Steuer- und Regelungstechnik

4. Übung

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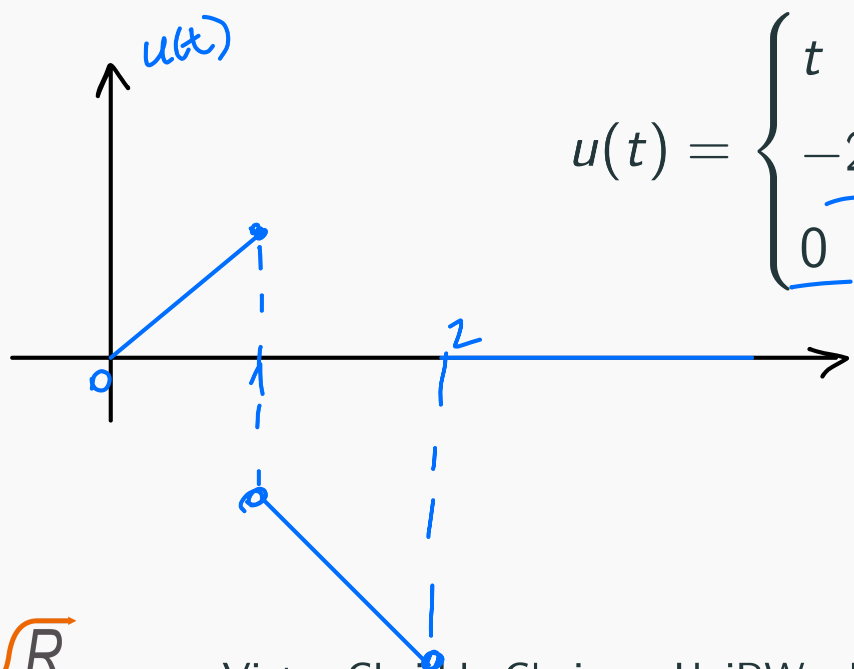
Berechnen Sie die Lösung für das Zustandssystem $\dot{x} = x + u$, das den skalaren Fall darstellt, mit der Anfangsbedingung $x_0 = -1$ und der folgenden stückweise linearen Funktion für den Eingang $u(t)$, wobei $t \in \mathbb{R}$:



$$u(t) = \begin{cases} t & \text{falls } 0 \leq t \leq 1 \\ -2t & \text{falls } 1 < t \leq 2 \\ 0 & \text{andernfalls.} \end{cases}$$

t

$\varphi(t, x_0, u) = ?$

Aufgabe 3.2

$0 \le t \le 1$: $\quad \underbrace{x_0}^{x(0)} = -1 \quad | \quad u(t) = t \quad , \quad \dot{x} = x + u = ax + bu$

$a = 1$ und $b = 1$

$$\varphi(t, x_0, u) = e^{at} x_0 + \int_0^t e^{a(t-\tau)}\, b\, u(\tau)\, d\tau$$

$$\varphi_1(t, x_0, u_1) = e^{t} \cdot x_0 + \int_0^t e^{(t-\tau)}\, u_1(\tau)\, d\tau = -e^{t} + \int_0^t e^{(t-\tau)} \cdot \tau \cdot d\tau$$

$$= \underline{-e^{t}} + \underline{e^{t}} \underbrace{\int_0^t e^{-\tau} \cdot \tau \cdot d\tau}$$

Integration durch Teile:

$$\int \alpha\, d\beta = \alpha\beta - \int \beta\, d\alpha$$

$$\int e^{-\tau} \tau\, d\tau \Rightarrow \left\{ \begin{array}{ll} \alpha = \tau & \Rightarrow d\alpha = d\tau \\ d\beta = e^{-\tau} d\tau & \Rightarrow \beta = -e^{-\tau} \end{array} \right\}$$

$$\int e^{-\tau} \tau\, d\tau = \tau \cdot (-e^{-\tau}) - \int -e^{-\tau}\, d\tau = \underbrace{-e^{-\tau} \cdot \tau} - e^{-\tau} + c_1 = \underline{-e^{-\tau}(\tau + 1) + c_1}$$

$$\varphi_1(t, x_0, u_1) = -e^{t} + e^{t}\left(-e^{-\tau}(\tau + 1)\right)\Big|_0^t = -e^{t} + e^{t}\left(-e^{-t}(t+1) + 1\right)$$

$$= -\cancel{e^{t}} + (-\cancel{e^{t-t}}(t+1)) + \cancel{e^{t}} = -(t+1)$$

$$\varphi_1(t, x_0, u_1) = -(t+1)$$

Aufgabe 3.2

$\varphi_1(t, \underline{\underline{x_0}}, u_1) = -(t+1) \quad \Rightarrow x_1 = \varphi_1(1, x_0, u_1) = -(1+1) = -2$

