MATTHIAS GERDTS

A BRIEF INTRODUCTION INTO SOLVING DIFFERENTIAL EQUATIONS

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Chapter 1

Solving Differential Equations

Problem:
Determine a continuously differentiable function $y$ that satisfies the differential equation

$$\frac{dy(t)}{dt} = f(t,y(t)),$$  \hspace{1cm} (1.1)

for all $t$, where $f$ is a given function of the two variables $t$ and $y$. $t$ is called independent variable and $y$ is called dependent variable.

The derivative $dy/dt$ is often abbreviated by $y'$ or $\dot{y}$, that is

$$\frac{dy(t)}{dt} = y'(t) = \dot{y}(t).$$

1.1 Solution of a Simple Differential Equation

Consider the differential equation

$$\frac{dy(t)}{dt} = ay(t) - b = a \cdot \left( y(t) - \frac{b}{a} \right)$$  \hspace{1cm} (1.2)

with given constants $a \neq 0$ and $b$.

If $a \neq 0$ and $y(t) \neq b/a$ (this would be the equilibrium solution), we can rewrite (1.2) as

$$\frac{1}{y(t) - \frac{b}{a}} \cdot \frac{dy(t)}{dt} = a.$$  \hspace{1cm} (1.3)

Observe that

$$\frac{d}{dt} \left( \ln \left| y(t) - \frac{b}{a} \right| \right) = \frac{1}{y(t) - \frac{b}{a}} \cdot \frac{dy(t)}{dt}$$

by the chain rule. With other words $\ln |y - b/a|$ is the antiderivative of the left side in (1.3). Hence, by integrating both sides in (1.3) w.r.t. $t$, we have

$$\ln \left| y(t) - \frac{b}{a} \right| = at + C,$$  \hspace{1cm} (1.4)

where $C$ is the constant of integration. Taking the exponential of both sides in (1.4) and solving for $y(t)$, we obtain

$$y(t) = \frac{b}{a} + c \cdot \exp(at),$$  \hspace{1cm} (1.5)
where \( c = \pm \exp(C) \) is arbitrary. For \( c = 0 \) the equilibrium solution \( y(t) = b/a \) arises. Subsuming, if \( a \neq 0 \), then all solutions of the differential equation (1.2) are given by

\[
y(t) = \frac{b}{a} + c \cdot \exp(at), \quad c \in \mathbb{R}. \tag{1.6}
\]

This is called general solution.

If we are interested in a particular solution, we have to fix the parameter \( c \). Most often, this is done implicitly by providing an initial condition

\[
y(0) = y_0. \tag{1.7}
\]

The initial condition (1.7) filters out that solution among (1.6) which passes through \( y_0 \) at \( t = 0 \):

\[
y_0 = y(0) = \frac{b}{a} + c \cdot \exp(a \cdot 0) = \frac{b}{a} + c \quad \Rightarrow \quad c = y_0 - \frac{b}{a}. \tag{1.8}
\]

Thus, the solution that fulfills (1.2) and (1.7) is given by

\[
y(t) = \frac{b}{a} + \left( y_0 - \frac{b}{a} \right) \cdot \exp(at).
\]

**Definition 1.1.1** (Initial Value Problem)

A differential equation together with an initial condition form an initial value problem:

\[
\frac{dy(t)}{dt} = f(t, y(t)), \quad y(a) = y_a. \tag{1.9}
\]

1.2 First Order Differential Equations

We investigate explicit ordinary equations of first order

\[
y'(t) = f(t, y(t)). \tag{1.10}
\]

We will describe several methods for solving special subclasses of (1.10). These subclasses are

- linear equations;
- separable equations;
- exact equations.
1.2.1 Linear Equations with Variable Coefficients

We consider the first order linear equation

\[ y'(t) + p(t)y(t) = g(t), \tag{1.11} \]

where \( p \) and \( g \) are given functions of \( t \).

In Section 1.1 we solved a special first order linear differential equation by integration. Therefore, it was necessary to recognize the antiderivative of a certain function.

