7. Übung 28.02.22, Victor Chuidde Chaim

7.1 Aufgabe : $H(s) = \dfrac{1}{ms^2 + \gamma s + k}$, $m, k > 0$

i) $\gamma > 2\sqrt{km}$

- Regelungsnormalform - SISO : $H(s) = \dfrac{1/m}{s^2 + \frac{\gamma}{m}s + k/m}$

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [b_0 - b_2 a_0, \; b_1 - b_2 a_1] = [1/m \;\; 0], \quad D = b_2 = 0$$

- Gewichtsfunktion: $g(t) = \sigma(t) \cdot C \cdot \exp(At) \cdot B + \delta(t) D$

$\exp(At) \Rightarrow$ Eigenwerte (A): $\det(A - \lambda\, id) = 0$

$EW(A)$: $\det\begin{pmatrix} -\lambda & 1 \\ -k/m & -\gamma/m - \lambda \end{pmatrix} = -\lambda(-\gamma/m - \lambda) + k/m = 0$

$\lambda^2 + \lambda\, \gamma/m + k/m = 0 \rightarrow \lambda_{1,2} = \dfrac{-\gamma/m \pm \sqrt{\gamma^2/m^2 - 4k/m}}{2}$

$\lambda_{1,2} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$ //

Fall $\gamma > 2\sqrt{km}$ $\therefore$ $\gamma^2 - 4km > 4km - 4km = 0$

$\gamma^2 - 4km > 0 \rightarrow$ zwei verschiedene reelle Wurzeln

und $Re(\lambda_{1,2}) < 0$, da $\gamma > 0$ ($m, k > 0$ und $\gamma > 2\sqrt{km}$

- Zerlegung in der Diagonalform :

$A = T \Lambda T^{-1}$ $\quad \Lambda = diag.(\lambda_1, \lambda_2)$

Eigenvektoren :

$v_1 \rightarrow A v_1 = \lambda_1 v_1 \quad (A - \lambda_1\, id) v_1 = 0$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -k/m & -\lambda_1 - \gamma/m \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \quad \begin{cases} \lambda_1 v_{11} = v_{12} \\ -\frac{k}{m} v_{11} = v_{12}(\lambda_1 + \gamma/m) \end{cases}$$

- $\lambda_1 v_{11} = v_{12}$ und $-\frac{k}{m} v_{11} = v_{12}(\lambda_1 + \frac{\gamma}{m})$ sind äquivalent

↳ Überprüfung: $-\frac{k}{m} v_{11} = v_{12}\left(\frac{-\gamma + (\gamma^2-4km)^{1/2}}{2m} + \frac{\gamma}{m}\right)$

$$\frac{-k \cdot m \cdot 2}{(2\gamma - \gamma + (\gamma^2 - 4km)^{1/2})} \cdot \frac{v_{11}}{m} = v_{12} \rightarrow \frac{-2km((\gamma^2-4km)^{1/2} - \gamma)}{m \cdot (\gamma + (\gamma^2-4km)^{1/2})((\gamma^2-4km)^{1/2} - \gamma)} v_{11} = v_{12}$$

$$\frac{-2\cancel{km}((\gamma^2-4km)^{1/2} - \gamma)}{m \cdot (\cancel{\gamma^2} - 4\cancel{km} - \cancel{\gamma^2})} \cdot v_{11} = v_{12}$$

$$\frac{-\gamma + (\gamma^2 - 4km)^{1/2}}{2m} \cdot v_{11} = v_{12} \Rightarrow \lambda_1 \cdot v_{11} = v_{12}.$$

Für $v_{11} = 1 \rightarrow v_{12} = \lambda_1 \rightarrow v_1 = [1 \quad \lambda_1]^T$ //

$v_2 \rightarrow A v_2 = \lambda_2 v_2 \rightarrow (A - \lambda_2 id) v_2 = 0$

$$\begin{bmatrix} -\lambda_2 & 1 \\ -k/m & -\lambda_2 - \gamma/m \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \quad \begin{cases} v_{21} \cdot \lambda_2 = v_{22} \\ -\frac{k}{m} v_{21} = v_{22}(\lambda_2 + \gamma/m) \end{cases}$$

wie zuvor: $\lambda_2 v_{21} = v_{22}$ und $-\frac{k}{m} v_{21} = v_{22}(\lambda_2 + \frac{\gamma}{m})$ sind äquivalent.

