Steuer- und Regelungstechnik

3. Übung

Victor Cheidde Chaim

31. Januar 2022

Universität der Bundeswehr München, LRT-15 Institut für Steuer- und Regelungstechnik

Aufgabe 2.1

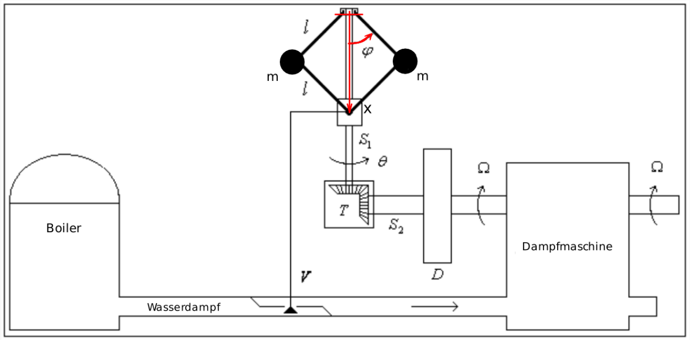


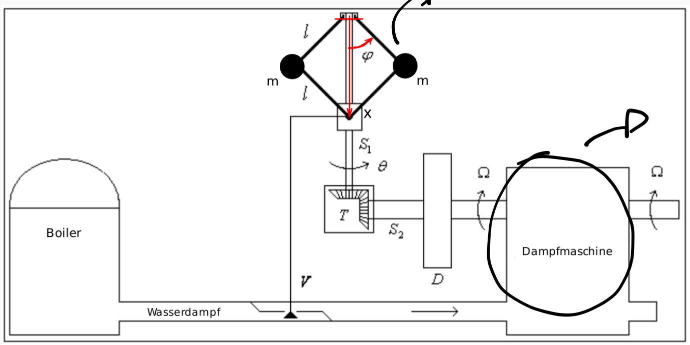
Abbildung 1: System: Dampfmaschine und Fliehkraftregler.
Quelle (bearbeitet von V. Chaim): *Bifurcation Analysis of the Watt Governor System*, Sotomayor, J.; Mello, L. F.; Braga, D. C.; Computational and Applied Mathematics, Vol. 26, N.1, pp 19-44, 2007.

- Drehmoment-Dämpfung
  $M_r = -b\dot{\varphi}(t);$
- Getriebeverhältnis
  $\theta(t) = n\Omega(t);$
- Eingang $u(t) = k(x(t) - x_{ref})$.

↳ Drehmoment

Aufgabe 2.1

$$\theta \rightarrow \boxed{f_m} \rightarrow \varphi, \dot{\varphi}, \ddot{\varphi}$$



$$u \rightarrow \boxed{I_D} \rightarrow \Omega, \dot{\Omega}$$

Aufgabe 2.2

1) Modellierungsziel: Zusammenhang Kraft F, $x(t)$ und $\theta(t)$ → Zustand: $x(t)$, $\dot{x}(t)$, $\theta(t)$, $\dot{\theta}(t)$

2) Blöcke: 2 Massen

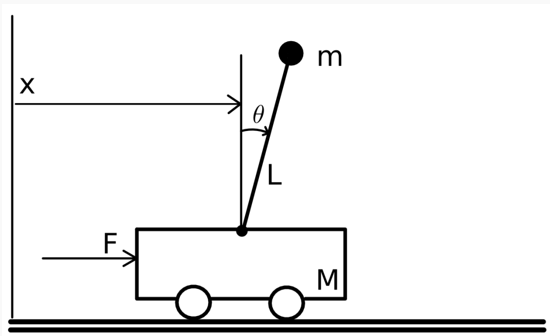
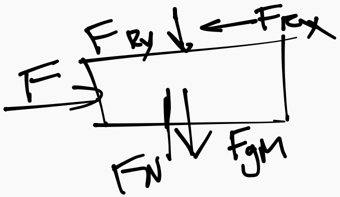


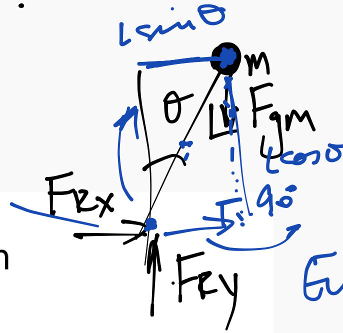
Abbildung 2: System: Umgekehrtes Pendel auf einem Wagen.

- die Kraft F = Eingang;
- keine Reibung zwischen dem Wagen und dem Boden.

