3. Übung Victor C. Chaim

Aufgabe 3.1: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\exp(At)$, $\exp(Bt)$, $\exp((A+B)t)$?

- $\exp(At) = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = id + At + \frac{A^2t^2}{2} + \dots$

$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \underset{m \geq 2}{A^m = 0}$ (nilpotent)

$\therefore \exp(At) = id + At = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ //

- $\exp(Bt) = \sum_{n=0}^{\infty} \frac{(Bt)^n}{n!} = id + Bt + \frac{B^2t^2}{2} + \dots$

$B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B \Rightarrow \underset{m \geq 1}{B^m = B}$

$\exp(Bt) = id + B \sum_{n=1}^{\infty} \frac{t^n}{n!} = id - B + B \sum_{n=0}^{\infty} \frac{t^n}{n!}$

$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ $\therefore$ $\exp(Bt) = id - B + Be^t$

$\exp(Bt) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$ //

- $\exp((A+B)t) = \sum_{n=0}^{\infty} \frac{(A+B)^n t^n}{n!}$

$(A+B) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow (A+B)^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
$= (A+B)$

$(A+B)^n = (A+B) \rightarrow n \geq 1$

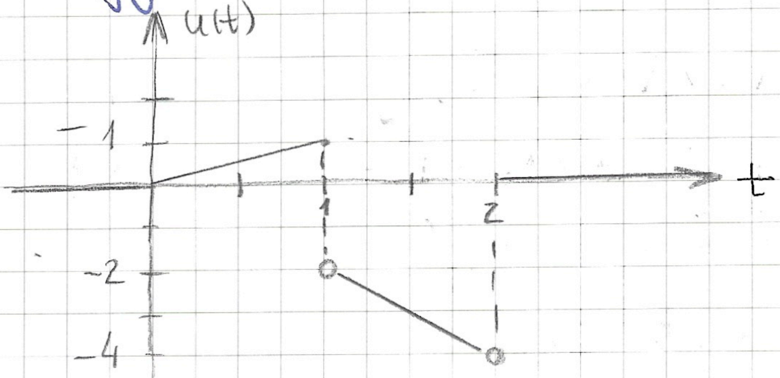
$\exp((A+B)t) = \sum_{n=0}^{\infty} \frac{(A+B)^n t^n}{n!} = I + (A+B) \sum \frac{t^n}{n!} =$

$= I - (A+B) + (A+B)e^t = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$

$$\exp(At)\exp(Bt) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & t \\ 0 & 1 \end{pmatrix} \neq \exp((A+B)t)$$

* Wenn $AB = BA$, dann gilt: $\exp(A+B) = \exp(A)\exp(B)$

Aufgabe 3.2 $\Rightarrow$ $\dot{x} = x + u$, $x_0 = -1$ $\quad u(t) = \begin{cases} t & , 0 \leq t \leq 1 \\ -2t & , 1 < t \leq 2 \\ 0 & \text{andernfalls} \end{cases}$



$\dot{x} = ax + bu$:

$$\varphi(t, x_0, u) = e^{at}x_0 + \int_0^t e^{a(t-\tau)} b\, u(\tau)\, d\tau$$

$\cdot\ a = 1,\ b = 1.$

$$\varphi(t, x_0, u) = \begin{cases} \varphi_1(t, x_0, u_1), & 0 \leq t \leq 1 \\ \varphi_2(t, x_1, u_2), & 1 < t \leq 2 \\ \varphi_3(t, x_3, u_3) & \text{andernfalls} \end{cases}$$

$$\varphi_1(t, x_0, u_1) = e^t x_0 + \int_0^t e^{(t-\tau)} u_1(\tau)\, d\tau = -e^t + e^t \int_0^t e^{-\tau}\tau\, d\tau$$

Integration durch Teile: $\int u\, dv = uv - \int v\, du$

$$\int e^{-\tau}\tau\, d\tau \to \left\langle \begin{matrix} u = \tau \\ dv = e^{-\tau} d\tau \end{matrix} \middle/ \begin{matrix} du = d\tau \\ v = -e^{-\tau} \end{matrix} \right\rangle$$

$$\int e^{-\tau}\tau\, d\tau = -e^{-\tau}\tau + \int e^{-\tau} d\tau = -e^{-\tau}\tau - e^{-\tau} = -e^{-\tau}(\tau + 1) + K$$

$$\varphi_1(t, x_0, u_1) = -e^t + e^t\left(\left(-e^{-\tau}(\tau+1)\right)\Big|_0^t\right) =$$

$$= -e^t + e^t\left(-e^{-t}(t+1) + e^0(0+1)\right)$$

$$= -e^t + e^t\left(-e^{-t}(t+1) + 1\right)$$

$$= \cancel{-e^t} + \cancel{e^t} - (t+1) = -(t+1)$$

$$x_1 = \varphi_1(1, x_0, u_1) = -(1+1) = -2$$

$$\varphi_2(t,x_1,u_2) = e^{(t-1)}\cdot x_1 + \int_1^t e^{(t-\tau)} u_2(\tau)\,d\tau$$