Für $v_{21} = 1 \rightarrow v_{22} = \lambda_2 \rightarrow v_2 = [1 \quad \lambda_2]^T$ //

$\therefore \quad T = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \quad T^{-1} = \frac{1}{(\lambda_2 - \lambda_1)} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$

$$\exp(At) = \sum_{n=0}^{\infty} \frac{A^n \cdot t^n}{n!} = \sum_{n=0}^{\infty} \frac{(T \Lambda T^{-1})^n \cdot t^n}{n!} = T \sum_{n=0}^{\infty} \frac{\Lambda^n \cdot t^n}{n!} \cdot T^{-1}$$

$$\exp(At) = T \begin{bmatrix} \sum_{n=0}^{\infty} \frac{\lambda_1^n t^n}{n!} & 0 \\ 0 & \sum^{\infty} \frac{\lambda_2^n t^n}{n!} \end{bmatrix} T^{-1} = T \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} T^{-1}$$

$$\exp(At) = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \cdot \frac{1}{(\lambda_2 - \lambda_1)}$$

$$\exp(At) = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \frac{1}{(\lambda_2 - \lambda_1)}$$

$$\exp(At) = \begin{bmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ \lambda_1 \lambda_2 (e^{\lambda_1 t} - e^{\lambda_2 t}) & \lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t} \end{bmatrix} \frac{1}{(\lambda_2 - \lambda_1)} \quad //$$

$$g(t) = [1/m \quad 0] \exp(At) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \sigma(t) = \frac{\sigma(t)}{m(\lambda_2-\lambda_1)} (e^{\lambda_2 t} - e^{\lambda_1 t})$$

Sprungantwort: $y(t) = \psi(t, 0, \sigma(t)) = (g * \sigma)(t)$

$$y(t) = (g * \sigma)(t) = \int_{-\infty}^{\infty} g(\tau)\, \sigma(t-\tau)\, d\tau =$$

$$y(t) = \frac{1}{m(\lambda_2-\lambda_1)} \int_{-\infty}^{\infty} \underbrace{\sigma(\tau)}_{0 \text{ für } \tau<0} (e^{\lambda_2 \tau} - e^{\lambda_1 \tau}) \underbrace{\sigma(t-\tau)}_{0 \text{ für } \tau>t} d\tau$$

$$y(t) = \frac{\sigma(t)}{m(\lambda_2-\lambda_1)} \int_0^t (e^{\lambda_2 \tau} - e^{\lambda_1 \tau})\, d\tau$$

$$y(t) = \frac{\sigma(t)}{m(\lambda_2-\lambda_1)} \left(\frac{e^{\lambda_2 \tau}}{\lambda_2} \Big|_0^t - \frac{e^{\lambda_1 \tau}}{\lambda_1} \Big|_0^t \right)$$

$$y(t) = \frac{\sigma(t)}{m(\lambda_2-\lambda_1)} \left(\frac{e^{\lambda_2 t}}{\lambda_2} - \frac{1}{\lambda_2} - \frac{e^{\lambda_1 t}}{\lambda_1} + \frac{1}{\lambda_1} \right)$$

$$y(t) = \frac{\sigma(t)}{m(\lambda_2-\lambda_1)} \left(\frac{\lambda_1 e^{\lambda_2 t} - \lambda_1 + \lambda_2 - \lambda_2 e^{\lambda_1 t}}{\lambda_2 \lambda_1} \right)$$

$$y(t) = \frac{\sigma(t)}{m \lambda_2 \lambda_1} \left(\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t} + (\lambda_2 - \lambda_1)}{(\lambda_2 - \lambda_1)} \right)$$

$$\hookrightarrow \lambda_2 \lambda_1 = \frac{\cancel{\rho^2} - \cancel{\rho^2} + 4km}{4m^2} = \frac{k}{m}$$

$$y(t) = \frac{\sigma(t)}{\cancel{m} \cdot k/\cancel{m}} \left(\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{(\lambda_2 - \lambda_1)} + 1 \right)$$

$$y(t) = \frac{\sigma(t)}{k} \left(\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)} + 1 \right) //$$

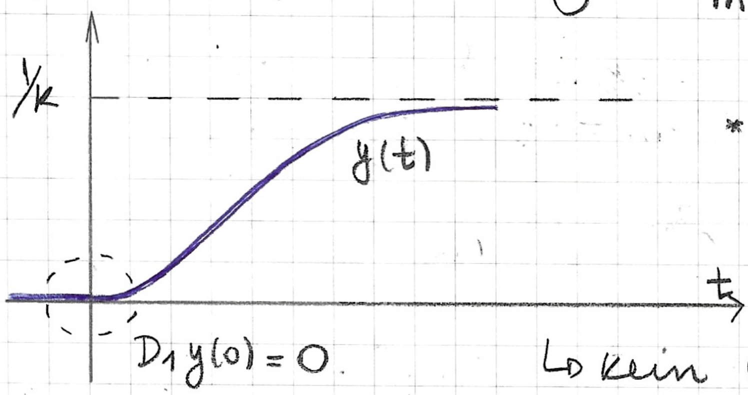
Anfangswert: $y(0) = \frac{\sigma(0)}{k} \left(\frac{\lambda_2 - \lambda_1}{\lambda_1 - \lambda_2} + 1 \right) = \frac{1}{k} (-1+1) = 0$