3)



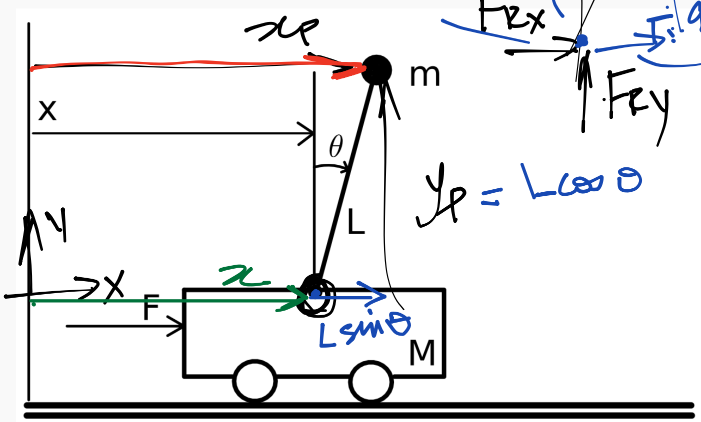
Newton: x: $M\cdot\ddot{x} = F - F_{Rx}$



x: $m\cdot\ddot{x}_p = F_{Rx}$

y: $m\cdot\ddot{y}_p = F_{Ry} - F_{gm}$

Euler: $\underbrace{\cancel{I_m}}_{0}\cdot\ddot{\theta} = F_{Rx}\cdot L\cdot\cos\theta - F_{Ry}\cdot L\cdot\sin\theta$



$$F_{Ry} = F_{Rx}\cdot\frac{\cos\theta}{\sin\theta}$$

$x_p = x + L\cdot\sin\theta \;\rightarrow\; \dot{x}_p = \dot{x} + \dot{\theta}L\cos\theta$

$y_p = L\cos\theta \qquad\qquad \dot{y}_p = -\dot{\theta}L\sin\theta$

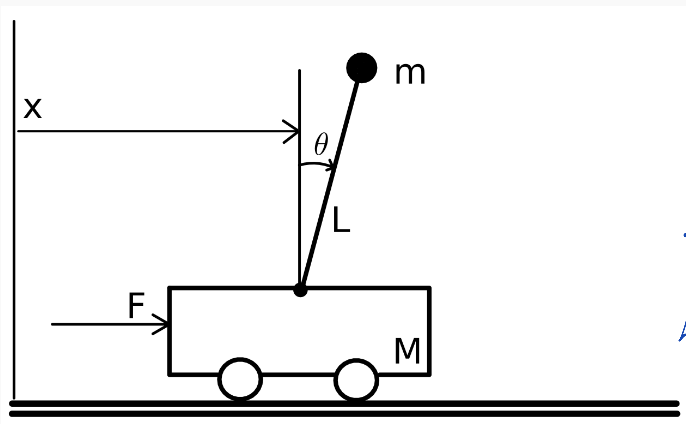
$\rightarrow \ddot{x}_p = \ddot{x} + L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$

$\ddot{y}_p = -L(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

$$\begin{cases} \ddot{x}_p = \ddot{x} + L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\ \ddot{y}_p = L(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta) \end{cases}$$

$$\begin{cases} M\ddot{x} = -F_{Rx} + F \\ m\ddot{x}_p = F_{Rx} \\ m\ddot{y}_p = F_{Ry} - F_{gm} \end{cases}$$

$$F_{Ry} = F_{Rx}\frac{\cos\theta}{\sin\theta}$$

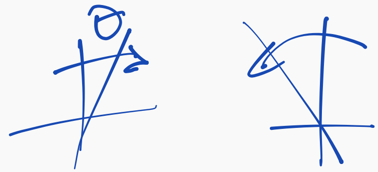


$$\begin{cases} M\ddot{x} = -m\ddot{x}_p + F \\ \cancel{m}\ddot{y}_p = \cancel{m}\ddot{x}_p\frac{\cos\theta}{\sin\theta} - \cancel{m}g \end{cases}$$

$$\boxed{M\ddot{x} = -m(\ddot{x} + L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta))} \quad (\mathrm{I})$$

Aufgabe 2.2

$$\ddot{y}_p = \ddot{x}_p \frac{\cos\theta}{\sin\theta} - g$$



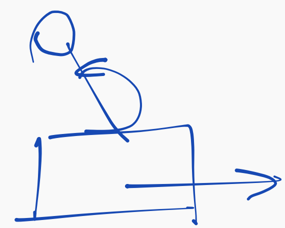
$\sin\theta$

$$L(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta) = \left(\ddot{x} + L(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\right)\frac{\cos\theta}{\sin\theta} - g$$

$$L(-\ddot{\theta}\sin^2\theta - \cancel{\dot{\theta}^2\sin\theta\cos\theta}) = \ddot{x}\cdot\cos\theta + L(\ddot{\theta}\cos^2\theta - \cancel{\dot{\theta}^2\sin\theta\cdot\cos\theta} - g\sin\theta$$

$$L(-\ddot{\theta}(\underbrace{\sin^2 + \cos^2}_{1})) = \ddot{x}\cos\theta - g\sin\theta$$

