2. Übung, 24.01.22, Victor C. Chaim

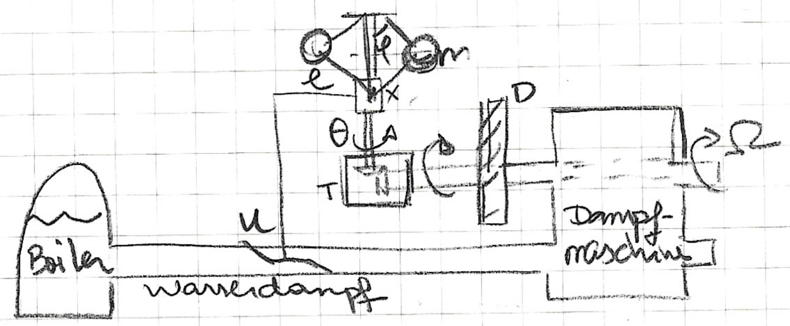
Aufgabe 2.1 : $M_r = -b\dot{\varphi}(t)$ $\quad \theta(t) = n\Omega(t)$

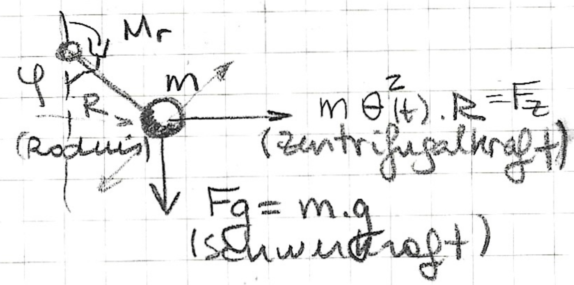
$u(t) = k(x(t) - x_{ref})$

1) Modellierungsziel: Zusammenhang zw. $\varphi(t)$ und $\theta(t)$ ($\Omega(t)$).

2) Blöcke: Masse, Trägheitsmoment, Dämpfer, Getriebeverhältnis

Verbindende Phänomene:





F_g, F_z, M_r, I_D, u

3) a) Verhalten der Blöcke:

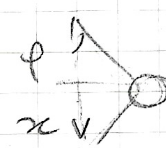
Masse: $M_I \overset{(I)}{=} I_m \cdot \varphi(t)$, $M_{Fg} \overset{(II)}{=} \sin(\varphi(t)) \cdot l \cdot mg$, $M_{Fz} \overset{(III)}{=} R m \theta^2(t) \cos(\varphi) l$

Dämpfer: $M_r = -b\dot{\varphi}(t)$ (IV)

Getriebeverhältnis: $\theta(t) = n\Omega(t)$ (V)

Trägheitsmoment: $M_D = u = k(x(t) - x_{ref})$ (VI)

3) b) Verbindungen der Blöcke:



$M_I = M_{Fz} - M_{Fg} - M_r$ (VII)

$M_D = k(x(t) - x_{ref}) = k(2 \cdot l \cdot \cos(\varphi(t)) - x_{ref})$ (VIII)

3) c) Vereinfachung und Elimination:

(VII) $M_I = I_m \ddot{\varphi} = R m \theta^2 \cos\varphi \cdot l - \sin\varphi \cdot l \cdot m g - b\dot{\varphi}$

$R = l \cdot \sin\varphi$ und $I_m = ml^2$

$ml^2 \ddot{\varphi} = l^2 m \theta^2 \cos\varphi \sin\varphi - \sin\varphi \, l m g - b\dot{\varphi}$

(IX): $\ddot{\varphi} = \theta^2 \cos\varphi \sin\varphi - \sin\varphi \frac{g}{l} - b^* \dot{\varphi}$, $\quad b^* = b/ml^2$

(VIII): $I_D \cdot \dot{\Omega} = k(2 l \cos\varphi - x_{ref})$

$(\text{V}) \rightarrow (\text{VIII})$: $\dfrac{\dot\theta}{n} I_D = k(2l\cos\varphi - x_{ref})$

$$\dot\theta = \frac{k\cdot n\cdot 2}{I_D}(l\cos\varphi) - \frac{x_{ref}\cdot k\cdot n}{I_D} = k^*\cdot\cos\varphi - k_1$$

wobei: $k^* = \dfrac{k\cdot n\cdot 2\cdot l}{I_D}$, $k_1 = \dfrac{x_{ref}\cdot k\cdot n}{I_D}$

$$x = [\varphi \quad \dot\varphi \quad \theta]^T$$

$$\dot x = \begin{pmatrix} \dot\varphi \\ \theta^2\cos\varphi\sin\varphi - \sin\varphi\frac{g}{l} - b^*\dot\varphi \\ k^*\cos\varphi - k_1 \end{pmatrix}$$

