6. Übung 21.02.22, Victor Cheidde Chaim

6.1 Aufgabe $A = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} \beta \\ 2 \end{pmatrix}$, $C = (\gamma \ 1)$, $D = \alpha$

* Ist das ZS asympt. stabil?

Eigenwerte von A: $(1-\lambda)(1-\lambda) = 0 \lessdot \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -1 \end{matrix}$

$Re(\lambda_1) > 0$ $\rightarrow$ nicht stabil!

* Satz 7.9 (5): BIBO-Stabil $\Leftrightarrow$ $Re(p) < 0$

$H(s) = C(s\,id - A)^{-1} \cdot B + D$ $\rightarrow$ D spielt keine Rolle bei BIBO-Stabilität. wir setzen $D = 0$

$H^*(s) = C(s\,id - A)^{-1} \cdot B$

$$(s\,id - A) = \begin{bmatrix} s-1 & -\alpha \\ 0 & s+1 \end{bmatrix}$$

$$(s\,id - A)^{-1} = \frac{1}{\det(s\,id - A)} \begin{bmatrix} s+1 & \alpha \\ 0 & s-1 \end{bmatrix} = \frac{1}{(s+1)(s-1)} \begin{bmatrix} s+1 & \alpha \\ 0 & s-1 \end{bmatrix}$$

$$H^*(s) = C(s\,id - A)^{-1} \cdot B = \frac{1}{(s+1)(s-1)} \cdot [\gamma \ 1] \begin{bmatrix} s+1 & \alpha \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} \beta \\ 2 \end{bmatrix}$$

$$H^*(s) = [\gamma \ 1] \begin{bmatrix} \beta(s+1) + 2\alpha \\ 2(s-1) \end{bmatrix} \cdot \frac{1}{(s+1)(s-1)}$$

$$H^*(s) = \frac{\gamma\beta(s+1) + 2\alpha\gamma + 2(s-1)}{(s-1)(s+1)}$$

Wir müssen den instabilen Pol 1 eliminieren:

Zähler $= 0$ oder Zähler $= k(s-1)$

1) Zähler $= k(s-1)$: $\rightarrow \boxed{\gamma = 0}$ $\rightarrow H^*(s) = \frac{2}{(s+1)}$

$\downarrow$

BIBO-stabil.

oder $s(2+\gamma\beta)+(\gamma\beta+2\gamma\alpha-2)=k(s-1)$

$$\begin{cases} 2+\gamma\beta = k \\ \gamma\beta+2\gamma\alpha-2=-k \end{cases} \quad \begin{matrix}\gamma\beta = -2+k \\ \gamma\alpha = -k+2\end{matrix} \Rightarrow \boxed{\alpha=-\beta}$$

2) Zähler $=0$: $\begin{cases} 2+\gamma\beta=0 \\ \gamma\beta+2\gamma\alpha-2=0 \end{cases} \quad \begin{matrix}\gamma\beta=-2 \\ \gamma\alpha=2\end{matrix} \Rightarrow \boxed{\alpha=-\beta}$

BIBO stabil für $\alpha=-\beta$ oder $\gamma=0$ oder ($\gamma\beta=-2+k$ und $\gamma\alpha=-k+2$, $k\in\mathbb{R}$).

$\Rightarrow$ Überprüfung:

- $\alpha=-\beta$: $H^*(s)=\dfrac{\gamma\beta(s+1)-2\gamma\beta+2(s-1)}{(s-1)(s+1)}=\dfrac{(2+\gamma\beta)(s-1)}{(s-1)(s+1)}$

$$H^*(s)=\frac{(2+\gamma\beta)}{(s+1)} \Rightarrow \text{BIBO-stabil } p=-1<0.$$

- $\gamma=0$: $H^*(s)=\dfrac{2(s-1)}{(s-1)(s+1)}=\dfrac{2}{(s+1)}$ BIBO-stabil

- $\gamma\beta=-2+k$ und $\gamma\alpha=-k+2$:

$$H^*(s)=\frac{(-2+k)(s+1)+2(2-k)+2(s-1)}{(s+1)(s-1)}=$$

$$=\frac{-2s+ks\cancel{-2}+k+\cancel{4}-2k+2s\cancel{-2}}{(s+1)(s-1)}=$$

$$=\frac{s(\cancel{-2}+k\cancel{+2})-k}{(s+1)(s-1)}=\frac{k(\cancel{s-1})}{(\cancel{s-1})(s+1)}=\frac{k}{s+1} \quad \text{BIBO stabil}$$

6.2 Aufgabe $A=-2$, $B=1$, $C=1$, $D=1$

i) $\phi(t)=\exp(At)=\sum_{n=0}^{\infty}\dfrac{(-2t)^n}{n!}=e^{-2t}$ //

ii) $H(s)=C(s\,id-A)^{-1}B+D=1\cdot(s-(-2))^{-1}\cdot 1+1=$

$$=\frac{1}{s+2}+1=\frac{1+s+2}{s+2}=\frac{s+3}{s+2} \quad //$$

iii) $g(t) = \sigma(t) . C . \exp(At) B + \delta(t) . D$

$$g(t) = \sigma(t) . 1 . e^{-2t} . 1 + \delta(t) . 1 = \sigma(t) e^{-2t} + \delta(t)$$

iv) Sprungantwort.

