Steuer- und Regelungstechnik

3. Übung

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Aufgabe 2.1

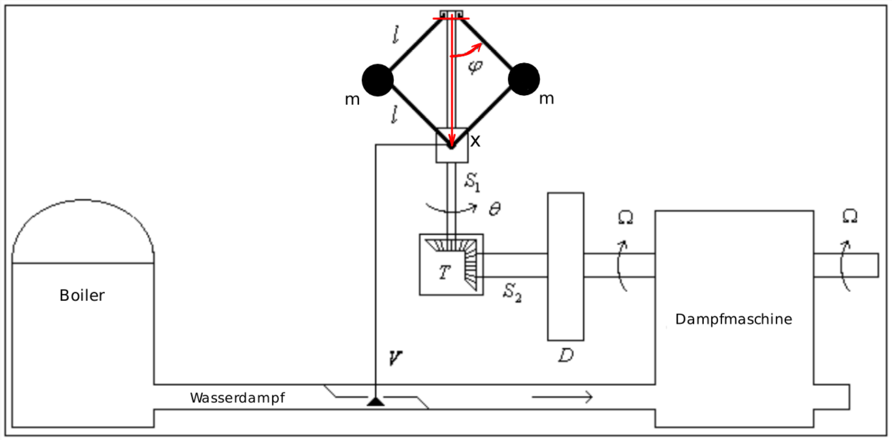


Abbildung 1: System: Dampfmaschine und Fliehkraftregler.
Quelle (bearbeitet von V. Chaim): *Bifurcation Analysis of the Watt Governor System*, Sotomayor, J.; Mello, L. F.; Braga, D. C.; Computational and Applied Mathematics, Vol. 26, N.1, pp 19-44, 2007.

- Drehmoment-Dämpfung
  $M_r = -b\dot{\varphi}(t)$;
- Getriebeverhältnis
  $\theta(t) = n\Omega(t)$;
- Eingang $u(t) = k(x(t) - x_{ref})$.

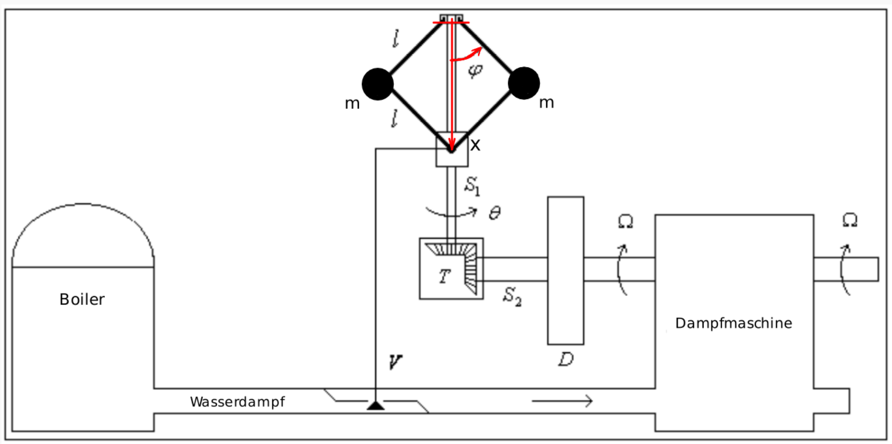
$B \rightarrow \frac{\partial f(x,u)}{\partial u}$

Aufgabe 2.1

$$\dot{x} = f(x,u) = \begin{pmatrix} \dot{\varphi} \\ \theta^2 \cos\varphi \sin\varphi - \sin\varphi \frac{g}{l} - b^* \dot{\varphi} \\ k^* \cos\varphi - k_1 \end{pmatrix}, \quad b^* = \frac{b}{ml^2}, \quad k^* = \frac{k n 2 l}{I_D}, \quad k_1 = \frac{x_{0} g \cdot k \cdot n}{I_D}$$

$$x = [\varphi \ \ \dot{\varphi} \ \ \theta]^T$$

3e) Linearisierung: Ruhelage: $\dot{x} = 0$ ($f(x_0, u_0) = 0$)



$$\begin{cases} \dot{\varphi}_0 = 0 & (I) \\ \theta_0^2 \cos\varphi_0 \sin\varphi_0 - \sin\varphi_0 \frac{g}{l} - b^* \dot{\varphi}_0 = 0 & (II) \\ k^* \cdot \cos\varphi_0 - k_1 = 0 & (III) \end{cases}$$