Equation (1.11) can not be solved directly as it was possible in the example in Section 1.1. But we can try to modify (1.11) such that it becomes integrable. In order to solve (1.11) Leibniz introduced a certain function \( \mu(t) \) – the integrating factor – and multiplied (1.11) with \( \mu(t) \). The integrating factor \( \mu(t) \) must be chosen in such a way that the resulting equation is easily integrable.

**Construction of an integrating factor:**

Step 1: Multiply (1.11) by a yet unknown function \( \mu(t) \):

\[ \mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)g(t). \tag{1.12} \]

Step 2: Observe that

\[ \frac{d}{dt} (\mu(t)y(t)) = \mu(t)y'(t) + \mu'(t)y(t). \tag{1.13} \]

Step 3: Notice, that the left side in (1.12) is equal to the right side in (1.13), if

\[ \mu'(t) = p(t)\mu(t) \tag{1.14} \]

holds. This is a linear differential equation for \( \mu \)!

Step 4: If we assume that \( \mu(t) > 0 \) holds for all \( t \), then it follows from (1.14) that

\[ \frac{d}{dt} \ln(\mu(t)) = \frac{\mu'(t)}{\mu(t)} = p(t). \tag{1.15} \]

By integration we obtain

\[ \ln(\mu(t)) = \int p(t)dt + k \]

with arbitrary constant \( k \). By choosing \( k = 0 \) we obtain the simplest possible positive function for \( \mu \):

\[ \mu(t) = \exp \left( \int p(t)dt \right). \tag{1.16} \]

Notice, that \( \mu \) in fact is positive!
Computing the solution:

From the construction of the integrating factor, that is Equation (1.14), it follows from (1.12) and (1.13) that

\[ \frac{d}{dt} (\mu(t)y(t)) = \mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)g(t). \]

Hence, by integration

\[ \mu(t)y(t) = \int \mu(s)g(s)ds + c. \]

Together with (1.16) we obtain the

**General Solution of (1.11):**

\[
y(t) = \int \frac{\mu(s)g(s)ds + c}{\mu(t)}, \quad \mu(t) = \exp\left(\int p(t)dt\right). \quad (1.17)
\]

**Example 1.2.1**

Solve the initial value problem

\[ ty'(t) + 2y(t) = 4t^2, \quad y(1) = 2. \]

Equivalent for \( t \neq 0 \):

\[
y'(t) + \frac{2}{t} y(t) = 4t, \quad y(1) = 2.
\]

Integrating factor:

\[
\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int 2/tdt\right) = \exp(2 \ln |t|) = t^2.
\]

General solution:

\[
y(t) = \int \frac{\mu(s)g(s)ds + c}{\mu(t)} = \int \frac{s^2 \cdot 4 \cdot sds + c}{t^2} = \frac{t^4 + c}{t^2} = t^2 + \frac{c}{t^2}.
\]
1.2. FIRST ORDER DIFFERENTIAL EQUATIONS

Initial condition:

\[ y(1) = 2 \quad \Rightarrow \quad 2 = 1 + \frac{c}{1} \quad \Rightarrow \quad c = 1. \]

Solution of initial value problem:

\[ y(t) = t^2 + \frac{1}{t^2}, \quad t > 0. \]

Notice, that the solution is unbounded for \( t \to 0 \) and can not be continued beyond \( t = 0 \).

1.2.2 Separable Equations

**Definition 1.2.2** (Separable Equations)

A separable nonlinear differential equation is of the form

\[ M(t) + N(y(t)) \frac{dy(t)}{dt} = 0, \tag{1.18} \]

where \( M \) and \( N \) are given functions of \( t \) and \( y \) respectively.

We will see how separable differential equations can be solved. Therefore, let \( H_1 \) and \( H_2 \) be functions such that

\[ \frac{dH_1}{dt}(t) = M(t), \quad \frac{dH_2}{dy}(y) = N(y), \]

that is \( H_1 \) is an antiderivative of \( M \) w.r.t. \( t \) and \( H_2 \) is an antiderivative of \( N \) w.r.t. \( y \).