$$\boxed{-L\ddot{\theta} = \ddot{x}\cos\theta - g\sin\theta}$$



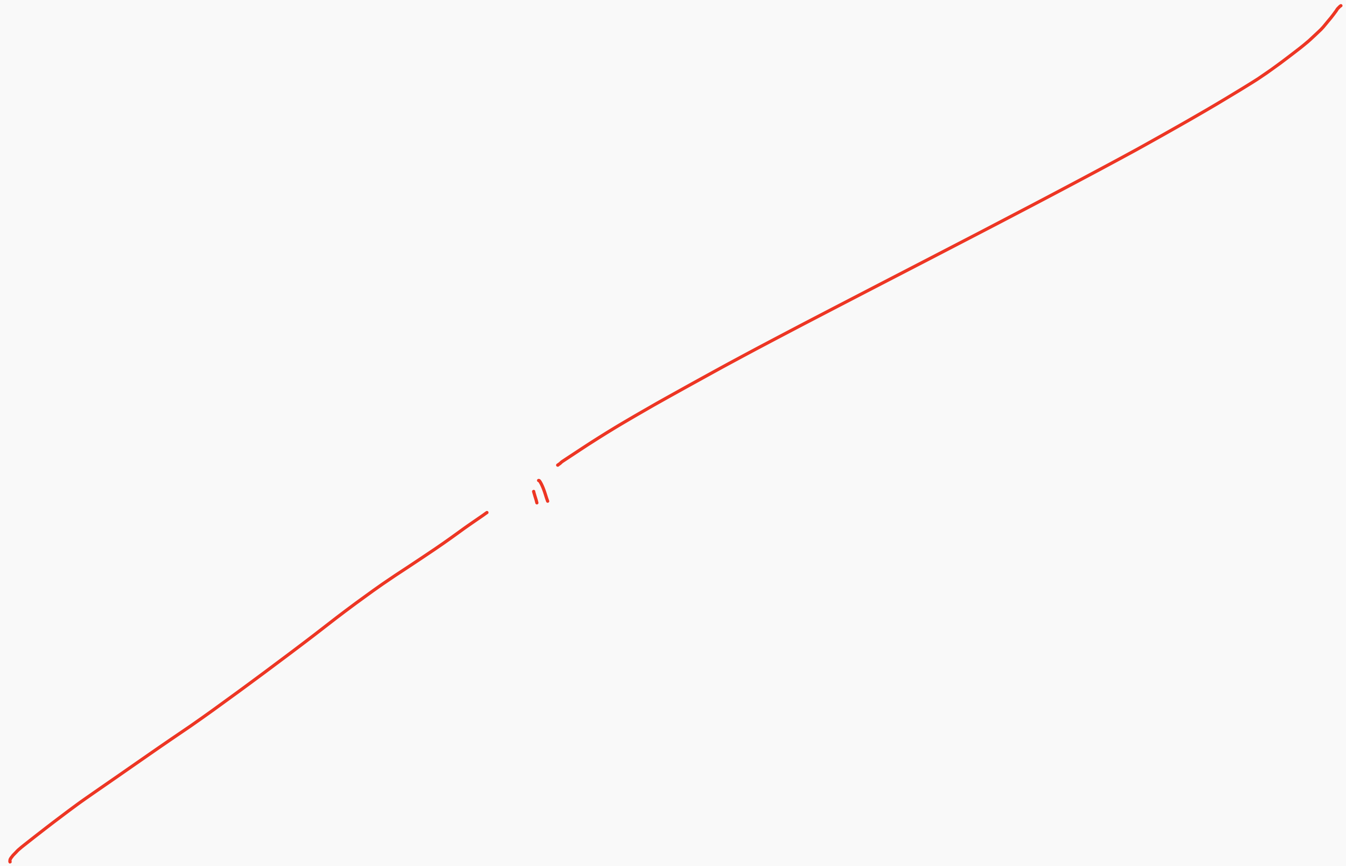
$$\ddot{\theta} = \frac{\ddot{x}\cos\theta - g\sin\theta}{-L} \rightarrow f_1(\ddot{x}) \rightarrow \text{die Linearisierung}$$

