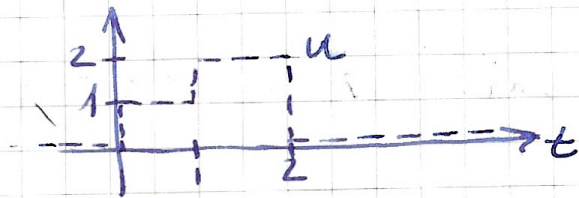


4. Übung

Victor C. Chaim

4.1 Aufgabe:
$$\begin{cases} \ddot{x} = -9x - 20\dot{x} + u \\ y = x \\ x_0 = [0 \ 0]^T \end{cases}$$



$y = ?$

$$\rightarrow \begin{cases} \ddot{x} = -9x - 20\dot{x} + u \\ y = x \end{cases} \xrightarrow{ZS} \begin{cases} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \ 0] \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + [0] u \end{cases} \quad D=0$$

Impulsantwort: $g(t) = \varphi(t) C e^{At} B + \delta(t) D$
 ↳ Sprungfunktion ↳ Deltafunktion

$g(t) = \varphi(t) C e^{At} B$, $\varphi(t) = \begin{cases} 1, & \text{falls } t \geq 0 \\ 0, & \text{sonst.} \end{cases}$

$g(t) = \varphi(t) [1 \ 0] \exp(At) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \rightarrow$ Eigenwerte: $\det(A - \lambda) = \begin{bmatrix} -\lambda & 1 \\ -20 & -9-\lambda \end{bmatrix}$

$\det(A - \lambda) = \lambda(9 + \lambda) + 20 = \lambda^2 + 9\lambda + 20 \rightarrow \begin{cases} \lambda_1 = -4 \\ \lambda_2 = -5 \end{cases}$

Eigenwerte $\neq 0$, A ist nicht nilpotent.

Diagonal transformation: $A = T \Lambda T^{-1}$, $T = [v_1 \ v_2]$

$v_1 \rightarrow Av_1 = \lambda_1 v_1 \rightarrow (A - \lambda_1) v_1 = 0$

$(A - \lambda_1) v_1 = \begin{bmatrix} 4 & 1 \\ -20 & -5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \Rightarrow \begin{cases} v_{11} = -v_{12}/4 \\ v_1 = [1 \ -4]^T \end{cases}$

$v_2: (A - \lambda_2) v_2 = \begin{bmatrix} 5 & 1 \\ -20 & -4 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \Rightarrow \begin{cases} v_{21} = -v_{22}/5 \\ v_2 = [1 \ -5]^T \end{cases}$

$T = [v_1 \ v_2] = \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix} //$

$T^{-1} = \frac{1}{\det(T)} \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} = (-1) \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix} //$

$$A = T \Lambda T^{-1} = \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix} //$$

$$\text{und } A^n = (T \Lambda T^{-1})^n = T \Lambda^n T^{-1} //$$

$$\begin{aligned} \exp(At) &= \exp(T \Lambda T^{-1} t) = \sum_{n=0}^{\infty} \frac{(T \Lambda T^{-1})^n t^n}{n!} = \\ &= \sum_{n=0}^{\infty} T \Lambda^n T^{-1} \frac{t^n}{n!} = T \sum_{n=0}^{\infty} \frac{\Lambda^n t^n}{n!} T^{-1} = \\ &= T \begin{bmatrix} \sum_{n=0}^{\infty} \lambda_1^n \frac{t^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \lambda_2^n \frac{t^n}{n!} \end{bmatrix} T^{-1} = \\ &= T \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} T^{-1} = T \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{5t} \end{bmatrix} T^{-1} // \end{aligned}$$

$$\text{Impulsantwort: } g(t) = v(t) \begin{bmatrix} 1 & 0 \end{bmatrix} \exp(At) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} g(t) &= v(t) \begin{bmatrix} 1 & 0 \end{bmatrix} T \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{5t} \end{bmatrix} T^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \\ &= v(t) \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \\ &= v(t) \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= v(t) \begin{bmatrix} e^{-4t} & e^{-5t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v(t) (e^{-4t} - e^{-5t}) // \end{aligned}$$

3 Fällen: $0 < t < 1$, $1 < t < 2$, $2 < t$

$$\text{(aus der Abbildung): } u(t) = \underset{u_1}{v(t)} + \underset{u_2}{v(t-1)} - 2 \underset{u_3}{v(t-2)}$$

$$\text{Linearität: } y(t) = \psi(t, x_0, u) = \psi(t, x_0, u_1) + \psi(t, x_0, u_2) - 2 \psi(t, x_0, u_3)$$

$$\begin{aligned} \text{Kausalität: } & \psi(t, x_0, u_1) = (g_x \cdot v)(t) \\ \text{zeitinvariant} & \psi(t, x_0, u_2) = (g_x \cdot v)(t-1) \\ & \psi(t, x_0, u_3) = (g_x \cdot v)(t-2) \end{aligned} \quad , x_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$\Rightarrow y = (g * v)_{\infty}(t) + (g * v)_{\infty}(t-1) - 2(g * v)_{\infty}(t-2)$$

$$(g * v)_{\infty}(t) = \int_{-\infty}^{\infty} g(z) v(t-z) dz = \int_{-\infty}^{\infty} v(z) e^{-4z-5z} v(t-z) dz =$$

$$= \int_0^t v(z) (e^{-4z} - e^{-5z}) v(t-z) dz = v(t) \int_0^t (e^{-4z} - e^{-5z}) dz =$$

$$= v(t) \left(\left(-\frac{e^{-4z}}{4} \right) \Big|_0^t - \left(-\frac{e^{-5z}}{5} \right) \Big|_0^t \right) =$$

$$= v(t) \left(-\frac{e^{-4t}}{4} + \frac{1}{4} + \frac{e^{-5t}}{5} - \frac{1}{5} \right) =$$

$$= v(t) \left(\frac{e^{-5t}}{5} - \frac{e^{-4t}}{4} + \frac{1}{20} \right) //$$

$$\therefore y(t) = v(t) \left(\frac{e^{-5t}}{5} - \frac{e^{-4t}}{4} + \frac{1}{20} \right) + v(t-1) \left(\frac{e^{-5(t-1)}}{5} - \frac{e^{-4(t-1)}}{4} + \frac{1}{20} \right) -$$

$$- 2v(t-2) \left(\frac{e^{-5(t-2)}}{5} - \frac{e^{-4(t-2)}}{4} + \frac{1}{20} \right) //$$