Then Equation (1.18) can be rewritten as

\[ \frac{dH_1}{dt}(t) + \frac{dH_2}{dy}(y(t)) \cdot \frac{dy}{dt}(t) = 0. \tag{1.19} \]

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By the chain rule it holds
\[
\frac{dH_2}{dt}(y(t)) = \frac{dH_2}{dy}(y(t)) \cdot \frac{dy}{dt}(t).
\]
Consequently, Equation 1.19 becomes
\[
\frac{d}{dt} [H_1(t) + H_2(y(t))] = 0. \tag{1.20}
\]
Integrating yields
\[
H_1(t) + H_2(y(t)) = c,
\]
where \( c \) is an arbitrary constant.

**Conclusion:**
Any differentiable function \( y \) that satisfies
\[
H_1(t) + H_2(y(t)) = c, \tag{1.21}
\]
is a solution of the separable differential equation \( (1.18) \).

**Remark 1.2.3**

*Unfortunately, Equation \( (1.21) \) defines the solution \( y \) implicitly and not explicitly. Sometimes it is possible to solve \( (1.21) \) for \( y \). Otherwise, numerical methods are needed to approximate \( y(t) \) for given \( t \).*

A given initial condition \( y(t_0) = y_0 \) implies
\[
H_1(t_0) + H_2(y_0) = c.
\]
Substituting \( c \) in \( (1.21) \) gives
\[
[H_1(t) - H_1(t_0)] + [H_2(y(t)) - H_2(y_0)] = 0.
\]
The Fundamental Theorem of Calculus yields
\[
\int_{t_0}^{t} M(s)ds + \int_{y_0}^{y} N(s)ds = 0.
\]

**Conclusion:**
Any differentiable function \( y \) that satisfies
\[
\int_{t_0}^{t} M(s)ds + \int_{y_0}^{y} N(s)ds = 0 \tag{1.22}
\]
solves the initial value problem given by \( (1.18) \) and \( y(t_0) = y_0 \).
Example 1.2.4

Solve the initial value problem

\[
\frac{dy(t)}{dt} = \frac{3t^2 + 4t + 2}{2(y(t) - 1)}, \quad y(0) = -1
\]

and determine the interval in which the solution exists.

Rewrite the equation as

\[
\begin{align*}
-3t^2 - 4t - 2 &= M(t) + 2(y(t) - 1) \quad \text{\(d\text{y}(t)\)} \\
= &N(y(t)) \quad \text{\(d\text{t}\)}
\end{align*}
\]

Evaluating (1.20):

\[
H_1(t) = \int M(t)dt + c_1 = -t^3 - 2t^2 - 2t + c_1
\]

\[
H_2(y) = \int N(y)dy + c_2 = y^2 - 2y + c_2
\]

and

\[-t^3 - 2t^2 - 2t + y(t)^2 - 2y(t) = c,
\]

where \(c = -c_1 - c_2\). Incorporating the initial value gives \(1 + 2 = c\). Hence any solution \(y(t)\) of the initial value problem is implicitly given by

\[
y(t)^2 - 2y(t) = t^3 + 2t^2 + 2t + 3.
\]

Since this equation is quadratic in \(y\) we can solve it for \(y\):

\[
y(t) = 1 \pm \sqrt{t^3 + 2t^2 + 2t + 4}.
\]

Observe, that only the solution with ‘-’ satisfies the initial condition. Therefore, the overall solution is

\[
y(t) = 1 - \sqrt{t^3 + 2t^2 + 2t + 4},
\]

which is defined for all \(t\) with \(t^3 + 2t^2 + 2t + 4 \geq 0\) (otherwise the argument of the root function would be negative). It turns out that this condition is fulfilled for \(t \geq -2\).
For $t \to -2$ the derivative of $y$ tends to infinity.