(I) → (II) : $\theta_0^2 \cos\varphi_0 \sin\varphi_0 - \sin\varphi_0 \frac{g}{l} = 0 \;|\; \sin\varphi_0 \left(\theta_0^2 \cos\varphi_0 - \frac{g}{l} \right) = 0$

$$\theta_0^2 \cos\varphi_0 = \frac{g}{l} \quad (IV)$$

(III) : $\boxed{\cos\varphi_0 = \frac{k_1}{k^*}}$ → $\theta_0^2 \cdot \frac{k_1}{k^*} = \frac{g}{l}$ → $\boxed{\theta_0^2 = \frac{g}{l} \cdot \frac{k^*}{k_1}}$

Aufgabe 2.1

$$\dot{x} = f(x,u) = \begin{pmatrix} \dot\varphi \quad = f_1 \\ \theta^2 \cos\varphi \sin\varphi - \sin\varphi \frac{g}{\ell} - b^* \dot\varphi \quad = f_2 \\ k^* \cos\varphi - k_1 \quad = f_3 \end{pmatrix}, \qquad \dot\varphi_0 = 0, \quad \theta_0^2 = \frac{g}{\ell}\cdot\frac{k^*}{k_1}$$

$$\cos\varphi_0 = \frac{k_1}{k^*}$$

$\varphi = x_1$, $\dot\varphi = x_2$, $\theta = x_3$

$$A = D_1 f(x_0, u_0)$$

$$A_{11} = \frac{\partial f_1(x_0,u_0)}{\partial \varphi} = 0, \quad A_{12} = \frac{\partial f_1(x_0,u_0)}{\partial \dot\varphi} = 1, \quad A_{13} = \frac{\partial f_1}{\partial \theta} = 0$$

$$A_{21} = \frac{\partial f_2(x_0,u_0)}{\partial \varphi} = \theta_0^2 \big(\underbrace{-\sin^2\varphi_0 + \cos^2\varphi_0}_{\cos 2\varphi_0}\big) - \cos\varphi_0 \cdot \frac{g}{\ell}$$

$$A_{22} = \frac{\partial f_2}{\partial \dot\varphi} = -b^*, \quad A_{23} = \frac{\partial f_2}{\partial \theta} = 2\,\theta_0 \cos\varphi_0 \sin\varphi_0 = \theta_0 \sin 2\varphi_0$$

$$A_{31} = \frac{\partial f_3}{\partial \varphi} = -k^* \sin\varphi, \quad A_{32} = \frac{\partial f_3}{\partial \dot\varphi} = 0, \quad A_{33} = \frac{\partial f_3}{\partial \theta} = 0$$

Aufgabe 2.1

$$\dot{x} = A x$$

$$\underbrace{\begin{bmatrix} \dot{\varphi} \\ \ddot{\varphi} \\ \dot{\theta} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \theta_0^2 \cos 2\varphi_0 - \cos\varphi_0 \frac{g}{l} & -b^* & \theta_0 \sin 2\varphi_0 \\ -k^* \sin\varphi_0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varphi \\ \dot{\varphi} \\ \theta \end{bmatrix}}_{x}$$

Aufgabe 2.2

1) Modellierungsziel: Zusammenhang zw. Kraft und Wagen. F und variablen $x(t)$ und $\theta(t)$.