$u_2 = -2t \quad (1 < t \leq 2)$

$\varphi_2(t, \underline{\underline{x_1}}, u_2)$

$$\varphi_2(t, x_1, u_2) = e^{(t-1)} \cdot x_1 + e^{t} \int_1^t e^{-\tau}(-2\tau)\,d\tau =$$

$$= -2e^{(t-1)} - 2e^{t} \int_1^t \underline{e^{-\tau} \cdot \tau \, d\tau}$$

$$= -2e^{(t-1)} - 2e^{t}\left(-e^{-\tau}(\tau+1)\right)\Big|_1^t = -2e^{(t-1)} - 2\underline{e^{t}}\left(-e^{-t}(t+1) + e^{-1} \cdot 2\right)$$

$$= -2e^{(t-1)} + 2\cancel{e^{(t-t)}}(t+1) - 4\underline{e^{-1} \cdot e^{t}} = e^{(t-1)}$$

$$\varphi_2(t, x_1, u_2) = -6e^{(t-1)} + 2(t+1) \,//$$

Aufgabe 3.2

$$\varphi_2(t, x_1, u_2) = -6e^{(t-1)} + 2(t+1) \rightarrow x_2 = \varphi_2(2, x_1, u_2) = 6(1+e)$$

$t > 2$

$$\varphi_3(t, x_2, u_3) = e^{(t-2)} \cdot x_2 + \cancelto{0}{\int_2^t e^{(t-\tau)} u_3(\tau)\, d\tau} = \underline{\underline{e^{(t-2)} \cdot x_2}}$$

$$\varphi_3(t, x_2, u_3) = 6(1-e) \cdot e^{(t-2)}$$

$$\varphi(t, x_0, u) = \begin{cases} \overbrace{-(t+1)}^{=\varphi_1}, & 0 \leq t \leq 1 \\ \overbrace{-6e^{(t-1)} + 2(t+1)}^{=\varphi_2}, & 1 < t \leq 2 \\ \overbrace{6(1-e)e^{(t-2)}}^{=\varphi_3}, & \text{andernfalls.} \end{cases}$$

Aufgabe 4.1

Betrachten Sie das folgende Differentialgleichungssystem

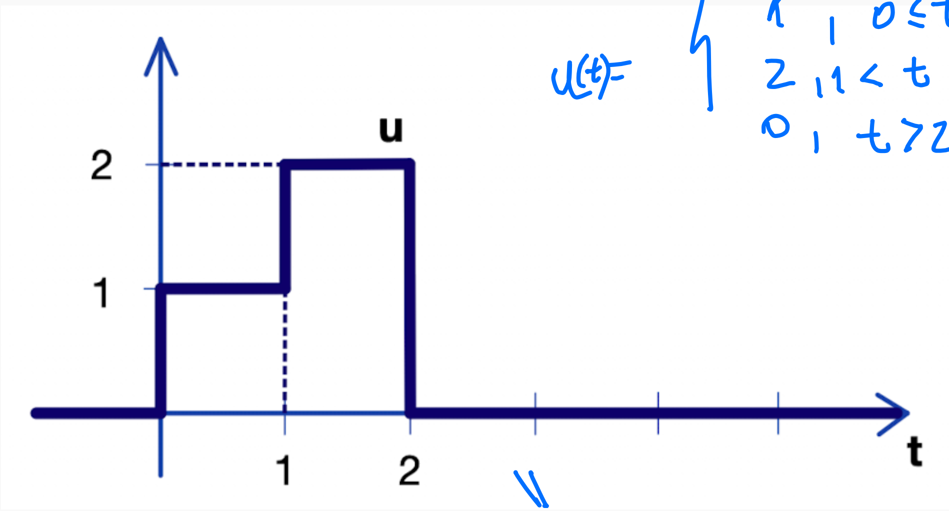
Zustandform:

$$\ddot{x}(t) = -9\dot{x}(t) - 20x(t) + u(t)$$

$$y(t) = x(t)$$

$$\underline{\underline{x(0)}} = 0 \quad \text{und} \quad \underline{\underline{\dot{x}(0)}} = 0$$

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t \le 1 \\ 2, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$$



$$\rightarrow u(t) = \sigma(t) + \sigma(t-1) - 2\sigma(t-2)$$

Berechnen Sie das Ausgangssignal y.