1.2.3 Exact Equations

**Definition 1.2.5** (Exact Equations)

The differential equation

$$M(t, y(t)) + N(t, y(t))y'(t) = 0 \quad (1.23)$$

is called **exact**, if there is a function $\psi(t, y)$ such that

$$\frac{\partial \psi}{\partial t}(t, y) = M(t, y), \quad \frac{\partial \psi}{\partial y}(t, y) = N(t, y). \quad (1.24)$$

For exact differential equations (1.23) it holds

$$0 = M(t, y(t)) + N(t, y(t))y'(t) = \frac{\partial \psi}{\partial t}(t, y(t)) + \frac{\partial \psi}{\partial y}(t, y(t)) \cdot \frac{dy(t)}{dt} = \frac{d}{dt} [\psi(t, y(t))].$$

Thus, the differential equation (1.23) is equivalent with

$$\frac{d}{dt} [\psi(t, y(t))] = 0. \quad (1.25)$$

Integration yields

$$\psi(t, y(t)) = c, \quad (1.26)$$

where $c$ is an arbitrary constant. Similar to separable equations, Equation (1.26) defines $y$ implicitly.
**Conclusion:**
Any differentiable function $y$ that satisfies
\[ \psi(t, y(t)) = c, \]
solves the differential equation (1.23).

A more convenient characterization of exact differential equations is provided by

**Theorem 1.2.6**
Let the functions $M, N, M_y$ and $N_t$ (subscripts denote the respective partial derivatives) be continuous in the rectangular region
\[ R := \{(t, y)^\top \in \mathbb{R}^2 \mid \alpha < t < \beta, \gamma < y < \delta\}. \]
Then Equation (1.23) is an exact differential equation in $R$ if and only if
\[ M_y(t, y) = N_t(t, y) \quad (1.27) \]
at each point $(t, y)^\top \in R$. That is, there exists a function $\psi$ satisfying (1.24) if and only if $M$ and $N$ satisfy (1.27).

**Example 1.2.7**
Solve the differential equation
\[ \left( y(t) \cos t + 2te^{y(t)} \right) + \left( \sin t + t^2e^{y(t)} - 1 \right) y'(t) = 0. \quad (1.28) \]
Observe that the equation is exact, because
\[ M_y(t, y) = \cos t + 2te^y = N_t(t, y). \]
Thus there is a function $\psi(t, y)$ such that
\[ \psi_t(t, y) = y \cos t + 2te^y = M(t, y), \]
\[ \psi_y(t, y) = \sin t + t^2e^y - 1 = N(t, y). \]
Integrating the first equation w.r.t. $t$, we obtain
\[ \psi(t, y) = y \sin t + t^2e^y + h(y), \quad (1.29) \]
where $h$ is an arbitrary function of $y$ (not of $t$!). Setting $\psi_y = N$ gives
\[ \psi_y = \sin t + t^2e^y + h'(y) = \sin t + t^2e^y - 1. \]
Thus \( h'(y) = -1 \) and \( h(y) = -y \) (the constant of integration can be neglected, because we only need one function \( h \) and not all).

Introducing \( h \) in (1.29) gives

\[
\psi(t, y) = y \sin t + t^2 e^y - y.
\]

The solutions of (1.28) are given implicitly by

\[
y \sin t + t^2 e^y - y = c.
\]

\[\blacksquare\]

**Integrating Factors**

It is sometimes possible to convert a differential equation that is not exact into an exact equation by introducing an integrating factor similar as for linear equations.

**Construction of an integrating factor:**

Step 1: Multiply (1.23) by a yet unknown function \( \mu(t, y) \):

\[
\mu(t, y)M(t, y)dt + \mu(t, y)N(t, y)dy = 0. \tag{1.30}
\]

Step 2: (1.30) is exact if and only if

\[
(\mu M)_y = (\mu N)_t \iff M \mu_y - N \mu_t + (M_y - N_t)\mu = 0. \tag{1.31}
\]

This is a first order partial differential equation for \( \mu \)!