$$= -2e^{(t-1)} + e^t \int_1^t e^{-\tau}(-2\tau)\,d\tau$$

$$= -2e^{(t-1)} - 2e^t \int_1^t e^{-\tau}\tau\,d\tau$$

$$= -2e^{(t-1)} - 2e^t\left(-e^{-\tau}(\tau+1)\right)\Big|_1^t$$

$$= -2e^{(t-1)} - 2e^t\left(-e^{-t}(t+1) + e^{-1}(2)\right)$$

$$= -2e^{(t-1)} - 4e^{(t-1)} + 2(t+1)\cancel{e^{(t-t)}}$$

$$= -6e^{(t-1)} + 2(t+1) \;/\!/$$

$$x_2 = \varphi_2(2,x_1,u_2) = -6e + 2.3 = 6(1-e)$$

$$\varphi_3(t,x_2,u_3) = e^{(t-2)}\cdot x_2 + \int_2^t \cancel{e^{(t-\tau)} u_3(\tau)}\,d\tau = x_2 e^{(t-2)}$$

$$\varphi_3(t,x_2,u_3) = e^{(t-2)}\cdot x_2 = 6(1-e)e^{(t-2)} \;/\!/$$

$$\varphi(t,x_0,u) = \begin{cases} -(t+1), & 0 \le t \le 1 \\ -6e^{(t-1)} + 2(t+1), & 1 < t \le 2 \\ 6(1-e)e^{(t-2)}, & \text{andernfalls.} \end{cases}$$

Aufgabe 3.3

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = u \\ x_1(0) = 0, \quad x_2(0) = 1 \end{cases}, \quad u = \begin{cases} t, & 0 \le t \le 1 \\ -2t, & 1 < t \le 2 \\ 0, & \text{andernfalls} \end{cases}$$

$$\dot{x} = Ax + Bu \rightarrow x = [x_1 \;\; x_2]^T$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\varphi(t,x_0,u) = \begin{cases} \varphi_1(t,x_0,u_1), & 0 \le t \le 1 \\ \varphi_2(t,x_1,u_2), & 1 < t \le 2 \\ \varphi_3(t,x_2,u_3) & \text{andernfalls} \end{cases}$$

$0 \le t \le 1: \quad u_1(t) = t$

$$\varphi_1(t, x_0, u_1) = e^{At} \cdot x_0 + \int_0^t e^{A(t-\tau)} B \cdot \tau \, d\tau$$

$\exp\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} t\right)$

Aufgabe 3.1

$$= \begin{pmatrix} e^t & e^t-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} + e^{At}\int_0^t e^{-A\tau} B \tau \, d\tau$$

$$= \begin{pmatrix} e^t & e^t-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix} e^t & e^t-1 \\ 0 & 1\end{pmatrix}\int_0^t \begin{pmatrix} e^{-\tau} & e^{-\tau}-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\tau \, d\tau$$

$$= \begin{pmatrix} e^t-1 \\ 1\end{pmatrix} + \begin{pmatrix} e^t & e^t-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\int_0^t (e^{-\tau}-1)\tau \, d\tau \\ \int_0^t (1)\tau \, d\tau\end{pmatrix} \rightarrow$$

$$\rightarrow \int_0^t (e^{-\tau}-1)\tau \, d\tau = \int_0^t e^{-\tau}\tau \, d\tau - \int_0^t \tau \, d\tau = -e^{-\tau}(\tau+1)\Big|_0^t - \frac{\tau^2}{2}\Big|_0^t$$

$$= -e^{-t}(t+1) + 1 - \frac{t^2}{2} \quad //$$

Integration durch Teile: Aufgabe 3.2

$$\varphi_1(t, x_0, u_1) = \begin{pmatrix} e^t-1 \\ 1\end{pmatrix} + \begin{pmatrix} e^t & e^t-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 - t^2/2 - e^{-t}(t+1) \\ t^2/2\end{pmatrix}$$

$$= \begin{pmatrix} e^t-1 \\ 1\end{pmatrix} + \begin{pmatrix} e^t - \cancel{t^2/2\, e^t} - (t+1) + \cancel{e^t\, t^2/2} - t^2/2 \\ t^2/2\end{pmatrix}$$

$$= \begin{pmatrix} e^t-1 \\ 1\end{pmatrix} + \begin{pmatrix} e^t - t^2/2 - t + 1 \\ t^2/2\end{pmatrix} = \begin{pmatrix} 2e^t - t^2/2 - t - 2 \\ 1 + t^2/2\end{pmatrix} //$$

$$x_1 = \varphi_1(1, x_0, u_1) = \begin{pmatrix} 2e - 1/2 - 1 - 2 \\ 1 + 1/2\end{pmatrix} = \begin{pmatrix} 2e - 7/2 \\ 3/2\end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{12}\end{pmatrix}$$