Endwert: $y(\infty) = \lim_{t\to\infty} \left(\frac{\sigma(t)}{k} \left(\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{(\lambda_1-\lambda_2)} + 1\right)\right)$

$$y(\infty) = \frac{1}{k} \lim_{t\to\infty} \left(\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{(\lambda_1-\lambda_2)} + 1\right) = \frac{1}{k},$$

weil $\lambda_1 < 0$ und $\lambda_2 < 0$, dann $\lim_{t\to\infty} e^{\lambda_{1,2} t} = 0$ //

$D_1\, y(0) = D_1\, \psi(0,0,\sigma) = g(0) = \frac{1}{m(\lambda_2-\lambda_1)} (e^{\lambda_2 0} - e^{\lambda_1 0}) = 0$ //



* Aperiodischer Fall
zwei reele Eigenwerte
und $\lambda_1 \neq \lambda_2$.
↳ kein Überschwingen.

ii) $0 < \gamma < 2\sqrt{k \cdot m}$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

Fall $0 < \gamma < 2\sqrt{km}$ $\therefore$ $\gamma^2 - 4km < 4km - 4km = 0$

$\gamma^2 - 4km < 0$ → konjugiertes komplexes Wurzelpaar mit positivem Realteil, da $\gamma > 0$.

Die Berechnung für den ersten Fall ($\gamma > 2\sqrt{km}$) ist gültig. Daher:

Impulsantwort: $g(t) = \frac{\sigma(t)}{m(\lambda_2-\lambda_1)} (e^{\lambda_2 t} - e^{\lambda_1 t})$

$\lambda_1 = -\frac{\gamma}{2m} + i \cdot a$, $\lambda_2 = -\frac{\gamma}{2m} - ia$, wobei $a = \frac{\sqrt{4km - \gamma^2}}{2m}$

$$\lambda_2 - \lambda_1 = -\cancel{\frac{\gamma}{2m}} - ia - \left(-\cancel{\frac{\gamma}{2m}} + ia\right) = -2 \cdot i \cdot a$$

$$e^{\lambda_1 t} = e^{(-\frac{\gamma}{2m} + ia)\cdot t} = e^{-\frac{\gamma}{2m} t} \cdot e^{iat}$$

$$e^{\lambda_2 t} = e^{(-\frac{\gamma}{2m} - ia) t} = e^{-\frac{\gamma}{2m} t} \cdot e^{-iat}$$

$$e^{\lambda_2 t} - e^{\lambda_1 t} = e^{-\frac{\gamma}{2m} t} (e^{-iat} - e^{iat})$$

$$g(t) = \frac{v(t)}{m(-2ia)} \cdot e^{-\frac{t\gamma}{2m}} \left(e^{-iat} - e^{iat} \right)$$

$$e^{iat} = \cos(at) + i\sin(at)$$
$$e^{-iat} = \cos(at) - i\sin(at)$$
$$\Big\rangle\ e^{-iat} - e^{iat} = -2i\sin(at)$$

$$g(t) = -\frac{v(t)}{m\cancel{2}\cancel{i}\cdot a} e^{-\frac{t\gamma}{2m}} \left(-\cancel{2}\cancel{i}\sin(at)\right) = \frac{v(t)}{m\cdot a} \cdot e^{-\frac{t\gamma}{2m}} \sin(at)$$

$$g(t) = \frac{v(t)}{\cancel{m}} \cdot \frac{2\cancel{m}}{(4km-\gamma^2)^{1/2}} \cdot e^{-t\gamma/2m} \cdot \sin\left(\frac{(4km-\gamma^2)^{1/2}}{2m} \cdot t\right)$$

$$g(t) = \frac{v(t)\cdot 2}{(4km-\gamma^2)^{1/2}} e^{-t\gamma/2m} \cdot \sin\left(\left(\frac{k}{m} - \frac{\gamma^2}{4m^2}\right)^{1/2} t\right) \quad //$$

Sprungantwort: $y(t) = \frac{v(t)}{k}\left(\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + 1\right)$

$$y(t) = \frac{v(t)}{k}\left(e^{-\frac{t\gamma}{2m}}\left(\frac{\cos(at)(\lambda_2-\lambda_1) + i\sin(at)(\lambda_2+\lambda_1)}{(\lambda_1-\lambda_2)}\right) + 1\right)$$

$$y(t) = \frac{v(t)}{k}\left(e^{-\frac{t\gamma}{2m}}\left(-\cos(at) - \frac{\gamma}{m} \cdot \frac{\cancel{i}\sin(at)}{2\cancel{i}a}\right) + 1\right)$$

$$y(t) = \frac{v(t)}{k}\left(1 - e^{-\frac{t\gamma}{2m}}\left(\frac{\gamma}{\cancel{m}(4km-\gamma^2)^{1/2}/\cancel{m}} \cdot \sin(at) + \cos(at)\right)\right)$$

$$y(t) = \frac{v(t)}{k}\left(1 - e^{-\frac{t\gamma}{2m}}\left(\frac{\gamma}{(4km-\gamma^2)^{1/2}} \cdot \sin(at) + \cos(at)\right)\right)$$