**Consequence:**

If a function \( \mu(t, y) \) that satisfies the partial differential equation (1.31) can be found, then equation (1.30) will be exact and can be solved as described before.

**Problem:** In general it is a hard task to solve (1.31).

**Special case:**

If \( \mu \) is a function of only one of the variables \( t \) or \( y \) (not of both!) it becomes possible to determine \( \mu \) under certain circumstances.

Assume that \( \mu \) is a function of \( t \) only. Then, \( (\mu M)_y = \mu M_y \) and

\[
(\mu M)_y = (\mu N)_t \iff \mu M_y - N \frac{d\mu}{dt} + M_y - N_t \mu = 0 \iff \frac{d\mu}{dt} = \frac{M_y - N_t}{N} \mu.
\]

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If it turns out that the expression \((M_y - N_t)/N\) is a function of \(t\) only, then the last equation is a \textbf{linear and separable first order ordinary differential equation} for \(\mu\) and can be solved as before.

\textbf{Remark 1.2.8}

A similar procedure can be used in the case that \(\mu\) is a function of \(y\) only.

\textbf{Example 1.2.9}

Find an integrating factor for the equation

\[(3ty + y^2) + (t^2 + ty) y' = 0.\]  
(1.32)

Notice, that this equation is not exact. Check, if it has an integrating factor \(\mu\) that depends only on \(t\):

\[
\frac{M_y(t, y) - N_t(t, y)}{N(t, y)} = \frac{3t + 2y - (2t + y)}{t^2 + ty} = \frac{1}{t}.
\]

Thus the integrating factor is

\[\mu'(t) = \frac{\mu(t)}{t} \Rightarrow \mu(t) = t.\]

Multiplying (1.32) with \(\mu\) yields

\[(3t^2y + ty^2) + (t^3 + t^2y) y' = 0,
\]

which is exact. Its solution is given implicitly by

\[t^3y + \frac{1}{2}t^2y^2 = c.\]

\section*{1.3 Second Order Linear Equations}

We consider

\begin{table}[h]
\begin{tabular}{|l|}
\hline
\textbf{second order linear equations:}  \hline
\begin{align*}
y''(t) + p(t)y'(t) + q(t)y(t) &= g(t) \quad \text{(1.33)} \\
\text{or} \\
P(t)y''(t) + Q(t)y'(t) + R(t)y(t) &= G(t) \quad \text{(1.34)}
\end{align*}
\hline
\end{tabular}
\end{table}

where \(p, q, g, P, Q, R\) and \(G\) are given functions of \(t\). If \(P(t) \neq 0\), Eq. (1.33) results from (1.34) by division of \(P(t)\).
Definition 1.3.1

Equation (1.33) is called **homogeneous**, if \( g(t) = 0 \) for all \( t \). Otherwise (1.33) is called nonhomogeneous.

Correspondingly, (1.34) is called **homogeneous**, if \( G(t) = 0 \) for all \( t \), and nonhomogeneous otherwise.

1.3.1 Homogeneous Equations with Constant Coefficients

We investigate homogeneous equations (1.34) where \( P, Q \) and \( R \) are constants, that is

\[
ay''(t) + by'(t) + cy(t) = 0, \quad a, b, c \text{ constant.} \quad (1.35)
\]

Example 1.3.2

Consider

\[
y''(t) - y(t) = 0. \quad (1.36)
\]

\((a = 1, b = 0, c = -1 \text{ in } (1.35)). \) Hence, we have to find functions which have the property, that their second derivative is the same as the function itself.

We know from calculus, that \( y_1(t) = \exp(t) \) is such a function. The same is true for the function \( y_2(t) = \exp(-t). \) Both are solutions of (1.36).

Furthermore, for arbitrary constants \( c_1 \) and \( c_2 \) the functions

\[
\begin{align*}
c_1y_1(t) &= c_1 \exp(t), \\
c_2y_2(t) &= c_2 \exp(-t)
\end{align*}
\]

are solutions of (1.36) as well (check it!).