Aufgabe 4.1

$$\begin{cases} \ddot{x} = -9\dot{x} - 20x + u \\ y = x \end{cases} \xrightarrow{ZS} \begin{cases} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \cdot u \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \cdot \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{[0]}_{D} u \end{cases}$$

Impulsantwort: $g(t) = \sigma(t) \cdot C e^{At} \cdot B + \delta(t) \cdot D$

$$g(t) = \sigma(t)\, C e^{At} B + \cancel{\delta(t) D}^{\,D=0} = \sigma(t) C e^{At} B \,, \quad \sigma(t) = \begin{cases} 1, & \text{falls } t \geq 0 \\ 0, & \text{sonst.} \end{cases}$$

$$g(t) = \sigma(t) \begin{bmatrix} 1 & 0 \end{bmatrix} e^{At} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \rightarrow \text{Eigenwerte: } \det(A - \lambda\, id) = 0 \rightarrow \det \begin{bmatrix} -\lambda & 1 \\ -20 & -9-\lambda \end{bmatrix} =$$

$$= \lambda(9+\lambda) + 20 = \lambda^2 + 9\lambda + 20 \rightarrow \begin{matrix} \lambda_1 = -4 \\ \lambda_2 = -5 \end{matrix}$$

Aufgabe 4.1

Eigenwerte $\neq 0$, A ist nicht nilpotent. $A = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}$

Diagonaltransformation: $A = T \Lambda T^{-1}$, $T = [v_1 \quad v_2]$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$

$v_1 \rightarrow A v_1 = \lambda_1 v_1 \rightarrow (A - \lambda_1) v_1 = 0$

$(A - id\lambda_1) v_1 = \begin{bmatrix} 4 & 1 \\ -20 & -5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} v_{11} = -v_{12}/4, \quad v_{11} = \underline{1}, \quad v_{12} = -4 \\ v_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \end{cases}$

$\rightarrow 4v_{11} + v_{12} = 0$

$-20 v_{11} - 5 v_{12} = 0 \quad (4v_{11} + v_{12} = 0)$

$(A - id\lambda_2) v_2 = \begin{bmatrix} 5 & 1 \\ -20 & -4 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} v_{21} = -v_{12}/5, \quad v_{21} = \underline{1}, \quad v_{22} = -5 \\ v_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \end{cases}$

$\rightarrow 5v_{21} + v_{22} = 0$

$-20 v_{21} - 4 v_{22} = 0 \quad (5v_{21} + v_{22} = 0)$

$T = [v_1 \quad v_2] = \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix}$, $T^{-1} = \frac{1}{\det(T)} \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} = (-1) \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix}$

$\det(T) = -5 - (-4) = -1$

Aufgabe 4.1

$$\therefore \quad A = T \Lambda T^{-1} = \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix}$$

$$\text{und} \quad A^n = (T \Lambda T^{-1})^n = \underline{T \Lambda^n T^{-1}} \longrightarrow \Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

$$\exp(At) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(T \Lambda T^{-1})^n \cdot t^n}{n!} = \sum_{n=0}^{\infty} \frac{T \Lambda^n T^{-1} \cdot t^n}{n!} = T \left(\sum_{n=0}^{\infty} \frac{\Lambda^n \cdot t^n}{n!} \right) T^{-1}$$

$$= T \left(\sum_{n=0}^{\infty} \frac{\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \cdot t^n}{n!} \right) T^{-1} = T \begin{pmatrix} \sum_{n=0}^{\infty} \frac{\lambda_1^n \cdot t^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{\lambda_2^n \cdot t^n}{n!} \end{pmatrix} T^{-1}$$

$$\sum_{n=0}^{\infty} \frac{\lambda_1^n \cdot t^n}{n!} \Rightarrow e^{\lambda_1 t} \qquad \sum_{n=0}^{\infty} \frac{\lambda_2^n \cdot t^n}{n!} = e^{\lambda_2 t}$$

$$= T \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} T^{-1} = T \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-5t} \end{bmatrix} T^{-1}$$

Aufgabe 4.1

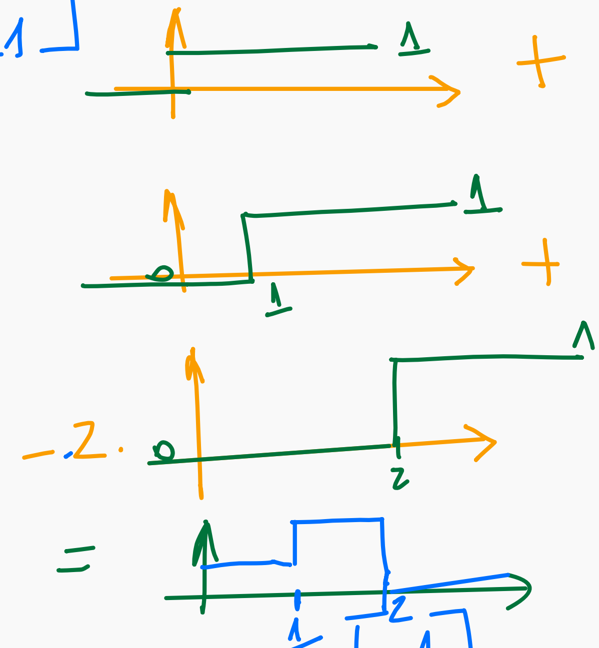
Impulsantwort: $g(t) = \sigma(t)[1\ \ 0]\exp(At)\begin{bmatrix}0\\1\end{bmatrix}$

