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Numerical Optimal Control
– Numerical Experiments –

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Contents

1 Optimal Control Problems

Try to implement the following optimal control problems in `OCPID-DAE1` and do experiments:

- Choose different grids.
- Choose different integration methods with and without automatic step-size selection.
- Choose different control approximations.
- Have a look at the adjoint estimates.
- Compute sensitivities.

1.1 Minimum Energy Problem

Minimize

$$\frac{1}{2}\int_0^1 u(t)^2 \; dt$$

subject to the differential equations

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= u(t), \end{aligned}$$

the boundary conditions

$$x_1(0) = x_1(1) = 0, \qquad x_2(0) = -x_2(1) = 1,$$

and the state constraint

$$x_1(t) - x_{max} \leq 0 \qquad (x_{max} = 1/9).$$

1.2 Brachistochrone Problem

Minimize the final time

$$t_f$$

subject to the differential equations

$$\begin{aligned} x'(t) &= \sqrt{2gy(t)}\cos\gamma(t), && x(0) = 0,\ x(t_f) = 5 \\ y'(t) &= \sqrt{2gy(t)}\sin\gamma(t), && y(0) = 1.2 \end{aligned}$$

and the state constraint

$$y(t) - x(t)\tan(\theta) - h \leq 0$$

with $h = 1.5$, $\theta = 20^\circ = 20\frac{2\pi}{360}[rad]$.

1.3 Vertical Ascent of a Rocket (Goddard Problem)

Maximize

$$x_1(t_f) \qquad (t_f \text{ free})$$

subject to the constraints

$$\begin{aligned} x_1' &= x_2, & x_1(0) &= 1 \\ x_2' &= \frac{u-d}{x_3} - \frac{1}{x_1^2}, & x_2(0) &= 0 \\ x_3' &= -\frac{u}{c}, & x_3(0) &= 1,\ x_3(t_f) = 0.6 \end{aligned}$$

and

$$0 \leq u \leq 3.5,$$

where $c = 0.5$, and

$$d = 310 x_2^2 \exp(500(1 - x_1)).$$

1.4 Earth-Mars Transfer

Maximize

$$x_4(t_f) \qquad (t_f \text{ free})$$

subject to the constraints

$$\begin{aligned} x_1' &= x_2, & & x_1(0) = 1, x_1(t_f) = 1.525 \\ x_2' &= \frac{x_3^2}{x_1} - \frac{1}{x_1^2} + \frac{c u_1 \sin u_2}{x_4}, & & x_2(0) = x_2(t_f) = 0, \\ x_3' &= -\frac{x_2 x_3}{x_1} + \frac{c u_1 \cos u_2}{x_4}, & & x_3(0) = 1,\ x_3(t_f) = \tfrac{1}{\sqrt{1.525}} \\ x_4' &= -u_1, & & x_4(0) = 1 \end{aligned}$$

and $c = 1.872$ and

$$0 \leq u_1 \leq 0.075, \quad -10 \leq u_2 \leq 10.$$

1.5 Water Boxes

Maximize

$$\int_0^{10} (10-t)u_1(t) + tu_2(t)dt$$

subject to the differential equations

$$\begin{aligned} x_1'(t) &= -u_1(t), \\ x_2'(t) &= u_1(t) - u_2(t), \end{aligned}$$

the initial conditions

$$x_1(0) = 4, \qquad x_2(0) = 4,$$

the state constraints

$$x_i(t) \geq 0, \qquad \forall t \in [0,10],\ i = 1,2,$$

and the control constraints

$$0 \leq u_i(t) \leq 1, \qquad t \in [0,10],\ i = 1,2.$$

1.6 Osmolovskii Example

Minimize

$$\int_0^{t_f} x(t)dt \qquad (t_f = 5)$$

subject to the constraints

$$\begin{aligned} x'''(t) &= u(t), \qquad x(0) = 2,\ x'(0) = 0,\ x''(0) = 0, \\ -1 &\leq u(t) \leq 1, \\ x(t) &\geq 1, \end{aligned}$$

Optimal solution has a cluster point of contact points for the state constraint.

1.7 NoName

Minimize

$$t_f + \int_0^{t_f} u(t)^2 dt$$

subject to the constraints

$$
\begin{aligned}
x_1'(t) &= x_2(t), \qquad x_1(0) = 0,\ x_1(t_f) = 1,\\
x_2'(t) &= u(t)^3, \qquad x_2(0) = 0,\ x_2(t_f) = 0,\\
-1 &\le u(t) \le 1
\end{aligned}
$$

1.8 Lotka-Volterra Fishing Problem

source: http://mintoc.de by Sebastian Sager

Minimize

$$\int_0^{12} (x_1(t) - 1)^2 + (x_2(t) - 1)^2 dt$$

subject to

$$
\begin{aligned}
x_1'(t) &= x_1(t) - x_1(t)x_2(t) - 0.4x_1(t)v(t)\\
x_2'(t) &= -x_2(t) + x_1(t)x_2(t) - 0.2x_2(t)v(t)\\
v(t) &\in [0, 1]\\
x(0) &= (0.5, 0.7)^\top
\end{aligned}
$$

1.9 Rayleigh Problem

Minimize

$$\int_0^{4.5} u(t)^2 + x_1(t)^2 dt$$

subject to the constraints

$$
\begin{aligned}
x_1' &= x_2, \qquad x_1(0) = -5,\\
x_2' &= -x_1 + x_2\,(1.4 - 0.14x_2^2) + 4u, \qquad x_2(0) = -5
\end{aligned}
$$

and

$$u + \frac{1}{6}x_1 \le 0.$$

1.10 Drug Displacement

source: Maurer, H. and Wiegand, M.: *Numerical solution of a drug displacement problem with bounded state variables*, Optimal Control Applications and Methods, Vol. 13 (1), pp. 43-55, 1992.