Konstante

$1 < t \le 2: \quad u_2(t) = -2t$

$$\varphi_2(t, x_1, u_2) = e^{A(t-1)} \cdot x_1 + \int_1^t e^{A(t-\tau)} B (-2\tau) \, d\tau$$

$$= \begin{pmatrix} e^{(t-1)} & e^{(t-1)}-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12}\end{pmatrix} + e^{At}\int_1^t e^{-A\tau} B (-2\tau) \, d\tau$$

$$= \begin{pmatrix} (x_{11}+x_{12}) e^{(t-1)} - x_{12} \\ x_{12}\end{pmatrix} + \begin{pmatrix} e^t & e^t-1 \\ 0 & 1\end{pmatrix}\int_1^t \begin{pmatrix} e^{-\tau} & e^{-\tau}-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}(-2\tau) \, d\tau$$

$$= \begin{pmatrix} (x_{11}+x_{12}) e^{(t-1)} - x_{12} \\ x_{12}\end{pmatrix} - 2\begin{pmatrix} e^t & e^t-1 \\ 0 & 1\end{pmatrix}\int_1^t \begin{pmatrix} e^{-\tau} & e^{-\tau}-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\tau \, d\tau$$

$$= \begin{pmatrix} (x_{11}+x_{12})e^{(t-1)} - x_{12} \\ x_{12} \end{pmatrix} - 2 \begin{pmatrix} e^t & e^t-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \int_1^t (e^{-\tau}-1)\tau \, d\tau \\ \int_1^t \tau \, d\tau \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}+x_{12})e^{(t-1)} - x_{12} \\ x_{12} \end{pmatrix} - 2 \begin{pmatrix} e^t & e^t-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -e^{-\tau}(\tau+1)\big|_1^t - \tau^2/2 \big|_1^t \\ \tau^2/2 \big|_1^t \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}+x_{12})e^{(t-1)} - x_{12} \\ x_{12} \end{pmatrix} - 2 \begin{pmatrix} e^t & e^t-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -e^{-t}(t+1) + e^{-1}(1+1) - t^2/2 - 1/2 \\ t^2/2 - 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}+x_{12})e^{(t-1)} - x_{12} \\ x_{12} \end{pmatrix} - 2 \begin{pmatrix} e^t & e^t-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -e^{-t}\cdot t - e^{-t} + e^{-1}\cdot 2 - t^2/2 - 1/2 \\ t^2/2 - 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}+x_{12})e^{(t-1)} - x_{12} \\ x_{12} \end{pmatrix} - 2 \begin{pmatrix} -t - 1 + e^t e^{-1}\cdot 2 \cancel{- e^t\cdot t^2/2} - e^t/2 \cancel{+ e^t t^2/2} - e^t/2 - \frac{t^2}{2} + 1/2 \\ t^2/2 - 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}+x_{12})e^{(t-1)} - x_{12} \\ x_{12} \end{pmatrix} - 2 \begin{pmatrix} -t - 1/2 + e^{(t-1)}\cdot 2 - e^t - t^2/2 \\ t^2/2 - 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}+x_{12})e^{(t-1)} - x_{12} \\ x_{12} \end{pmatrix} + \begin{pmatrix} 2t + 1 - 4e^{(t-1)} + 2e^t + t^2 \\ -t^2 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}+x_{12}-4)e^{(t-1)} + 2e^t + t^2 + 2t + (1-x_{12}) \\ -t^2 + (1+x_{12}) \end{pmatrix} \quad // \qquad \text{Konstante}$$

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$$x_2 = \varphi_2(2, x_1, u_2) = \begin{pmatrix} (x_{11}+x_{12}-4)e^1 + 2e^2 + 4 + 4 + 1 - x_{12} \\ -4 + 1 + x_{12} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$t > 2$: $\quad u_3(t) = 0$

$$\varphi_3(t, x_2, u_3) = e^{A(t-2)}\cdot x_2 + \overset{0}{\cancel{\int_2^t e^{A(t-2)} B(0)\, d\tau}} = e^{A(t-2)}\cdot x_2$$

$$\varphi_3(t, x_2, u_3) = \begin{pmatrix} e^{(t-2)} & e^{(t-2)}-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} (x_{21}+x_{22})e^{(t-2)} - x_{22} \\ x_{22} \end{pmatrix} \quad //$$

$\therefore$

$$\varphi(t, x_0, u) = \begin{cases} \begin{pmatrix} 2e^t - t^2/2 - t - 2 \\ 1 + t^2/2 \end{pmatrix}, & 0 \leq t \leq 1 \\ \begin{pmatrix} (x_{11}+x_{12}-4)e^{(t-1)} + 2e^t + t^2 + 2t + (1-x_{12}) \\ -t^2 + (1+x_{12}) \end{pmatrix}, & 1 < t \leq 2 \\ \begin{pmatrix} (x_{21}+x_{22})e^{(t-2)} - x_{22} \\ x_{22} \end{pmatrix}, & \text{andernfalls} \end{cases} \quad //$$

wobei: $\begin{cases} x_{11} = 2e - 7/2 \\ x_{21} = 3/2 \end{cases}$ und $\begin{cases} x_{21} = (x_{11}+x_{12}-4)e + 2e^2 + 9 - x_{12} \\ x_{22} = -3 + x_{12} \end{cases}$

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