3e) Linearisierung:

Ruhelage: $\dot\varphi = 0$, $\theta = \theta_0$, $\varphi = \varphi_0$

$\dot\theta = 0$, $\ddot\varphi = 0$

$\rightarrow k^*\cos\varphi_0 = k_1$, $\cos\varphi_0 = \dfrac{k_1}{k^*}$ //

$\rightarrow \theta_0^2\cos\varphi_0\cancel{\sin\varphi_0} - \cancel{\sin\varphi_0}\cdot\frac{g}{l} - \cancel{b^*\dot\varphi_0} = 0$

$$\theta_0^2 = \frac{g}{l}\frac{1}{\cos(\varphi_0)} = \frac{g}{l}\cdot\frac{k^*}{k_1} \text{ //} \quad (\cos\varphi_0 \neq 0)$$

$A = D_1 f(x_0, u_0)$

$A_{11} = \dfrac{\partial f_1}{\partial\varphi} = 0$, $A_{12} = \dfrac{\partial f_1}{\partial\dot\varphi} = 1$, $A_{13} = \dfrac{\partial f_1}{\partial\theta} = 0$

$A_{21} = \dfrac{\partial f_2}{\partial\varphi} = \theta_0^2\underbrace{(\cos^2\varphi_0 - \sin^2\varphi_0)}_{=\cos 2\varphi_0} - \cos\varphi_0\frac{g}{l}$

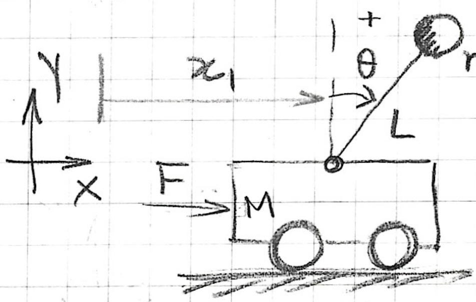
$A_{22} = \dfrac{\partial f_2}{\partial\dot\varphi} = -b^*$, $A_{23} = \dfrac{\partial f_2}{\partial\theta} = \theta_0\underbrace{2\cos\varphi_0\sin\varphi_0}_{=\sin 2\varphi_0}$

$A_{31} = \dfrac{\partial f_3}{\partial\varphi} = -k^*\sin\varphi_0$, $A_{32} = \dfrac{\partial f_3}{\partial\dot\varphi} = 0$, $A_{33} = \dfrac{\partial f_3}{\partial\theta} = 0$

$\rightarrow \dot x = Ax$

$$\Rightarrow \begin{bmatrix}\dot\varphi \\ \ddot\varphi \\ \dot\theta\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \theta_0^2\cos 2\varphi_0 - \cos\varphi_0\frac{g}{l} & -b^* & \theta_0\sin 2\varphi_0 \\ -k^*\sin\varphi_0 & 0 & 0 \end{bmatrix}\begin{bmatrix}\varphi \\ \dot\varphi \\ \theta\end{bmatrix} \text{ //}$$

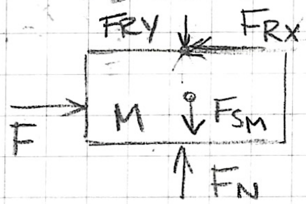
Aufgabe 2.2 $\quad x(t) \rightarrow$ Position des Wagens
$F \rightarrow$ Eingang



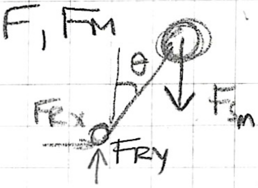
1) Modellierungsziel: Zusammenhang zw. Kraft auf Wagen und $x(t)$, $\theta(t)$

2) Blöcke: 2 Massen

3a)



Verhalten der Blöcke:
Masse M: F_{RX}, F_N, F_{Ry}, F_{SM}, F, F_M
Masse m: F_{RX}, F_{Ry}, F_{Sm}, F_m



3b) Verbindung der Blöcke:

X: $\quad F_{Mx} = M \cdot \ddot{x}_1 = F - F_{RX}$ (I)