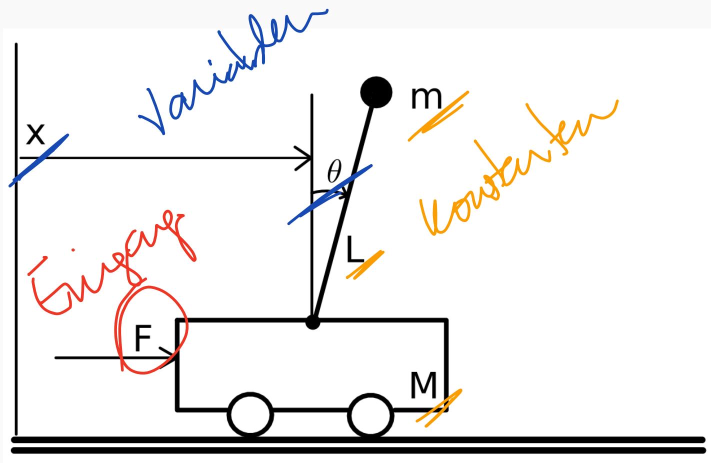


Abbildung 2: System: Umgekehrtes Pendel auf einem Wagen.

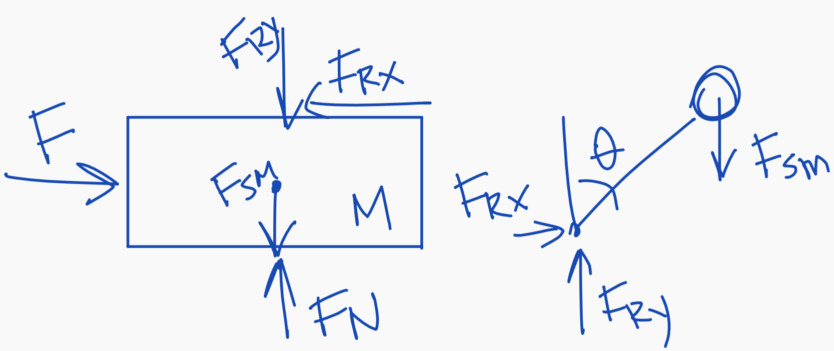
2) Blöcke: 2 Massen.

- die Kraft F = Eingang;
- keine Reibung zwischen dem Wagen und dem Boden.

3)a) Verhalten der Blöcke.

Masse M: F_{Rx}, F_N, F_{Ry}, F_{SM}, F, F_M

Masse m: F_{Rx}, F_{Ry}, F_{Sm}, F_m Newton



Aufgabe 2.2

3)a) Verhalten der Blöcke.

Masse M: F_{Rx}, F_N, F_{Ry}, F_{SM}, F, F_M

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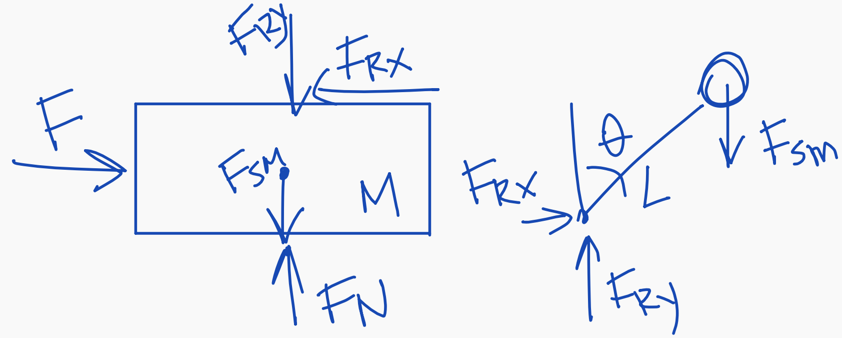
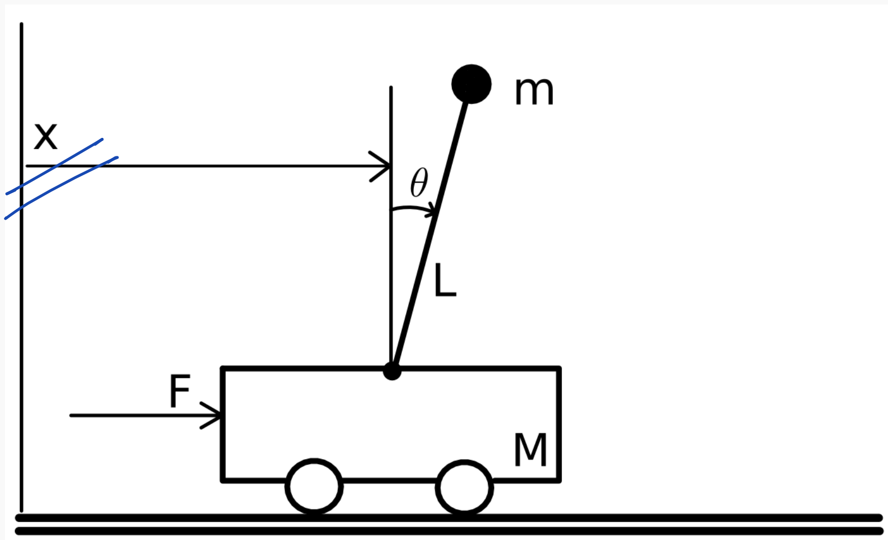