$f_2(\ddot{\theta})$

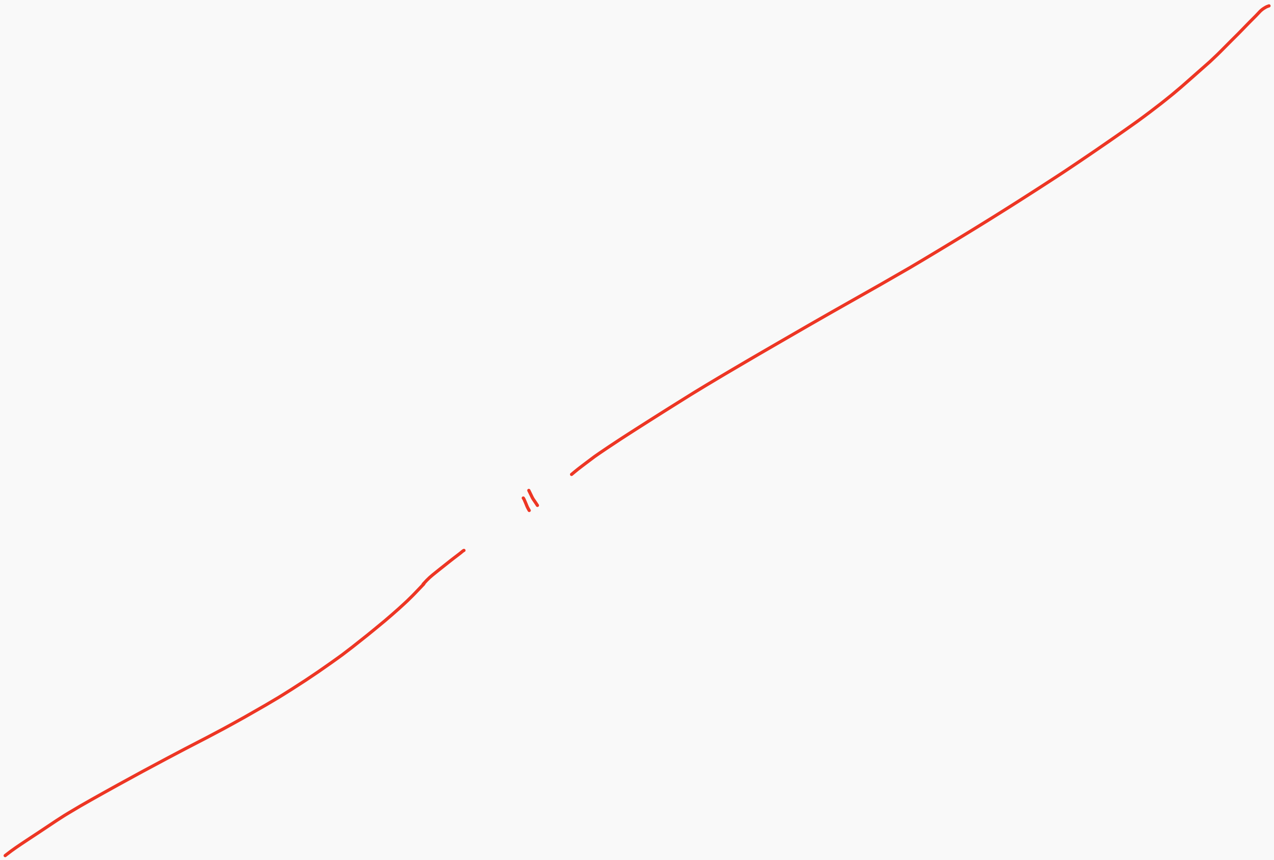
Aufgabe 2.2

$$
\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & \overset{0}{\cancel{A_{22}}} & -\frac{mg}{M} & 0 \\
0 & 0 & 0 & 1 \\
0 & \overset{0}{\cancel{A_{23}}} & \frac{g(m+M)}{LM} & 0
\end{bmatrix}
\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}
+
\begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(LM) \end{bmatrix} \cdot F
$$