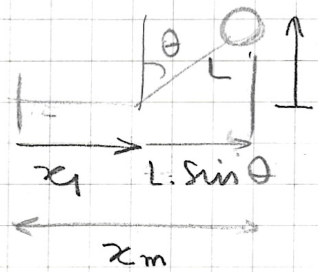
Drehmoment: $F_{RX} \cdot \cos\theta \cdot L = F_{Ry} \sin\theta \, L$ (II)

X: $\quad F_{mx} = m \cdot \ddot{x}_m = F_{RX}$ (III)

Y: $\quad F_{my} = m \cdot \ddot{y}_m = F_{Ry} - F_{Sm}$ (IV)

3)c) Vereinfachung und Elimination:

• kinematische Gleichungen:



$y_m = L \cos\theta$

(V) $\dot{x}_m = \dot{x}_1 + L\dot{\theta}\cos\theta$

(VI) $\dot{y}_m = -L\sin\theta \cdot \dot{\theta}$

(VII) $\ddot{x}_m = \ddot{x}_1 + L(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta)$

(VIII) $\ddot{y}_m = -L(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

• (VII) → (III) → (I): $M\ddot{x}_1 = F - m\ddot{x}_m = F - m(\ddot{x}_1 + L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$

$$\boxed{(M+m)\ddot{x}_1 = F - mL\cos\theta \cdot \ddot{\theta} - mL\dot{\theta}^2\sin\theta \quad \text{(IX)}}$$

• (IV) und (III) → (II): $\cancel{m}\ddot{x}_m \cos\theta \cancel{L} = (\cancel{m}\ddot{y}_m + \cancel{m} g) \cdot \sin\theta \cancel{L}$ (X)

• (VII) und (VIII) → (X): $(\ddot{x}_1 + L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta))\cos\theta =$
$= (-L(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) + g)\sin\theta$

$\ddot{x}_1\cos\theta + L\ddot{\theta}\cos^2\theta - \cancel{L\dot{\theta}^2\sin\theta\cos\theta} = g\sin\theta - L\ddot{\theta}\sin^2\theta - \cancel{L\dot{\theta}^2\sin\theta\cos\theta}$

$\ddot{x}_1\cos\theta + L\ddot{\theta}\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} - g\sin\theta = 0$

$$\Rightarrow \ddot{x}_1 \cos\theta + L\ddot{\theta} - g\sin\theta = 0 \quad (\mathrm{XI})$$

$$\boxed{\ddot{\theta} = -\frac{\ddot{x}_1 \cos\theta}{L} + \frac{g}{L}\sin\theta} \quad (\mathrm{XII})$$

$$(\mathrm{XII}) \to (\mathrm{IX}): \quad (M+m)\ddot{x}_1 = F - mL\cos\theta\left(\frac{g}{L}\sin\theta - \frac{\ddot{x}_1\cos\theta}{L}\right) - m L\dot{\theta}^2\sin\theta$$

$$(M+m-m\cos^2\theta)\ddot{x}_1 = F - m\cos\theta\sin\theta\, g - mL\dot{\theta}^2\sin\theta \quad (\mathrm{XIII})$$

$$(\mathrm{XIII}) \to (\mathrm{IX}): \quad \ddot{\theta} = -\left(\frac{F - mg\cos\theta\sin\theta - mL\dot{\theta}^2\sin\theta}{M + m(1-\cos^2\theta)}\right)\cdot\frac{\cos\theta}{L} + \frac{g}{L}\sin\theta$$

$$\ddot{\theta} = -\frac{(F - mg\cos\theta\sin\theta - mL\dot{\theta}^2\sin\theta)}{M + m\sin^2\theta}\cdot\frac{\cos\theta}{L} + \frac{g}{L}\sin\theta$$

$$x = [\theta, \dot{\theta}, x_1, \dot{x}_1]^T, \quad u = F$$

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ -\frac{(F - mg\cos\theta\sin\theta - mL\dot{\theta}^2\sin\theta)}{M + m\sin^2\theta}\frac{\cos\theta}{L} + \frac{g}{L}\sin\theta \\ \dot{x}_1 \\ \frac{F - m\cos\theta\sin\theta\, g - mL\dot{\theta}^2\sin\theta}{M + m\sin^2\theta} \end{pmatrix}$$

3e) Linearisierung:

$\to$ Ruhelage: $\dot{x} = 0, \quad \dot{\theta}_0 = 0, \quad \dot{x}_1 = 0, \quad x_1 = x_0, \quad u_0 = F_0$

$$\dot{x}_4 = 0 \to \dot{x}_2 = 0 = \frac{g}{L}\cdot\sin\theta \quad \therefore \quad \theta_0 = 0$$

$$A = D_1 f(x_0, u_0)$$

$$A_{11} = \frac{\partial f_1}{\partial\theta} = 0, \quad A_{12} = \frac{\partial f_1}{\partial\dot{\theta}} = 1, \quad A_{13} = 0, \quad A_{14} = 0$$

$$A_{21} = mg\left(\frac{\cos^2\theta_0 - \cancel{\sin^2\theta_0} - \cancel{2\sin\theta_0\cos\theta_0}}{(M + m\sin^2\theta_0)^2}\right)\cdot\frac{\cos\theta_0}{L} - \frac{mg\cos\theta_0\cancel{\sin\theta_0}}{(M + m\sin^2\theta_0)}\frac{\sin\theta_0}{L} +$$

$$+\left(\frac{\cancel{mL\dot{\theta}^2\cos\theta_0 - 2\sin\theta_0\cos\theta_0}}{(M + m\sin^2\theta_0)^2}\right)\frac{\cos\theta_0}{L} - \frac{mL\dot{\theta}^2\cancel{\sin\theta_0}}{\cancel{(M + m\sin^2\theta_0)^2}\cdot L} + \frac{g}{L}\cos\theta_0$$

$$A_{21} = \frac{mg}{ML} + \frac{g}{L} = \frac{g}{L}\left(\frac{M+m}{M}\right)$$

$$\cdot\ A_{22} = \frac{\partial f_2}{\partial \dot\theta} = \frac{mL\,2.\dot\theta_0 \cancel{\sin\theta_0}\,.\cos\theta}{L\,\cancel{M + m\sin^2\theta_0}} = 0$$

$$\cdot\ A_{23} = \frac{\partial f_2}{\partial x_1} = 0 \quad , \quad A_{24} = \frac{\partial f_2}{\partial \dot x_1} = 0$$

$$\cdot\ A_{31} = \frac{\partial f_3}{\theta} = 0 \quad , \quad A_{32} = \frac{\partial f_3}{\dot\theta} = 0 \quad , \quad A_{33} = \frac{\partial f_3}{x_1} = 0 \,,\ A_{34} = \frac{\partial f_3}{\dot x_1} = 1$$

$$\cdot\ A_{41} = \frac{\partial f_4}{\theta} = \frac{-m\,\underbrace{(\cos^2\theta_0 - \cancel{\sin^2\theta_0})}_{=1}.g - \cancel{mL\dot\theta^2\cos\theta_0}}{(M + m\cancel{\sin^2\theta_0})} -$$

$$- \frac{-2\sin\theta_0 . \cancel{\cos\theta_0}\, m . 2}{(M + \cancel{m\sin^2\theta_0})}$$

$$\cdot\ A_{41} = \frac{\partial f_4}{\theta}(x_0, u_0) = -\frac{mg}{M} \quad //$$

$$\cdot\ A_{42} = \frac{\partial f_4}{\dot\theta} = -\frac{mL2\dot\theta_0 \cancel{\sin\theta_0}}{\cancel{M + m\sin^2\theta_0}} = 0 \ //$$

$$\cdot\ A_{43} = \frac{\partial f_4}{x_1} = 0 \,, \quad A_{44} = \frac{\partial f_4}{\partial \dot x_1} = 0$$

$$B = D_2 f(x_0, u_0) \Rightarrow \quad B_{11} = \frac{\partial f_1}{\partial u} = 0 \,, \quad B_{21} = \frac{\partial f_2}{\partial u} = -\frac{1 \cdot \cos\theta}{\cancel{(M + m\sin^2\theta_0)}.L}$$

$$B_{21} = -\frac{1}{ML} \quad // \qquad B_{31} = \frac{\partial f_3}{\partial u} = 0 \,, \quad B_{41} = \frac{\partial f_4}{\partial u} = \frac{1}{M}$$

$$\begin{pmatrix} \dot\theta \\ \ddot\theta \\ \dot x_1 \\ \ddot x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{L}\frac{(M+m)}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -g\,m/M & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \dot\theta \\ x_1 \\ \dot x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/(ML) \\ 0 \\ 1/M \end{pmatrix} . u \qquad //$$

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