$$g(t) = \sigma(t)\underbrace{[1\ \ 0]}_{1\times 2}\underbrace{T}_{2\times 2}\begin{bmatrix}e^{-4t} & 0\\ 0 & e^{-5t}\end{bmatrix}T^{-1}\cdot\begin{bmatrix}0\\1\end{bmatrix}$$

$$T = \begin{bmatrix}1 & 1\\ -4 & -5\end{bmatrix} \quad \text{und} \quad T^{-1} = \begin{bmatrix}5 & 1\\ -4 & -1\end{bmatrix}$$

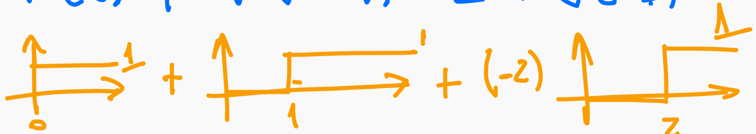
$$g(t) = \sigma(t)\,\underbrace{[1\ \ 1]\begin{bmatrix}e^{-4t} & 0\\ 0 & e^{-5t}\end{bmatrix}}\begin{bmatrix}1\\-1\end{bmatrix} = \sigma(t)\,[e^{-4t}\ \ e^{-5t}]\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\boxed{g(t) = \sigma(t)\left(e^{-4t} - e^{-5t}\right)}$$



3 Fällen: $0 < t < 1$, $1 < t < 2$, $2 < t$

$$u(t) = \sigma(t) + \sigma(t-1) - 2\sigma(t-2)$$



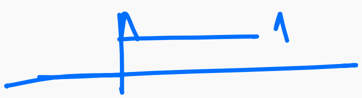
Aufgabe 4.1

$$\begin{cases} g(t) = \sigma(t)\left(e^{-4t} - e^{-5t}\right) \\ u(t) = \sigma(t) + \sigma(t-1) - 2\sigma(t-2) \quad \text{(aus der Abbildung)} \end{cases}$$

Linearität: $y(t) = \psi(t, x_0, u) = \psi(t, x_0, u_1) + \psi(t, x_0, u_2) - 2\psi(t, x_0, u_3)$

Kausalität :
Zeitinvariant

$$\psi(t, x_0, u_1) = (g * \sigma)(t)$$
$$\psi(t, x_0, u_2) = (g * \sigma)(t-2) \quad , x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\psi(t, x_0, u_3) = (g * \sigma)(t-2)$$



$$(g * \sigma)(t) = \int_{-\infty}^{\infty} g(\tau)\, \sigma(t-\tau)\, d\tau = \int_{-\infty}^{\infty} \underbrace{\sigma(\tau)}_{\text{,,0'' für } t<0} \left(e^{-4\tau} - e^{-5\tau}\right) \overbrace{\sigma(t-\tau)}^{=0 \text{ für } \tau > t} d\tau$$

$$= \int_0^t \sigma(\tau)\left(e^{-4\tau} - e^{-5\tau}\right)\sigma(t-\tau)\, d\tau$$

Aufgabe 4.1

$$(g * \sigma)(t) = \int_0^t \underbrace{\sigma(\tau)}_{=1} (e^{-4\tau} - e^{-5\tau}) \underbrace{\sigma(t-\tau)}_{=1} \, d\tau = \sigma(t) \int_0^t (e^{-4\tau} - e^{-5\tau}) \, d\tau$$

$$= \sigma(t) \left(\int_0^t e^{-4\tau} d\tau - \int_0^t e^{-5\tau} d\tau \right) = \sigma(t) \left(-\frac{e^{-4\tau}}{4} \Big|_0^t - \left(-\frac{e^{-5\tau}}{5} \right) \Big|_0^t \right)$$

$$= \sigma(t) \left(-\frac{e^{-4t}}{4} + \frac{1}{4} + \frac{e^{-5t}}{5} - \frac{1}{5} \right) = \sigma(t) \left(\frac{e^{-5t}}{5} - \frac{e^{-4t}}{4} + \frac{1}{20} \right) //$$

$$\therefore \; y(t) = \sigma(t) \cdot \left(\frac{e^{-5t}}{5} - \frac{e^{-4t}}{4} + \frac{1}{20} \right) + \sigma(t-1) \cdot \left(\frac{e^{-5(t-1)}}{5} - \frac{e^{-4(t-1)}}{4} + \frac{1}{20} \right) - 2 \cdot \sigma(t-2) \left(\frac{e^{-5(t-2)}}{5} - \frac{e^{-4(t-2)}}{4} + \frac{1}{20} \right)$$