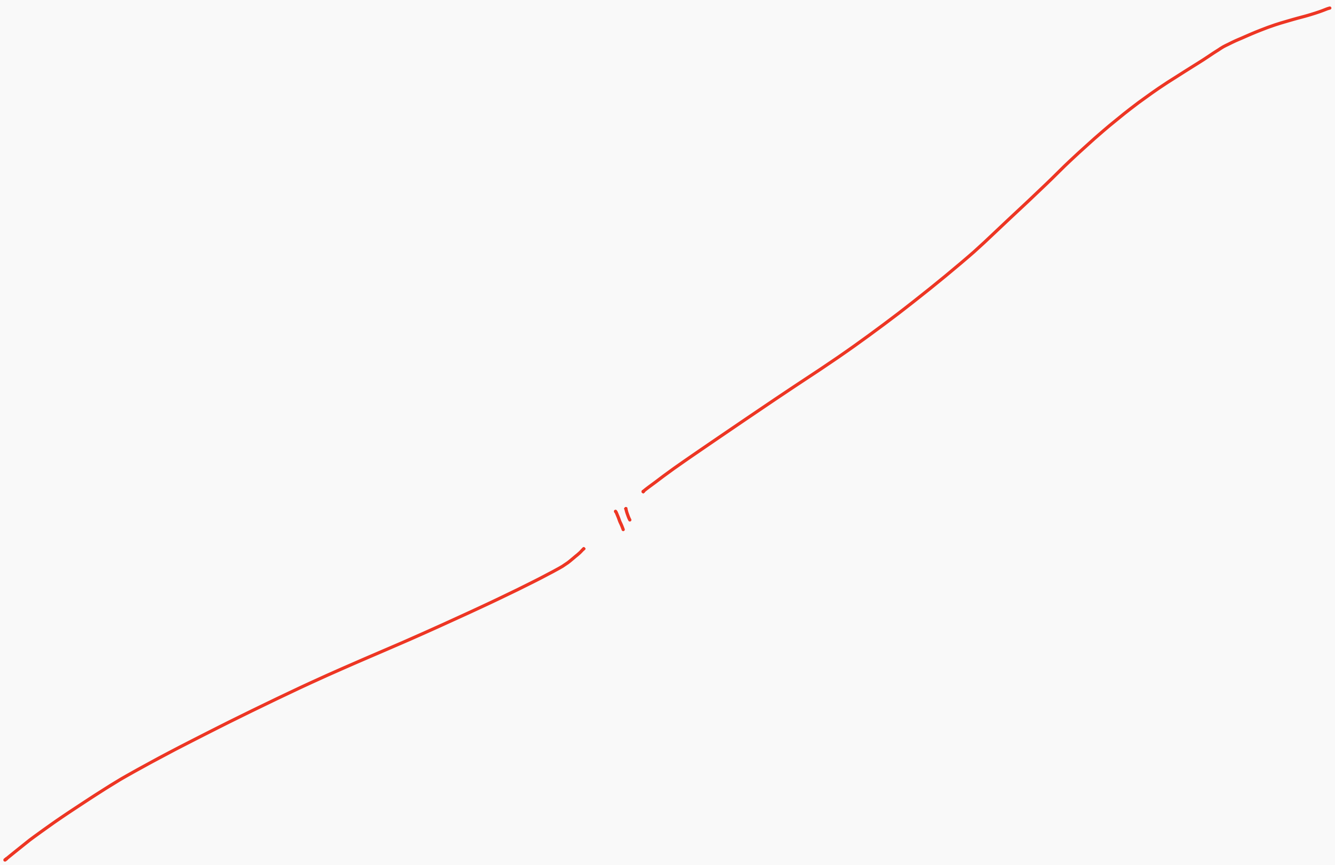
Aufgabe 2.2



Aufgabe 2.2



Aufgabe 2.2



Aufgabe 2.2

Aufgabe 3.1

$$\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Rightarrow \exp(At) = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

$$\hookrightarrow 3! = 3.2.1 = 6$$

Gegeben seien die Matrizen A und B durch

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Berechnen Sie $\exp(At)$, $\exp(Bt)$ und $\exp((A + B)t)$. Gilt hier $\exp(A + B) = \exp(A)\exp(B)$?

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \exp(At) = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \frac{id}{\underset{1}{0!}} + \frac{At}{\underset{1}{1!}} + \frac{A^2 t^2}{\underset{2}{2!}} + \dots$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow A^n = 0, \quad n \geq 2$$

nilpotent Matrix

$$\exp(At) = id + At = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

$$\exp(At) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \exp(Bt) = \sum_{n=0}^{\infty} \frac{(Bt)^n}{n!} = \underline{id} + \underbrace{Bt + \frac{B^2 t^2}{2} + \dots}_{n=1}$$

$$B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B^n = B \quad n \geq 1$$

$$B^3 = B \cdot B^2 = BB = B^2 = B$$

$$\exp(Bt) = id + B \underbrace{\sum_{n=1}^{\infty} \frac{t^n}{n!}}_{\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}} = id \; \underbrace{- B + B \underbrace{\sum_{n=0}^{\infty} \frac{t^n}{n!}}_{e^t}}_{= B \sum_{n=1}^{\infty} \frac{t^n}{n!}} = id - B + Be^t$$

$$\exp(Bt) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$$

$$\exp((A+B)t) = \sum_{n=0}^{\infty} \frac{((A+B)t)^n}{n!} = id + (A+B)t + \frac{(A+B)^2 t^2}{2} + \dots$$

$$(A+B)^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{(A+B)^n = (A+B) \quad n \geq 1}$$

$$\exp((A+B)t) = id + (A+B) \sum_{n=1}^{\infty} \frac{t^n}{n!} = id - (A+B) + (A+B) \underbrace{\sum_{n=0}^{\infty} \frac{t^n}{n!}}_{e^t}$$