Finally, any linear combination

\[
y(t) = c_1y_1(t) + c_2y_2(t) = c_1 \exp(t) + c_2 \exp(-t)
\]

of \( y_1 \) and \( y_2 \) with arbitrary constants \( c_1 \) and \( c_2 \) is a solution of (1.36) as well, because

\[
\begin{align*}
y'(t) &= c_1 \exp(t) - c_2 \exp(-t), \\
y''(t) &= c_1 \exp(t) + c_2 \exp(-t) = y(t).
\end{align*}
\]

Initial conditions:

By the way, now we can see, that we need two initial conditions to fix \( c_1 \) and \( c_2. \) For example, let

\[
y(0) = 2, \quad y'(0) = -1.
\]

This leads to the equations \( c_1 + c_2 = 2 \) and \( c_1 - c_2 = -1. \) The solution is given by \( c_1 = 1/2 \) and \( c_2 = 3/2 \) and the function

\[
y(t) = \frac{1}{2} \exp(t) + \frac{3}{2} \exp(-t)
\]
solves the initial value problem.

We adapt the method of the example to solve (1.35) and make the ansatz

\[ y(t) = \exp(rt), \quad r = \text{constant}. \]  

(1.37)

Introduction of the ansatz in (1.35) yields

\[ (ar^2 + br + c)\exp(rt) \neq 0. \]

Division by \( \exp(rt) \) yields

\[ ar^2 + br + c = 0. \]  

(1.38)

**Definition 1.3.3** (characteristic equation)

Equation (1.38) is called **characteristic equation** of the differential equation (1.35).

**Conclusion:**

\[ y(t) = \exp(rt) \] is a solution of (1.35), if and only if \( r \) is a root of the characteristic function (1.38).

The characteristic equation is a quadratic equation in \( r \) and has two roots \( r_1 \) and \( r_2 \) given by

\[ r_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]  

(1.39)

The following combinations may occur:

(i) Both roots are real numbers and different (\( \rightarrow b^2 - 4ac > 0 \), this section).

(ii) Both roots are real but identical (\( \rightarrow b^2 - 4ac = 0 \), section 1.3.3).

(iii) Both are complex conjugates (\( \rightarrow b^2 - 4ac < 0 \), section 1.3.2).

(the case that one root is real and the other is complex can not occur, since the complex conjugate is always a root, too)

**Case (i):**

Let \( r_1 \neq r_2 \) be real roots of the characteristic equation (1.38). Then,

\[ y_1(t) = \exp(r_1t), \quad y_2(t) = \exp(r_2t) \]

are solutions of (1.35). As in the example, all **linear combinations**

\[ y(t) = c_1y_1(t) + c_2y_2(t) = c_1\exp(r_1t) + c_2\exp(r_2t) \]  

(1.40)
of \(y_1\) and \(y_2\) are solutions as well (check it!).

**Initial values:**

We want to find the solution with

\[
y(t_0) = y_0, \quad y'(t_0) = y'_0.
\]  

(1.41)

Introducing these conditions in (1.40) we get

\[
y(t_0) = c_1 \exp(r_1 t_0) + c_2 \exp(r_2 t_0) = y_0, \\
y'(t_0) = c_1 r_1 \exp(r_1 t_0) + c_2 r_2 \exp(r_2 t_0) = y'_0.
\]

This linear equation system has the solution

\[
c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} \exp(-r_1 t_0), \quad c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} \exp(-r_2 t_0).
\]  

(1.42)

Since \(r_1 \neq r_2\) these values are well defined.

Subsuming, we have found so far, that:

(i) All functions \(y\) in (1.40) solve the differential equation (1.35).

(ii) The function \(y\) in (1.40) with constants as in (1.42) solves the initial value problem given by (1.35) and (1.41).