3)b) Verbindung der Blöcke:

$$y:\ F_N - F_{SM} - F_{Ry} = 0$$

$$x:\ F - F_{Rx} = \ddot{x} \cdot M \quad (I)$$

$$y:\ m \cdot \ddot{y}_m = F_{Ry} - F_{Sm} \quad (II)$$

$$x:\ m \cdot \ddot{x}_m = F_{Rx} \quad (III)$$

$$\text{Drehmoment: } F_{Rx} \cdot \cos\theta \cdot L = F_{Ry} \cdot \sin\theta \cdot L \quad (IV)$$

Aufgabe 2.2

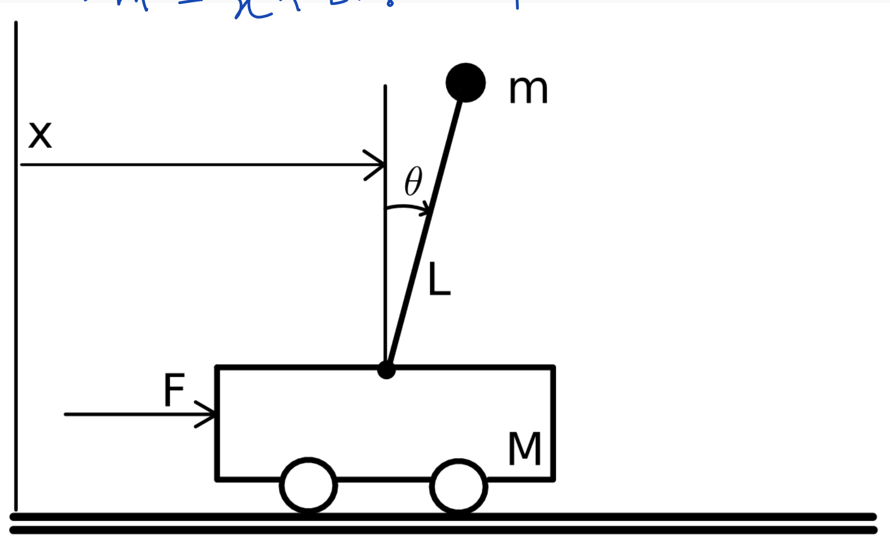
3) c) Vereinfachung und Elimination:

$L\cos\theta = y_m$,

$x_m = x + L\cdot\sin\theta$,

$\dot x_m = \dot x + L\cdot\dot\theta\cdot\cos\theta$ (V)

$\dot y_m = -L\sin\theta\cdot\dot\theta$ (VI)



$$\begin{cases} \ddot x_m = \ddot x + L(\ddot\theta\cos\theta - \dot\theta^2\sin\theta) & \text{(VII)} \\ \ddot y_m = -L(\ddot\theta\sin\theta + \dot\theta^2\cos\theta) & \text{(VIII)} \end{cases}$$

(VII) → (II) → (I):