$$\exp((A+B)t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot e^t$$

$$= \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix} \quad //$$

Aufgabe 3.1

$$\exp(At)\exp(Bt) = \exp((A+B)t) \; ?$$

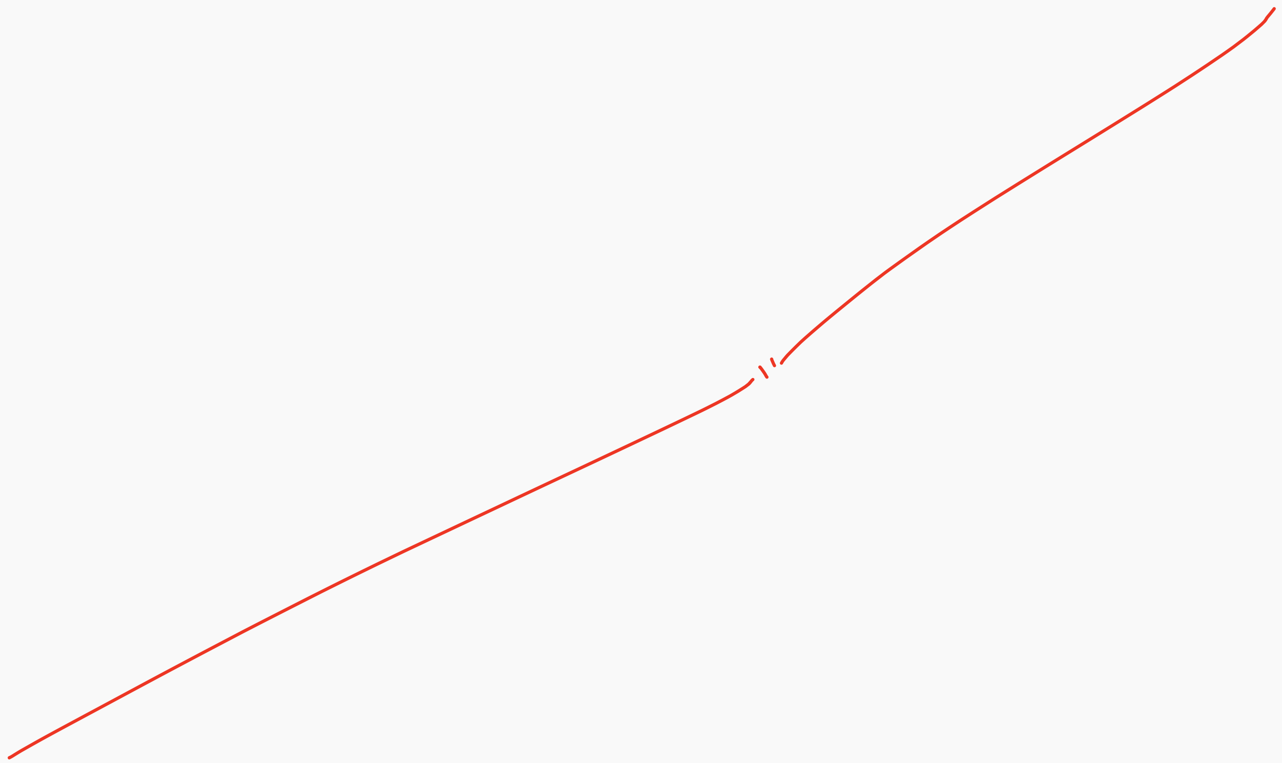
$$\Rightarrow \underline{AB = BA}$$

$$\exp(At)\exp(Bt) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & t \\ 0 & 1 \end{pmatrix}$$

$$\exp((A+B)t) = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$$

$$\exp(At)\exp(Bt) \neq \exp((A+B)t) \;\; \unicode{x21af}$$

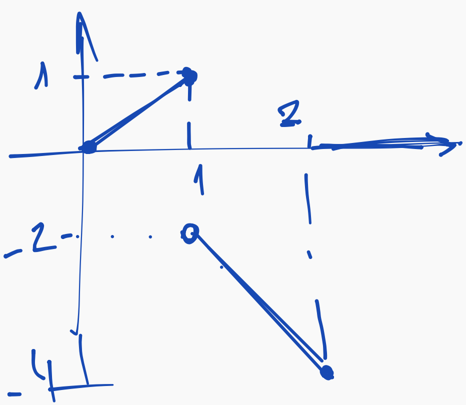
Aufgabe 3.1



Aufgabe 3.2

$$\dot{x} = ax + bu \qquad a = 1$$
$$\varphi(t, x_0, u) = e^{at} \cdot x_0 + \int_0^t e^{a(t-\tau)} u(\tau)\, d\tau \qquad b = 1$$

Berechnen Sie die Lösung für das Zustandssystem $\dot{x} = x + u$, das den skalaren Fall darstellt, mit der Anfangsbedingung $x_0 = -1$ und der folgenden stückweise linearen Funktion für den Eingang $u(t)$, wobei $t \in \mathbb{R}$:



$$u(t) = \begin{cases} t = u_1 & \text{falls } 0 \leq t \leq 1 \\ -2t = u_2 & \text{falls } 1 < t \leq 2 \\ 0 = u_3 & \text{andernfalls.} \end{cases}$$

$$\varphi(t, x_0, u) = \begin{cases} \varphi_1(t, x_0, u_1), & \text{falls } 0 \leq t \leq 1 \\ \varphi_2(t, x_1, u_2), & \text{falls } 1 < t \leq 2 \\ \varphi_3(t, x_2, u_3), & \text{andernfalls} \end{cases}$$

Aufgabe 3.2

$0 \le t \le 1: \quad u_1 = t, \quad x_0 = -1$

$$\varphi_1(t, x_0, u) = x_0 \cdot e^{t} + \int_0^t e^{(t-\tau)} u(\tau)\, d\tau = -e^{t} + e^{t} \int_0^t e^{-\tau} u(\tau)\, d\tau \quad \text{"} \tau \text{"}$$