The question is, whether (1.40) are actually all solutions of (1.35) or are there other solutions, which can not be expressed in the ansatz (1.37).

**Example 1.3.4**

*Find the solution of the initial value problem*

\[
y''(t) + 5y'(t) + 6y(t) = 0, \quad y(0) = 2, \quad y'(0) = 3.
\]

From the ansatz \(y(t) = \exp(rt)\) we have the characteristic equation

\[
r^2 + 5r + 6 = (r + 2)(r + 3) = 0 \quad \Rightarrow \quad r_1 = -2, r_2 = -3.
\]

The general solution is

\[
y(t) = c_1 \exp(-2t) + c_2 \exp(-3t).
\]

Initial conditions:

\[
c_1 + c_2 = 2, \quad -2c_1 - 3c_2 = 3 \quad \Rightarrow \quad c_1 = 9, c_2 = -7
\]

Solution:

\[
y(t) = 9 \exp(-2t) - 7 \exp(-3t).
\]

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1.3.2 Complex Roots of the Characteristic Function

In Section 1.3.1 we discussed homogeneous equations with constant coefficients (1.35)

\[ ay''(t) + by'(t) + cy(t) = 0, \quad a, b, c \text{ constant.} \]

By the ansatz \( y(t) = \exp(rt) \), cp. (1.37), we derived the characteristic equation (1.38)

\[ ar^2 + br + c = 0. \]

We observed that \( y(t) = \exp(rt) \) is a solution of (1.35), if and only if \( r \) is a root of the characteristic equation. In Section 1.3.1 we discussed the case that \( r_1 \neq r_2 \) are real-valued roots of the characteristic equation. In this case the general solution was given by

\[ y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t). \]

Now, we consider the case that the discriminant \( b^2 - 4ac \) is negative, cp. (1.39). Then roots \( r_1 \) and \( r_2 \) of (1.35) are \textbf{conjugate complex numbers}, say

\[ r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu, \quad \lambda, \mu \in \mathbb{R}. \quad (1.43) \]

The corresponding solutions are

\begin{align*}
    y_1(t) &= \exp((\lambda + i\mu)t) = \exp(\lambda t) \cdot \exp(i\mu t) \\
    &= \exp(\lambda t) (\cos(\mu t) + i \sin(\mu t)), \quad (1.44) \\
    y_2(t) &= \exp((\lambda - i\mu)t) = \exp(\lambda t) \cdot \exp(-i\mu t) \\
    &= \exp(\lambda t) (\cos(\mu t) - i \sin(\mu t)). \quad (1.45)
\end{align*}

Herein, we used

\begin{center}
\textbf{Euler’s formula:}
\end{center}

\[ \exp(ix) = \cos(x) + i\sin(x), \quad x \in \mathbb{R}. \quad (1.46) \]

Unfortunately, the solutions \( y_1 \) and \( y_2 \) in (1.44) and (1.45) are complex-valued functions. But since our original differential equation has real-valued coefficients, we are only interested in \textbf{real-valued solutions}. According to the Superposition Principle, cp. Theorem ???, any linear combination of \( y_1 \) and \( y_2 \) is a solution as well.

Consider two particular linear combinations:

\begin{align*}
    y_1(t) + y_2(t) &= \exp(\lambda t) (\cos(\mu t) + i \sin(\mu t)) + \exp(\lambda t) (\cos(\mu t) - i \sin(\mu t)) \\
    &= 2 \exp(\lambda t) \cos(\mu t)
\end{align*}
and

\[ y_1(t) - y_2(t) = \exp(\lambda t) (\cos(\mu t) + i \sin(\mu t)) - \exp(\lambda t) (\cos(\mu t) - i \sin(\mu t)) \]
\[ = -2i \exp(\lambda t) \sin(\mu t). \]

If we neglect the constant multipliers 2 and 2i, respectively, (this is possible, because the differential equation is homogeneous) we have obtained a pair of real-valued solutions:

\[ u(t) = \exp(\lambda t) \cos(\mu t), \]
\[ v(t) = \exp(\lambda t) \sin(\mu t). \]

These functions \( u \) and \( v \) are just the real and imaginary parts, respectively, of \( y_1 \).