$M\ddot x = F - m\ddot x_m =$

$M\ddot x = F - m(\ddot x + L(\ddot\theta\cos\theta - \dot\theta^2\sin\theta))$

$$\boxed{(M+m)\ddot x = F - mL\cos\theta\,\ddot\theta - mL\dot\theta^2\sin\theta} \quad \text{(IX)}$$

(III) → (II) → (IV):

$\cancel{m}\ddot x_m\cos\theta\cdot\cancel{L} = \cancel{m}(\ddot y_m + g)\sin\theta\cdot\cancel{L}$

$\ddot x_m\cos\theta = (\ddot y_m + g)\sin\theta$ (X)

(VII) und (VIII) → (X): $(\ddot x + L(\ddot\theta\cos\theta - \dot\theta^2\sin\theta))\cos\theta =$
$\sin\theta(g - L(\ddot\theta\sin\theta + \dot\theta^2\cos\theta))$

Aufgabe 2.2

$$\boxed{(M+m)\ddot{x} = F - mL\cos\theta\,\ddot{\theta} - mL\dot{\theta}^2\sin\theta} \quad (IX)$$

$$(\ddot{x} + L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta))\cos\theta = \sin\theta\,(g - L(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)) \Rightarrow \ddot{x}\cos\theta + L\ddot{\theta}\cos^2\theta - \cancel{L\dot{\theta}^2\sin\theta\cos\theta} = g\sin\theta - L\ddot{\theta}\sin^2\theta - \cancel{L\dot{\theta}^2\sin\theta\cos\theta}$$

$$\ddot{x}\cos\theta + L\ddot{\theta}\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} - g\sin\theta = 0 \qquad x = [x \;\; \dot{x} \;\; \theta \;\; \dot{\theta}]$$

$$\ddot{x}\cos\theta + L\ddot{\theta} - g\sin\theta = 0 \quad (X) \Rightarrow \boxed{\ddot{\theta} = -\frac{\ddot{x}\cos\theta}{L} + \frac{g}{L}\sin\theta} \quad (XII)$$

$$(XII) \rightarrow (IX) \Rightarrow (M+m)\ddot{x} = F - mL\cos\theta\left(-\frac{\ddot{x}\cos\theta}{L} + \frac{g}{L}\sin\theta\right) - mL\dot{\theta}^2\sin\theta \quad (XIII)$$

$$\ddot{\theta} = -\frac{(F - mg\cos\theta\sin\theta - mL\dot{\theta}^2\sin\theta)}{M + m\sin^2\theta}\frac{\cos\theta}{L} + \frac{g}{L}\sin\theta$$

$$XIII \rightarrow XII \Rightarrow \ddot{x} = \frac{F - mg\cos\theta\sin\theta - mL\dot{\theta}^2\sin\theta}{M + m\sin^2\theta}.$$

Aufgabe 2.2

$$\dot{x} = f(x,u) = \begin{pmatrix} \dot{x} = f_1 \\ \dfrac{F - mg\cos\theta\sin\theta - mL\dot{\theta}^2\sin\theta}{M + m\sin^2\theta} = f_2 \\ \dot{\theta} = f_3 \\ -\dfrac{(F - mg\cos\theta\sin\theta - mL\dot{\theta}^2\sin\theta)}{M + m\sin^2\theta}\dfrac{\cos\theta}{L} + \dfrac{g}{L}\sin\theta = f_4 \end{pmatrix}$$

$x = [x \; \dot{x} \; \theta \; \dot{\theta}]$

$u = F$

Ruhelage: $\dot{x} = 0 \rightarrow \boxed{\dot{x}_0 = 0}, \boxed{\dot{\theta}_0 = 0}$

$f_2 = 0$, $\dfrac{\overset{u}{F_0} - mg\cos\theta_0\sin\theta_0 - \cancel{mL\dot{\theta}^2\sin\theta_0}}{M + m\sin^2\theta_0} = 0$, $\overset{u_0}{F_0} = mg\cos\theta_0\sin\theta_0$