Minimize

$$t_f$$

subject to the constraints

$$\begin{aligned} x_1'(t) &= D^2(C_2(0.02 - x_1(t)) + 46.4x_1(t)(u(t) - 2x_2(t)))/C_3 \\ x_2'(t) &= D^2(C_1(u(t) - 2x_2(t)) + 46.4x_2(t)(0.02 - x_1(t)))/C_3 \\ 0 &\le u(t) \le u_{max} = 8, \\ x_1(t) &\le \alpha = 0.025 \end{aligned}$$

and

$$\begin{aligned} x_1(0) &= 0.02, \\ x_1(t_f) &= 0.02, \\ x_2(0) &= 0, \\ x_2(t_f) &= 2, \\ D &= 1 + 0.2x_1 + 0.2x_2, \\ C_1 &= D^2 + 232 + 46.4x_2, \\ C_2 &= D^2 + 232 + 46.4x_1, \\ C_3 &= C_1C_2 - (46.4)^2 x_1 x_2. \end{aligned}$$

Parameters: $\alpha = 0.026$ or $\alpha = 0.023$, $u_{max} = 8$.
The state constraint is active for $0.02 < \alpha \le 0.02870464$.

1.11 Flight Path Optimization

Minimize

$$\Phi(C_L, \mu, t_f) = -\left(\frac{\Lambda(t_f) - \Lambda(0)}{\Lambda(0)}\right)^2 - \left(\frac{\Theta(t_f) - \Theta(0)}{\Theta(0)}\right)^2$$

subject to differential equations for the velocity v, inclination γ, azimuth angle χ, altitude

h, latitude Λ, and longitude Θ,

$$
\begin{aligned}
v' &= -D(v,h;C_L)\frac{1}{m} - g(h)\sin\gamma + \\
&\quad + \omega^2\cos\Lambda(\sin\gamma\cos\Lambda - \cos\gamma\sin\chi\sin\Lambda)R(h), \\
\gamma' &= L(v,h;C_L)\frac{\cos\mu}{mv} - \left(\frac{g(h)}{v} - \frac{v}{R(h)}\right)\cos\gamma + \\
&\quad + 2\omega\cos\chi\cos\Lambda + \omega^2\cos\Lambda(\sin\gamma\sin\chi\sin\Lambda + \cos\gamma\cos\Lambda)\frac{R(h)}{v}, \\
\chi' &= L(v,h;C_L)\frac{\sin\mu}{mv\cos\gamma} - \cos\gamma\cos\chi\tan\Lambda\frac{v}{R(h)} + \\
&\quad + 2\omega(\sin\chi\cos\Lambda\tan\gamma - \sin\Lambda) - \omega^2\cos\Lambda\sin\Lambda\cos\chi\frac{R(h)}{v\cos\gamma}, \\
h' &= v\sin\gamma, \\
\Lambda' &= \cos\gamma\sin\chi\frac{v}{R(h)}, \\
\Theta' &= \cos\gamma\cos\chi\frac{v}{R(h)\cos\Lambda},
\end{aligned}
$$

with functions [1]

$$
\begin{aligned}
L(v,h,C_L) &= q(v,h)\;F\;C_L, & \rho(h) &= \rho_0\exp(-\beta h), \\
D(v,h,C_L) &= q(v,h)\;F\;C_D(C_L), & R(h) &= r_0 + h, \\
C_D(C_L) &= C_{D_0} + k\,{C_L}^2, & g(h) &= g_0(r_0/R(h))^2, \\
q(v,h) &= \tfrac{1}{2}\rho(h)v^2
\end{aligned}
$$

and parameters

$$
\begin{aligned}
F &= 305, & r_0 &= 6.371\cdot 10^6, & C_{D_0} &= 0.017, \\
k &= 2, & \rho_0 &= 1.249512, & \beta &= 1/6900, \\
g_0 &= 9.80665, & \omega &= 7.27\cdot 10^{-5}, & m &= 115000.
\end{aligned}
$$

The mass m is assumed to be constant. Box constraints for the controls C_L (lift coefficient) and μ (bank angle) are given by

$$
\begin{aligned}
0.01 \le\; & C_L \le 0.18326, \\
-\frac{\pi}{2} \le\; & \mu \le \frac{\pi}{2}.
\end{aligned}
$$

The initial value corresponds to a position above Bayreuth:

$$
\begin{pmatrix} v(0) \\ \gamma(0) \\ \chi(0) \\ h(0) \\ \Lambda(0) \\ \Theta(0) \end{pmatrix} = \begin{pmatrix} 2150.5452900 \\ 0.1520181770 \\ 2.2689279889 \\ 33900.000000 \\ 0.8651597102 \\ 0.1980948701 \end{pmatrix}.
$$

A terminal constraint is given by

$$h(t_f) = 500.$$

The terminal time t_f is free. Finally, the dynamic pressure constraint

$$q(v, h) \leq q_{max}$$

with $q_{max} = 60000$ $[N/m^2]$ has to be obeyed.