Integration durch Teile: $\int e^{-\tau} \cdot \tau \, d\tau$

$$\int u\, dv = uv - \int v\, du \Rightarrow \left\langle \begin{array}{l} u = \tau \\ dv = e^{-\tau} d\tau \end{array} \right. \Bigg| \begin{array}{l} du = d\tau \\ v = -e^{-\tau} \end{array}$$

$$\int e^{-\tau} \tau \, d\tau = -e^{-\tau} \cdot \tau - \int -e^{-\tau} d\tau = -e^{-\tau}\tau - e^{-\tau} + C$$

$$= \underline{-e^{-\tau}(1+\tau) + C}$$

$$\varphi_1(t, x_0, u) = -e^{t} + e^{t}\left(-e^{-\tau}(1+\tau)\Big|_0^t\right) = -(t+1) \quad //$$

Aufgabe 3.2

$1 < t \leq 2, \quad x_1 = \varphi_1(1, x_0, u_1) = -(1+1) = -2 \quad t \quad u_2 = -2t$

$$\varphi_2(t, x_1, u_2) = \underline{x_1} e^{(t-1)} + \int_1^t e^{(t-\tau)} u(\tau)\, d\tau = -2 e^{(t-1)} + e^t \int_1^t e^{-\tau} u(\tau)\, d\tau$$

$$= -2 e^{(t-1)} + e^t \int_1^t e^{-\tau} \cdot (-2\tau)\, d\tau =$$

$$= -2 e^{(t-1)} - 2 e^t \int_1^t e^{-\tau} \cdot \tau \cdot d\tau$$

$$= -2 e^{(t-1)} - 2 e^t \left(-e^{-\tau}(\tau+1) \Big|_1^t \right)$$

$$= -2 e^{(t-1)} - 2 e^t \left(-e^{-t}(t+1) + e^{-1}(1+1) \right) = -6 \cdot e^{(t-1)} + 2(t+1)$$

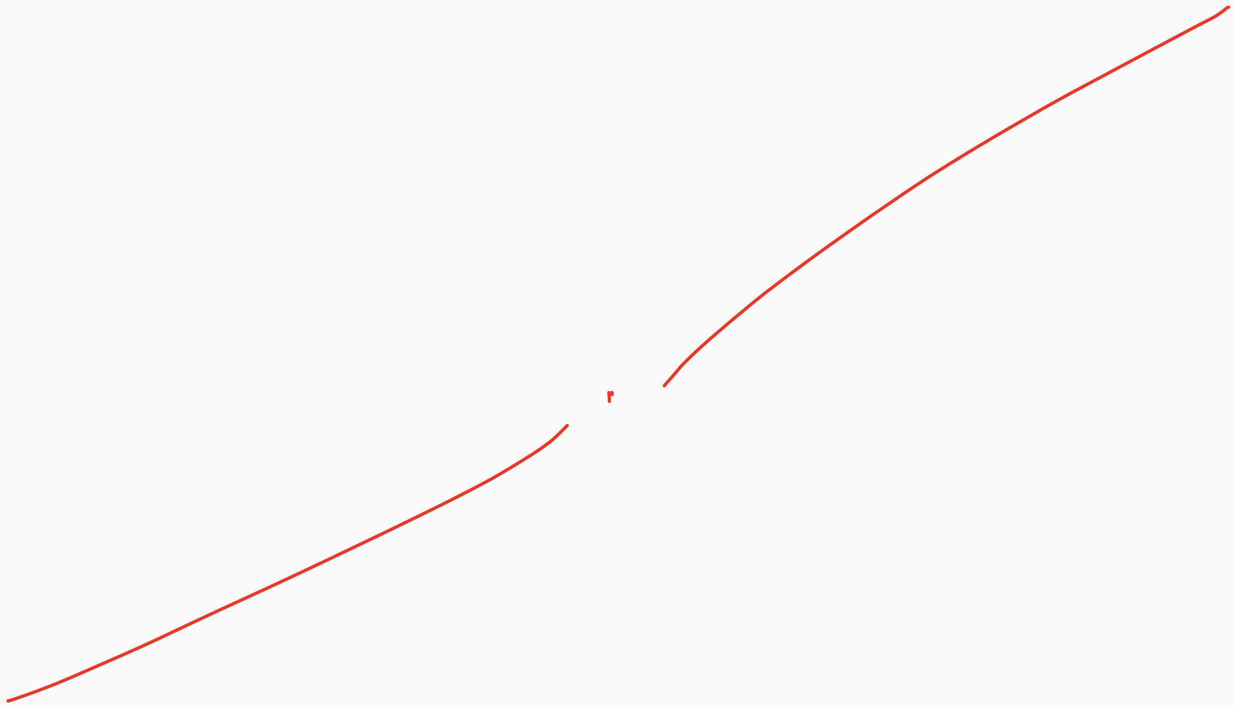
andernfalls: $\varphi_3(t, x_2, u_3)$, $\quad x_2 = \varphi_2(2, x_1, u_2) = -6e^{(2-1)} + 2(3)$