$$\psi(t,0,\sigma) = (g * \sigma)(t) = \sigma(t) \int_0^t e^{-2\tau} d\tau + \sigma(t) \overset{1}{\int_0^t \delta(\tau) d\tau}$$

$$\psi(t,0,\sigma) = \sigma(t) \left. \frac{e^{-2\tau}}{-2} \right|_0^t + \sigma(t) =$$

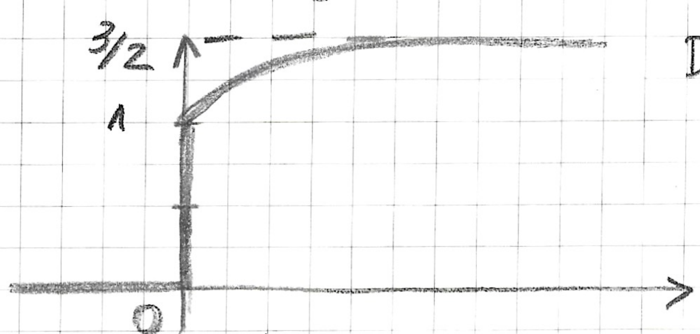
$$= \sigma(t) \left(1 + \frac{1}{2} - \frac{e^{-2t}}{2} \right) = \sigma(t) . \left(\frac{3}{2} - \frac{e^{-2t}}{2} \right)$$

Anfangswert: $\psi(0,0,\sigma) = 1 \left(\frac{3}{2} - \frac{1}{2} \right) = 1$ //

$\left(\text{oder } s \to \infty \quad H(\infty) = \lim_{s \to \infty} \frac{s+3}{s+2} = 1 \right)$

Endwert: $\psi(\infty,0,\sigma) = \lim_{t \to \infty} \sigma(t) \left(\frac{3}{2} - \frac{e^{-2t}}{2} \right) = \frac{3}{2}$

$\left(\text{oder } s \to 0 \quad H(0) = 3/2 \right)$



$D_1 \psi(0,0,\sigma) = \frac{-2}{-2} \overset{1}{\cancel{e^{-2 \cdot 0}}} = 1$ //

$D_1 \psi(0,0,\sigma) > 0.$

6.3 Aufgabe $\quad H(s) = \frac{1}{ms^2 + \gamma s + k} \quad m, \gamma, k > 0$

i) Regelungsnormalform $H(s)$:

$$h(s) = \frac{b_n s^n + \dots + b_1 . s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad C = (b_0 - b_n a_0, b_1 - b_n a_1, \dots), \quad D = b_n$$

$$H(s) = \frac{1}{ms^2+\gamma s+k} = \frac{1/m}{s^2+\frac{\gamma}{m}s+k/m} \qquad \begin{aligned} b_0 &= 1/m \\ a_0 &= k/m \\ a_1 &= \gamma/m \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [\,1/m \quad 0\,] \quad D = 0$$

ii) $G(s) = \dfrac{s^2}{ms^2+\gamma s+k}$, Regelungsnormalform

$$G(s) = \frac{s^2}{ms^2+\gamma s+k} = \frac{s^2/m}{s^2+\frac{\gamma}{m}s+k/m} \qquad \begin{aligned} b_2 &= 1/m \\ a_0 &= k/m \\ a_1 &= \gamma/m \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C' = [-k/m^2 \quad -\gamma/m^2] \quad D = 1/m$$

6.4 Aufgabe $\quad A = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $C = (1 \quad 0)$, $D = 0$

i) $H(s) = C(s\,id - A)^{-1}B + \cancel{D}^{\,0}$

$$(s\,id - A) = \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix}, \quad (s\,id-A)^{-1} = \frac{1}{(s+3)(s+1)}\begin{bmatrix} s+1 & 0 \\ 0 & s+3 \end{bmatrix}$$

$$H(s) = \frac{1}{(s+3)(s+1)}[1 \quad 0]\begin{bmatrix} s+1 & 0 \\ 0 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{(s+3)} \quad //$$

ii) $u(t) = \cos(3t) \qquad u(t) = Re(e^{3it})$, $\tilde{u} = e^{3it}$

$y(t) = Re(\tilde{y}(t))$, $\quad \tilde{y}(t) = H(\lambda)\,\tilde{u}(t)$

- Eigenwerte von A : $-1, -3 \neq 3i$ $\quad$ 1. Bedingung erfüllt ✓

$\tilde{y}(t) = H(3i)\,\tilde{u}(t) \Rightarrow H(3i) = \dfrac{1}{3i+3} = \dfrac{3i-3}{9i^2-9} = \dfrac{-1}{6}(i-1)$

$H(3i) = -1/6\,(e^{\frac{\pi}{2}i} - 1) \quad (e^{ki} = \cos(k) + i\sin(k))$.

$y(t) = Re(\tilde{y}(t)) = Re\left(\frac{-1}{6}(e^{\frac{\pi}{2}i}-1)\cdot e^{3it}\right) = Re\left(-\frac{1}{6}(e^{(\pi/2+3t)i} - e^{3it})\right)$

$$y(t) = \frac{1}{6}\left(\cos(3t) - \cos(3t+\pi/2)\right) \quad //$$