.

COROLLARY 6.7. *In the notation from above we have*

$$\pi_{\overline{G}}^G(\mathcal{K}_r^S(L/K, p)) \subseteq \mathcal{K}_r^S(\overline{L}/K, p)$$

and equality holds if $\overline{G}$ is abelian or if $p \nmid |H|$.

Proof. Choose an equivariant embedding $\phi_r : H_r^+(L) \rightarrowtail H^0(L/K)$ such that

$$\mathcal{K}_r^S(L/K,p) = \left(\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}((\mathrm{cok}\,\phi \otimes \mathbb{Z}_p)^\vee) \cdot (A_\phi^S)^{-1}\right)^\sharp.$$

As in Lemma 5.6 (ii), ϕ_r induces $\overline{\phi}_r : H_r^+(\overline{L}) \rightarrowtail H^0(\overline{L}/K)$. In fact, $\overline{\phi}_r = \phi_r^H$ and $\mathrm{cok}\,(\overline{\phi}_r) = (\mathrm{cok}\,\phi_r)^H$, as $H_r^+(L)$ is a c.t. G-module. Since Fitting invariants behave well under base change (cf. [Ni10], Lemma 5.5) and $\pi_{\overline{G}}^G(A^S_{\phi_r}) = A^S_{\overline{\phi}_r}$ by Corollary 6.5 (iii), we obtain

$$\begin{aligned}\pi_{\overline{G}}^G(\mathcal{K}_r^S(L/K,p)) &= \left(\mathcal{F}_{\mathbb{Z}_p\overline{G}}((\mathrm{cok}\,\phi_r \otimes \mathbb{Z}_p)^\vee_H) \cdot (A^S_{\overline{\phi}_r})^{-1}\right)^\sharp \\ &= \left(\mathcal{F}_{\mathbb{Z}_p\overline{G}}((\mathrm{cok}\,\phi_r^H \otimes \mathbb{Z}_p)^\vee) \cdot (A^S_{\overline{\phi}_r})^{-1}\right)^\sharp \\ &\subseteq \mathcal{K}_r^S(\overline{L}/K,p),\end{aligned}$$

where $\mathcal{F}_{\mathbb{Z}_p\overline{G}}((\mathrm{cok}\,\phi_r \otimes \mathbb{Z}_p)^\vee_H)$ is a Fitting invariant of $(\mathrm{cok}\,\phi_r \otimes \mathbb{Z}_p)^\vee_H$ over $\mathbb{Z}_p\overline{G}$, but maybe not the maximal one. Since Fitting invariants over commutative rings are unique, we have an equality if $\overline{G}$ is abelian. If p is prime to $|H|$, the idempotent $e_H := |H|^{-1}\sum_{h\in H} h$ belongs to $\mathbb{Z}_pG$ such that $\mathbb{Z}_p\overline{G}$ occurs as a direct factor of $\mathbb{Z}_pG$ and hence

$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_pG}(M) = \mathrm{Fitt}^{\max}_{\mathbb{Z}_p\overline{G}}(e_H \cdot M) \oplus \mathrm{Fitt}^{\max}_{(1-e_H)\mathbb{Z}_pG}((1-e_H)\cdot M)$$

for any $\mathbb{Z}_pG$-module M. □

Remark 7. Assume that $p \nmid |H|$. Then the above shows that if (15) holds for L/K, the same is true for $\overline{L}/K$.

Proof of Proposition 6.1. By Lemma 5.6 and Corollary 6.5 both sides of Conjecture 5.4 commute with induction and inflation of characters. This implies (i) and can be used to prove the other two assertions exactly as [RW97], Prop. 11 and [Nia], Cor. 2. Note that Proposition 5.5 shows that (i) is also a special case of [BF01], Prop. 4.2.b. □

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