$$= 6(1-e^1)$$

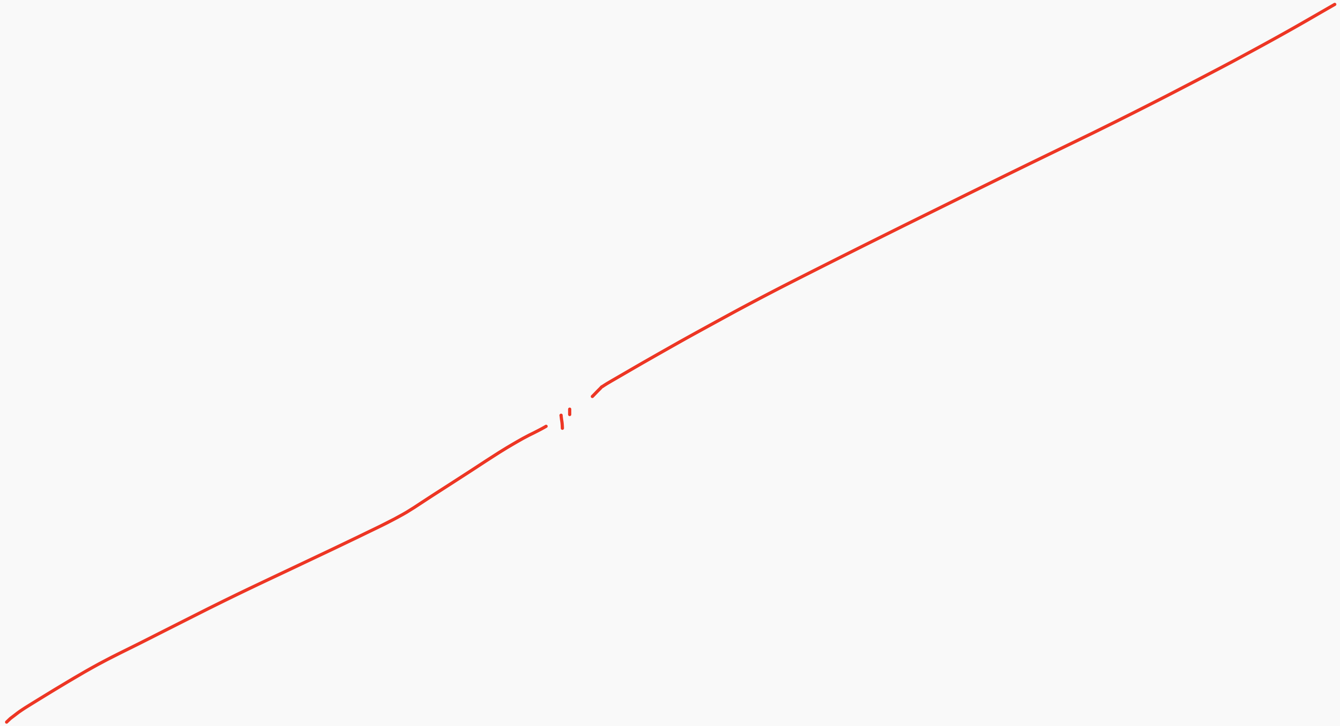
$$\varphi_3(t, x_2, u_3) = x_2 e^{(t-2)} + \cancelto{0}{\int_2^t e^{(t-\tau)} \underline{u(\tau)}\, d\tau} = 6(1-e^1)\, e^{(t-2)}$$

$$\varphi(t, x_0, u) = \begin{cases} -(t+1), & 0 \leq t \leq 1 \\ -6e^{(t-1)} + 2(t+1), & 1 < t \leq 2 \\ \underline{\underline{6(1-e^1) \cdot e^{(t-2)}}}, & \text{andernfalls } (t > 2) \end{cases}$$

Aufgabe 3.2



Aufgabe 3.2



Aufgabe 3.3

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad x = [x_1 \quad x_2]^T$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Betrachten Sie das folgende Zustandssystem

$$\dot{x}_1(t) = x_1(t) + x_2(t) \qquad , \qquad u(t) = \begin{cases} t & \text{falls } 0 \leq t \leq 1 \\ -2t & \text{falls } 1 < t \leq 2 \\ 0 & \text{andernfalls} \end{cases}$$
$$\dot{x}_2(t) = u(t)$$

wobei der Eingang $u(t)$ eine stückweise lineare Funktion ist und $t \in \mathbb{R}$. Berechnen Sie die Lösung des Systems unter der Annahme, dass die Anfangsbedingungen $x_1(0) = 0$ und $x_2(0) = 1$ sind.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u \qquad x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$0 \le t \le 1: \quad u(t) = t, \quad x_0 = [0 \quad 1]^T$$

$$\varphi_1(t, x_0, u) = \underbrace{e^{At} \cdot x_0}_{I} + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$\exp(At) = \exp\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} t\right) = \begin{pmatrix} e^t & e^t-1 \\ 0 & 1 \end{pmatrix}$$

$$\exp(At) \cdot x_0 = \begin{pmatrix} e^t & e^t-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t-1 \\ 1 \end{pmatrix}$$