Mit dem wissen: $K_1 \cdot \sin(\alpha) + K_2 \cos(\alpha) = (K_1^2 + K_2^2)^{1/2} \cdot \sin(\alpha + \arctan K_2/K_1)$

$$K_1 = \frac{\gamma}{(4km-\gamma^2)^{1/2}}, \quad K_2 = -1 \quad \therefore$$

$$y(t) = \frac{v(t)}{k}\left(1 - e^{-t\gamma/2m}\left(\frac{\gamma^2}{4km-\gamma^2} + 1\right)^{\frac{1}{2}} \cdot \sin\left(at + \tan^{-1}\left(\frac{(4km-\gamma^2)^{1/2}}{\gamma}\right)\right)\right)$$

$$y(t) = \frac{v(t)}{k}\left(1 - e^{-t\gamma/2m}\left(\frac{4km - \cancel{\gamma^2} + \cancel{\gamma^2}}{4km-\gamma^2}\right)^{1/2} \sin\left(at + \tan^{-1}\left(\frac{\left(1 - \frac{\gamma^2}{4km}\right)^{1/2}}{\frac{\gamma}{2\sqrt{km}}}\right)\right)\right)$$

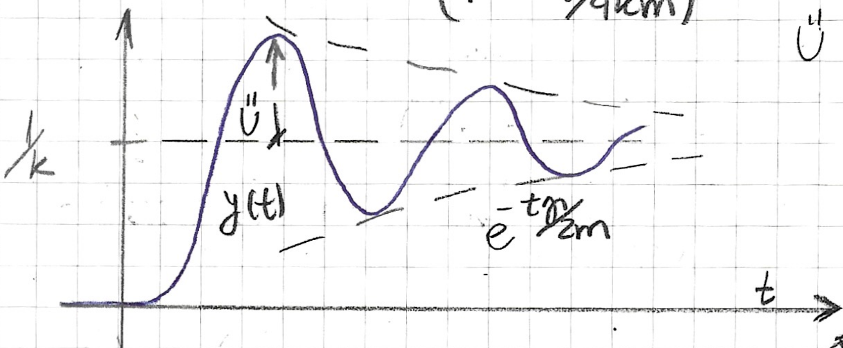
Mit der Kenntnis: $\arccos(\alpha) = \arctan\left(\frac{\sqrt{1-\alpha^2}}{\alpha}\right)$
für $0 < \alpha < 1$

$0 < \frac{\gamma}{2\sqrt{km}} < \frac{2\sqrt{km}}{2\sqrt{km}} < 1$ ✓ Bedingung erfüllt.

Dann können wir $y(t)$ umschreiben als:

$$y(t) = \frac{v(t)}{k}\left(1 - e^{-t\gamma/2m}\left(\frac{1}{\left(1-\frac{\gamma^2}{4km}\right)^{1/2}}\right)\cdot \sin\left(at + \underbrace{\arccos\left(\frac{\gamma}{2\sqrt{km}}\right)}_{\varphi}\right)\right)$$

$$y(t) = \frac{v(t)}{k}\left(1 - \frac{e^{-t\gamma/2m}}{\left(1-\gamma^2/4km\right)^{1/2}}\cdot \sin\left(\frac{\sqrt{4km-\gamma^2}}{2m}\cdot t + \varphi\right)\right) \quad //$$



Ü = Überschwingen

Periodischer Fall

$\ddot{U} = \frac{1}{k}\exp\left(\frac{-\pi\gamma}{(4km-\gamma^2)^{1/2}}\right)$

Endwert: $1/k$ //
$\left(\lim_{t\to\infty} e^{-t\gamma/2m} = 0\right)$

- Nachweis des Überschwingens:

lokales Extremum/Minimum: $\frac{dy(t_0)}{dt} = 0$

$$\frac{dy(t_0)}{dt} = \frac{1}{k}\frac{(\lambda_2\lambda_1 e^{\lambda_1 t_0} - \lambda_1\lambda_2 e^{\lambda_2 t_0})}{\lambda_1-\lambda_2} = 0$$

$$\Rightarrow e^{\lambda_1 t_0} = e^{\lambda_2 t_0} \Rightarrow \cancel{e^{-t_0\gamma/2m}}(\cancel{\cos(at_0)} + i\sin(at_0)) = \cancel{e^{-t_0\gamma/2m}}(\cancel{\cos(at)} - i\sin(at))$$

