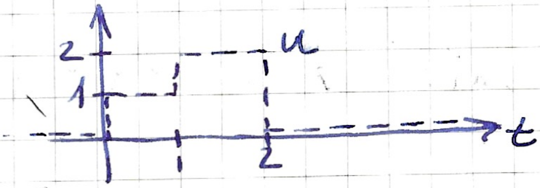
4. Übung, 07.02.22, Victor C. Chaim

4.1 Aufgabe: $\begin{cases} \ddot{x} = -9\dot{x} - 20x + u \\ y = x \\ x_0 = [0 \quad 0]^T \end{cases}$



$\cdot\ y = ?$

$\rightarrow \begin{cases} \ddot{x} = -9\dot{x} - 20x + u \\ y = x \end{cases} \xrightarrow{ZS} \begin{cases} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \quad 0] \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + [0]\, u \end{cases}$

Impulsantwort: $g(t) = \underbrace{\sigma(t)}_{\text{Sprungfunktion}} C e^{At} \cdot B + \cancel{\delta(t) D}$ ↳ Deltafunktion, $D = 0$

$g(t) = \sigma(t) C e^{At} B, \quad \sigma(t) = \begin{cases} 1, & \text{falls } t \geq 0 \\ 0, & \text{sonst.} \end{cases}$

$g(t) = \sigma(t) [1 \quad 0] \exp(At) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \rightarrow$ Eigenwerte: $\det(A - \lambda) = \begin{bmatrix} -\lambda & 1 \\ -20 & -9-\lambda \end{bmatrix}$

$\det(A - \lambda) = \lambda(9 + \lambda) + 20 = \lambda^2 + 9\lambda + 20 \rightarrow \begin{array}{l} \lambda_1 = -4 \\ \lambda_2 = -5 \end{array}$

Eigenwerte $\neq 0$, A ist nicht nilpotent.

Diagonaltransformation: $A = T \Lambda T^{-1}, \quad T = [v_1 \quad v_2]$

$v_1 \rightarrow A v_1 = \lambda_1 v_1 \rightarrow (A - \lambda_1) v_1 = 0$

$(A - \lambda_1) v_1 = \begin{bmatrix} 4 & 1 \\ -20 & -5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \rightarrow \begin{cases} v_{11} = -v_{12}/4 \\ v_1 = [1 \quad -4]^T \end{cases}$

$v_2 : (A - \lambda_2) v_2 = \begin{bmatrix} 5 & 1 \\ -20 & -4 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \rightarrow \begin{cases} v_{21} = -v_{22}/5 \\ v_2 = [1 \quad -5]^T \end{cases}$

$T = [v_1 \quad v_2] = \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix}$ //

$T^{-1} = \frac{1}{\det(T)} \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} = (-1) \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix}$ //

$$\therefore \quad A = T \Lambda T^{-1} = \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} \overset{=-4}{\lambda_1} & 0 \\ 0 & \underset{=-5}{\lambda_2} \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix}$$ //

und $A^n = (T \Lambda T^{-1})^n = T \Lambda^n T^{-1}$ //

$$\exp(At) = \exp(T \Lambda T^{-1} t) = \sum_{n=0}^{\infty} \frac{(T\Lambda T^{-1})^n t^n}{n!} =$$

$$= \sum_{n=0}^{\infty} T \Lambda^n T^{-1} \frac{t^n}{n!} = T \sum_{n=0}^{\infty} \frac{\Lambda^n t^n}{n!} T^{-1} =$$

$$= T \begin{bmatrix} \sum_{n=0}^{\infty} \lambda_1^n t^n / n! & 0 \\ 0 & \sum_{n=0}^{\infty} \lambda_2^n \frac{t^n}{n!} \end{bmatrix} T^{-1} =$$

$$= T \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} T^{-1} = T \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-5t} \end{bmatrix} T^{-1}$$ //

Impulsantwort: $g(t) = \sigma(t) [1 \; 0] \exp(At) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$g(t) = \sigma(t) [1 \; 0]\, T \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-5t} \end{bmatrix} T^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \sigma(t) [1 \; 0] \begin{bmatrix} 1 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \sigma(t) [1 \; 1] \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \sigma(t) [e^{-4t} \; e^{-5t}] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sigma(t) (e^{-4t} - e^{-5t})$$ //

3 Fällen: $0 < t < 1, \quad 1 < t < 2, \quad 2 < t$

(aus der Abbildung): $u(t) = \underset{u_1}{\underset{\shortparallel}{\sigma(t)}} + \underset{u_2}{\underset{\shortparallel}{\sigma(t-1)}} - 2 \cdot \underset{u_3}{\underset{\shortparallel}{\sigma(t-2)}}$

Linearität: $y(t) = \psi(t, x_0, u) = \psi(t, x_0, u_1) + \psi(t, x_0, u_2) - 2\psi(t, x_0, u_3)$

Kausalität, Zeitinvariant: $\psi(t, x_0, u_1) = (g * \sigma)(t)$
$\psi(t, x_0, u_2) = (g * \sigma)(t-1)$, $x_0 = [0 \; 0]^T$
$\psi(t, x_0, u_3) = (g * \sigma)(t-2)$

$$\Rightarrow y = (g*\sigma)(t) + (g*\sigma)(t-1) - 2(g*\sigma)(t-2)$$

$$(g*\sigma)(t) = \int_{-\infty}^{\infty} g(\tau)\sigma(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \underbrace{\sigma(\tau)}_{\overset{\shortparallel}{0} \text{ für } \tau<0}(e^{-4\tau}-e^{-5\tau})\underbrace{\sigma(t-\tau)}_{\overset{\shortparallel}{0} \text{ für } \tau>t}\,d\tau =$$

$$= \int_0^t \underbrace{\sigma(\tau)}_{\overset{\shortparallel}{1}}(e^{-4\tau}-e^{-5\tau})\underbrace{\sigma(t-\tau)}_{\overset{\shortparallel}{1}}\,d\tau = \sigma(t)\int_0^t (e^{-4\tau}-e^{-5\tau})\,d\tau =$$

$$= \sigma(t)\left(\left(-\frac{e^{-4\tau}}{4}\right)\Big|_0^t - \left(-\frac{e^{-5\tau}}{5}\right)\Big|_0^t\right) =$$

$$= \sigma(t)\left(-\frac{e^{-4t}}{4} + \frac{1}{4} + \frac{e^{-5t}}{5} - \frac{1}{5}\right) =$$

$$= \sigma(t)\left(\frac{e^{-5t}}{5} - \frac{e^{-4t}}{4} + \frac{1}{20}\right) \;//$$

$$\therefore\; y(t) = \sigma(t)\left(\frac{e^{-5t}}{5} - \frac{e^{-4t}}{4} + \frac{1}{20}\right) + \sigma(t-1)\left(\frac{e^{-5(t-1)}}{5} - \frac{e^{-4(t-1)}}{4} + \frac{1}{20}\right) -$$

$$- 2\sigma(t-2)\left(\frac{e^{-5(t-2)}}{5} - \frac{e^{-4(t-2)}}{4} + \frac{1}{20}\right) \;//$$