$u_0 = mg\cos\theta_0\sin\theta_0$

$f_4 = 0 \rightarrow -\dfrac{(\cancel{F_0} - \cancel{mg\cos\theta_0\sin\theta_0} - mL\cancel{\dot{\theta}_0^2}\sin\theta_0)}{M + m\sin^2\theta_0}\cdot\dfrac{\cos\theta_0}{L} + \dfrac{g}{L}\sin\theta_0 = 0$

$\sin\theta_0 = 0$, $\boxed{\theta_0 = 0} \rightarrow \boxed{u_0 = 0}$

Aufgabe 2.2

$$\begin{cases} A = \begin{pmatrix} A_{11} & A_{12} & \\ & \ddots & \\ & & \ddots \end{pmatrix}, & A_{11} = \dfrac{\partial f_1(x_0, u_0)}{\partial x} \; \dots \\ B = \begin{pmatrix} B_{11} \\ B_{12} \\ \vdots \end{pmatrix}, & B_{11} = \dfrac{\partial f_1(x_0, u_0)}{\partial u} \end{cases}$$

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{gm}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(M+m)}{M} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$= A \qquad = x$$

Aufgabe 3.1

Gegeben seien die Matrizen A und B durch

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Berechnen Sie $\exp(At)$, $\exp(Bt)$ und $\exp((A+B)t)$. Gilt hier $\exp(A+B) = \exp(A)\exp(B)$?

Aufgabe 3.1

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- $$\exp(At) = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = id + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \dots$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow A^m = 0$ (nilpotent) $\underline{\underline{m \geq 2}}$

$$\exp(At) = id + At + \cancel{\frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \dots} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot t$$

0, weil

$$\exp(At) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} //$$

Aufgabe 3.1

$$\exp(Bt) = \sum_{n=0}^{\infty} \frac{(B\cdot t)^n}{n!} = id + Bt + \frac{B^2t^2}{2} + \dots$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = B$$

$$B^3 = B^2 \cdot B = B \cdot B = B^2 = B \quad // \quad \Rightarrow \quad \boxed{B^m = B, \; m \geq 1}$$

$$\exp(Bt) = id + Bt + B\cdot\frac{t^2}{2} + \dots = id + B\sum_{n=1}^{\infty}\frac{t^n}{n!} = id - B + B\overbrace{\sum_{n=0}^{\infty}\frac{t^n}{n!}}^{e^t}$$

$$\exp(Bt) = id - B + B\cdot e^t = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{id} - \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{B} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^t = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}$$

Aufgabe 3.1

$$\exp((A+B)t) \rightarrow \quad A+B = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=A} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{=B} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\exp((A+B)t) = \sum_{n=0}^{\infty} \frac{(A+B)^n \cdot t^n}{n!} = id + (A+B)t + \frac{(A+B)^2 t^2}{2!} + \dots$$

$$(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = (A+B)$$

$$m \geq 1$$

$$\hookrightarrow (A+B)^m = (A+B)$$

$$\exp((A+B)t) = id + (A+B) \sum_{n=1}^{\infty} \frac{t^n}{n!} = id - (A+B) + (A+B) \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t \quad \checkmark$$

$$\exp((A+B)t) = id - (A+B) + (A+B)e^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} e^t & e^t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$$

Aufgabe 3.1

$$\underline{\exp(At)\exp(Bt) \overset{?}{=} \exp((A+B)t)}$$

$$\exp(At)\exp(Bt) = \underbrace{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}_{\exp(At)} \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}}_{\exp(Bt)} = \begin{bmatrix} e^t & t \\ 0 & 1 \end{bmatrix} \neq \underbrace{\exp((A+B)t)}_{\begin{bmatrix} e^t & e^t-1 \\ 0 & 1 \end{bmatrix}}$$

$\rightarrow$ Wenn $AB = BA$, dann gilt: $\exp(A+B) = \exp(A)\exp(B)$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ für unsere Aufgabe das gilt nicht weil: $AB \neq BA$.