$\Rightarrow \sin(at_0) = -\sin(at_0)$, für $at_0 = n\cdot\pi$, $n = 0, 1, 2, \ldots$

Für $at_0 = 0$, $t_0 = 0$, $y(0) = 0$, Minimum

Nehmen wir $t_0 = \pi/a$ an:

$$\frac{d^2y(t_0)}{dt^2} = \frac{1}{k}\frac{(\lambda_2\lambda_1^2 e^{\lambda_1 t_0} - \lambda_1\lambda_2^2 e^{\lambda_2 t_0})}{(\lambda_1-\lambda_2)} = \frac{1}{k}\frac{(\lambda_1 e^{\lambda_1 t_0} - \lambda_2 e^{\lambda_2 t_0})\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}$$

$$= \frac{\lambda_1\lambda_2}{k}\frac{e^{-t_0\gamma/2m}(\lambda_1 e^{iat_0} - \lambda_2 e^{-iat_0})}{(\lambda_1-\lambda_2)}$$

$$e^{iat_0} = \cos(at_0) + i\sin(at_0) \overset{at_0=\pi}{=} -1$$

$$e^{-iat_0} = \cos(at_0) - i\sin(at_0) = -1$$

$$\frac{d^2y(t_0)}{dt^2} = \frac{\lambda_1\lambda_2}{k} e^{-\frac{t_0\gamma}{2m}} \frac{(-\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} = -\frac{\lambda_1\lambda_2}{k} e^{-\frac{t_0\gamma}{2m}}$$

$$\lambda_1\lambda_2 = \left(-\frac{\gamma}{2m}+ia\right)\left(-\frac{\gamma}{2m}-ia\right) = \frac{\gamma^2}{4m^2} - i^2a^2$$

$$\lambda_1\lambda_2 = \frac{\gamma^2}{4m^2} + a^2 = \frac{\cancel{\gamma^2}+4km-\cancel{\gamma^2}}{4m^2} = \frac{k}{m} > 0 \quad (k,m>0)$$

$$\therefore \frac{d^2y(t_0)}{dt^2} = -\frac{1}{m}\cdot e^{-\frac{t_0\gamma}{2m}} < 0, \quad t_0 = \frac{\pi}{a} \text{ lokales Extremum}$$

Für gerade Werte von n und $n=0$:

$$y(0) = \frac{1}{k}\left(1 - \frac{e^0}{(1-\gamma^2/4km)^{1/2}} \sin(0+\varphi)\right) = 0$$

$$\sin(0+\varphi) = \sin(\varphi) = \sin(\arccos(\gamma/2\sqrt{km})) = \sqrt{1-\gamma^2/4km}$$

$$y(0) = \frac{1}{k}\left(1 - \frac{\sqrt{1-\gamma^2/4km}}{\sqrt{1-\gamma^2/4km}}\right) = 0 \quad \text{(Anfangswert)}$$

$$y(2\pi/a) = \frac{1}{k}\left(1 - \frac{e^{-\frac{2\pi\gamma}{2ma}}}{\sqrt{1-\gamma^2/4km}} \sin(2\pi+\varphi)\right)$$

$$\sin(2\pi+\varphi) = \sqrt{1-\gamma^2/4km}$$

$$y(2\pi/a) = \frac{1}{k}\left(1 - e^{-\frac{2\pi\gamma}{2ma}}\right)$$

$\vdots$

abnehmend für zunehmendes n

$$y(2n\pi/a) = \frac{1}{k}\left(1 - e^{-\frac{2\pi n\gamma}{2ma}}\right)$$

Globales Minimum

und $y(0) < y(2\pi/a) < y(4\pi/a) < \dots$ $\boxed{y(0)}$

Für ungerade Werte von n:

$$y(\pi/a) = \frac{1}{k}\left(1 + e^{-\frac{\pi\gamma}{2ma}}\right)$$

abnehmend für zunehmendes n

Globales Maximum

$\vdots$

$$y((2n+1)/a) = \frac{1}{k}\left(1 + e^{-\frac{(2n+1)\pi\gamma}{2ma}}\right)$$ und $\boxed{y(\pi/a)} > y(3\pi/a) > y(5\pi/a) > \dots$

Überschwingen : $\ddot{U} = y\left(\frac{\pi}{a}\right) - \frac{1}{k}$

$$\ddot{U} = \frac{1}{k} + \frac{\exp}{k}\left(\frac{-\pi\gamma}{2ma}\right) - \frac{1}{k} = \frac{\exp}{k}\left(\frac{-\pi\gamma}{2ma}\right)$$

$$\ddot{U} = \frac{\exp}{k}\left(\frac{-\pi\gamma}{\cancel{2m}\frac{\sqrt{4km-\gamma^2}}{\cancel{2m}}}\right) = \frac{1}{k}\cdot\exp\left(\frac{-\pi\gamma}{\sqrt{4km-\gamma^2}}\right) \;//$$

iii) $\gamma = 2\sqrt{km}$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{-\gamma \pm \sqrt{\cancel{4km} - \cancel{4km}}}{2m} = \frac{-\gamma}{2m},$$

$\lambda_1 = \lambda_2 = \lambda$, $\exp(At) \to$ Eigenwerte - Verschiebungstrick

- $A = (A - \lambda id) + \lambda id$, $\lambda id \to$ Diagonalmatrix

Eigenwerte von $(A - \lambda id)$: $A = \begin{bmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{bmatrix}$

$$\det\begin{bmatrix} \gamma/2m - p & 1 \\ -k/m & -\gamma/m + \gamma/2m - p \end{bmatrix} = \left(\frac{\gamma}{2m} - p\right)\left(-\frac{\gamma}{2m} - p\right) + \frac{k}{m} = 0$$

$$= p^2 - \frac{\gamma^2}{4m^2} + \frac{k}{m} = p^2 - \frac{(2\sqrt{km})^2}{4m^2} + \frac{k}{m} = p^2 - \frac{k}{m} + \frac{k}{m} = p^2 = 0$$

Eigenwerte von $(A - \lambda id)$, $p_{1,2} = 0$, $\to$ Nilpotent!

$$(A - \lambda id)^2 = (A - \lambda id)(A - \lambda id) = \begin{bmatrix} \gamma/2m & 1 \\ -k/m & -\gamma/2m \end{bmatrix}\begin{bmatrix} \gamma/2m & 1 \\ -k/m & -\gamma/2m \end{bmatrix} =$$

$$= \begin{bmatrix} \gamma^2/4m^2 - k/m & \cancel{\frac{\gamma}{2m}} - \cancel{\frac{\gamma}{2m}} \\ -\cancel{\frac{k\gamma}{2m^2}} + \cancel{\frac{k\gamma}{2m^2}} & -\frac{k}{m} + \gamma^2/4m^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^d = 0,\ d \geq 2$$

$$*\left(-\frac{k}{m} + \gamma^2/4m^2 = -k/m + \frac{(2\sqrt{km})^2}{4m^2} = -k/m + k/m = 0\right)$$

$$\exp(At) = \exp\left(((A - \lambda id) + \lambda id)\cdot t\right) =$$

$$= \exp((A - \lambda id)\cdot t)\exp(\lambda id\, t)$$

$$\exp(\lambda\, id\cdot t) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \end{bmatrix} = e^{\lambda t} \cdot id \;//$$

$$-\exp((A-\lambda\, id)t) = \sum_{n=0}^{\infty} \frac{(A-\lambda\, id)^n \cdot t^n}{n!} = id + (A-\lambda\, id)t \;//$$

$$\exp(At) = e^{\lambda t} id\, ((A-\lambda\, id)t + id) = e^{\lambda t}((A-id)t+id)$$

$$\exp(At) = e^{\lambda t} \begin{bmatrix} \gamma/2m \cdot t + 1 & t \\ -\frac{k}{m} \cdot t & -\frac{\gamma}{2m} t + 1 \end{bmatrix}$$

Impulsantwort: $g(t) = C \exp(At) B\, \sigma(t) + \cancel{D \cdot \delta(t)}^{\,0}$

$$g(t) = \sigma(t) \, [1/m \;\; 0] \exp(At) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{m} \cdot e^{\lambda t} \cdot t \cdot \sigma(t)$$

$$g(t) = \frac{1}{m} e^{-\gamma/2m \cdot t} \cdot t \cdot \sigma(t), \quad -\frac{\gamma}{2m} = -\frac{2\sqrt{km}}{2m} = \sqrt{\frac{k}{m}}$$

$$g(t) = \frac{t}{m} \cdot e^{-\sqrt{\frac{k}{m}} \cdot t} \cdot \sigma(t) \;// \qquad \text{Eigenfrequenz: } \omega_n = \sqrt{k/m} \;//$$

Sprungantwort: $y(t) = \varphi(t, 0, \sigma) = (g * \sigma)(t)$

$$y(t) = \frac{\sigma(t)}{m} \int_0^t \tau \cdot e^{-\omega_n \cdot \tau} d\tau \Rightarrow \int u\, dv = uv - \int v\, du$$

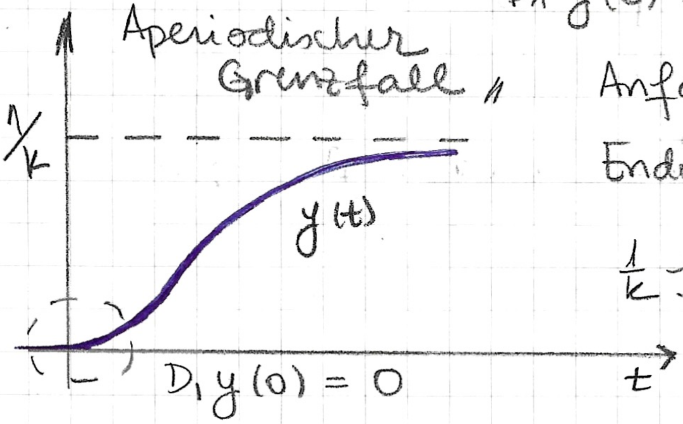
$$\int \tau e^{-\omega_n \tau} d\tau = \left\langle \begin{matrix} u = \tau \\ dv = e^{-\omega_n \tau} d\tau \end{matrix} \;\middle|\; \begin{matrix} du = d\tau \\ v = -\frac{e^{-\omega_n \tau}}{\omega_n} \end{matrix} \right\rangle = -\frac{\tau e^{-\omega_n \tau}}{\omega_n} - \frac{e^{-\omega_n \tau}}{\omega_n^2} + K$$

$$\int_0^t \tau e^{-\omega_n \tau} d\tau = -e^{-\omega_n \tau}\left(\frac{1 + \omega_n \tau}{\omega_n^2}\right)\Bigg|_0^t = -e^{-\omega_n \cdot t}\left(\frac{1+\omega_n \cdot t}{\omega_n^2}\right) + \left(\frac{1}{\omega_n^2}\right)$$

$$y(t) = \frac{\sigma(t)}{m} \cdot \frac{1}{\omega_n^2}\left(1 - e^{-\omega_n t}(1 + \omega_n \cdot t)\right)$$

$$y(t) = \frac{\sigma(t)}{\cancel{m}} \frac{\cancel{m}}{k}\left(1 - e^{-\sqrt{k/m} \cdot t}\left(1 + \sqrt{\frac{k}{m}} \cdot t\right)\right)$$

$$y(t) = \frac{\sigma(t)}{k}\left(1 - e^{-\sqrt{k/m} \cdot t}\left(1 + \sqrt{\frac{k}{m}} \cdot t\right)\right) \;//$$



$D_1\, y(0) = g(0) = \frac{1}{m} \cdot e^{0} \cdot 0 = 0$ //

Anfangswert: $y(0) = 0$

Endwert: $\lim_{t \to \infty} y(t)$

$$\frac{1}{k} \lim_{t\to\infty} 1 - e^{-w_n \cdot t}\left(1 + \sqrt{\frac{k}{m}} \cdot t\right) = \frac{1}{k} -$$

$$- \lim_{t\to\infty} \frac{1 + w_n t}{e^{w_n t}} \Rightarrow \text{L'Hospital}$$

$$\lim_{t\to\infty} y(t) = \frac{1}{k} - \lim_{t\to\infty} \cancel{\frac{w_n}{w_n e^{w_n t}}} \xrightarrow{} 0 = \frac{1}{k} \;//$$

iv) $\gamma = 0$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \overset{\gamma = 0}{=} \pm \frac{i\sqrt{4km}}{2m} = \pm i \sqrt{\frac{k}{m}} = \pm i w_n$$

$w_n = \sqrt{\frac{k}{m}}$, Eigenfrequenz.

$\lambda_{1,2} \to$ konjugiertes komplexes Wurzelpaar mit keinem Realteil, da $\gamma = 0$. $Re(\lambda_{1,2}) = 0$.

*Gleiche Lösung wie für den zweiten Fall, mit $\gamma \to 0$.

$$g(t) = \frac{2 \cdot \sigma(t)}{(4km - \gamma^2)^{1/2}} e^{-t\gamma/2m} \cdot \sin\left(\left(k/m - \gamma^2/4m^2\right)^{1/2} \cdot t\right)$$

Impulsantwort: $g(t) = \frac{\sigma(t)}{\sqrt{km}} \cdot \sin\left(\sqrt{k/m} \cdot t\right) = \frac{\sigma(t)}{\sqrt{km}} \sin(w_n \cdot t)$

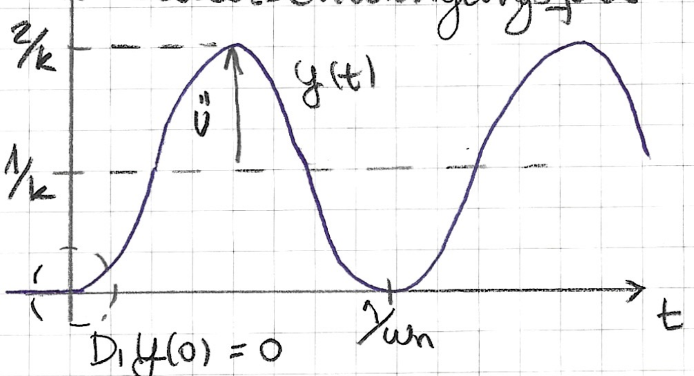
↳ kein exponentielles Abklingen, die Antwort ist rein harmonisch

$$y(t) = \frac{\sigma(t)}{k}\left(1 - \frac{e^{-t\gamma/2m}}{(1 - \gamma^2/4km)^{1/2}} \cdot \sin\left(t \frac{\sqrt{4km - \gamma^2}}{2m} + \arccos\left(\frac{\gamma}{2\sqrt{km}}\right)\right)\right)$$

Sprungantwort: $y(t) = \frac{\sigma(t)}{k}\left(1 - \sin(w_n \cdot t + \pi/2)\right)$

$y(t) = \frac{\sigma(t)}{k}\left(1 - \cos(w_n t)\right) \to$ kein exponentielles Abklingen, die Antwort ist rein harmonisch

Dauerschwingungsfall



· Anfangswert: $y(0) = 0$

· kein Endwert.

· $\ddot{u} = \frac{1}{k} \exp\left(\frac{-\pi\gamma}{(4km-\gamma^2)^{1/2}}\right) = \frac{1}{k}$

v) $\gamma < 0$

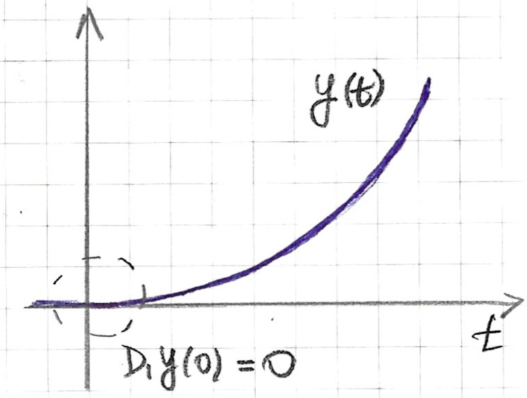
$$\lambda_1, \lambda_2 = \frac{\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}, \quad Re(\lambda_1, \lambda_2) > 0$$

Das System ist nicht asymptotisch stabil!

I) $\gamma < 0$ und $\lambda_1, \lambda_2 \in \mathbb{R}$:

$$g(t) = \frac{\sigma(t)}{m(\lambda_2 - \lambda_1)} \left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)$$

$$y(t) = \frac{\sigma(t)}{k} \left(\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + 1\right)$$



Anfangswert: $y(0) = 0$

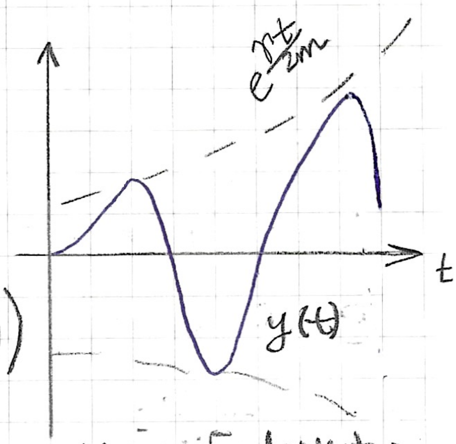
Endwert: $\lim_{t \to \infty} \frac{\sigma(t)}{k} \left(\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + 1\right) \to \infty$, da $\lambda_{1,2} > 0$

· kein Endwert

II) $\gamma < 0$ und $\lambda_1, \lambda_2 \in \mathbb{C}$:

$$g(t) = \frac{2\sigma(t)}{(4km - \gamma^2)^{1/2}} \cdot e^{\frac{\gamma t}{2m}} \sin\left(\left(\frac{k}{m} - \frac{\gamma^2}{4km}\right)^{1/2} \cdot t\right)$$

$$y(t) = \frac{\sigma(t)}{k} \left(1 - \frac{e^{\frac{\gamma t}{2m}}}{\left(1 - \frac{\gamma^2}{4km}\right)^{1/2}} \cdot \sin\left(\frac{\sqrt{4km - \gamma^2}}{2m} t + \varphi\right)\right)$$



Anfangswert: $y(0) = 0$

kein Endwert

Endwert: $\lim_{t \to \infty} \frac{\sigma(t)}{k} \left(1 - \frac{e^{\frac{\gamma t}{2m}}}{\left(1 - \frac{\gamma^2}{4km}\right)^{1/2}} \cdot \sin\left(\frac{\sqrt{4km - \gamma^2}}{2m} t + \varphi\right)\right) \to \pm\infty$, da $Re(\lambda_{1,2}) > 0$

* 7.2 Aufgabe $\subseteq$ 8.1 Aufgabe