**Summary:**
If the roots of the characteristic equation (1.38) are complex numbers \( \lambda \pm i\mu \) with \( \mu \neq 0 \) then the general solution of (1.35) is given by

\[ y(t) = c_1 \exp(\lambda t) \cos(\mu t) + c_2 \exp(\lambda t) \sin(\mu t) \quad (1.47) \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

**Example 1.3.5**

Solve

\[ y''(t) + y'(t) + y(t) = 0. \]

Characteristic equation and its roots:

\[ r^2 + r + 1 = 0, \quad r_{1/2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}. \]

General solution

\[ y(t) = c_1 \exp\left( -\frac{t}{2} \right) \cos\left( \frac{\sqrt{3}}{2} t \right) + c_2 \exp\left( -\frac{t}{2} \right) \sin\left( \frac{\sqrt{3}}{2} t \right). \]

Behavior of the solution: decays exponentially since \( \lambda = -1/2 < 0. \)

**Example 1.3.6**

Solve

\[ 16y''(t) - 8y'(t) + 145y(t) = 0, \quad y(0) = -2, \quad y'(0) = 1. \]

Characteristic equation and its roots:

\[ 16r^2 - 8r + 145 = 0, \quad r_{1/2} = \frac{1}{4} \pm i \frac{3}{\mu}. \]
General solution:

\[ y(t) = c_1 \exp\left(\frac{t}{4}\right) \cos(3t) + c_2 \exp\left(\frac{t}{4}\right) \sin(3t). \]

Initial conditions:

\[-2 = y(0) = c_1, \quad 1 = y'(0) = \frac{c_1}{4} + 3c_2 \Rightarrow c_2 = \frac{1}{2}.\]

Solution of initial value problem:

\[ y(t) = -2 \exp\left(\frac{t}{4}\right) \cos(3t) + \frac{1}{2} \exp\left(\frac{t}{4}\right) \sin(3t). \]

Behavior of the solution: grows exponentially since \( \lambda = 1/4 > 0. \)
1.3.3 Repeated Roots

We again consider the homogeneous equation with constant coefficients (1.35)

\[ ay''(t) + by'(t) + cy(t) = 0, \quad a, b, c \text{ constant} \]

and its characteristic equation (1.38)

\[ ar^2 + br + c = 0. \]

In this section we focus on the case, that the two roots \( r_1 \) and \( r_2 \) are real-valued and equal: \( r_1 = r_2 \). In this case that discriminant \( b^2 - 4ac \) is zero and we have

\[ r_1 = r_2 = -\frac{b}{2a}. \]

**Problem:** Both roots yield the same solution

\[ y_1(t) = \exp\left(-\frac{b}{2a}t\right) \]

of (1.35). Hence, we need to construct a second solution in order to obtain a fundamental set of solutions.

**Example 1.3.7**

**Differential equation:**

\[ y''(t) + 4y'(t) + 4y(t) = 0. \] \hspace{1cm} (1.48)

**Characteristic equation:**

\[ r^2 + 4r + 4 = (r + 2)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = -2. \]

**One solution:**

\[ y_1(t) = \exp(-2t). \]

Of course, all multiples of \( y_1(t) \) are also solutions:

\[ c \cdot y_1(t) \text{ are solutions.} \] \hspace{1cm} (1.49)

In order to construct a second linear independent solution we apply the **Method of d’Alembert.**

**Idea of d’Alembert:**

Replace the constant \( c \) in (1.49) by a function \( v(t) \) and try to determine \( v(t) \) such that \( v(t) \cdot y_1(t) \) is a solution of (1.35).

Following the idea of d’Alembert, we have

\[ y(t) = v(t) \cdot y_1(t) = v(t) \exp(-2t), \]

\[ y'(t) = v'(t) \exp(-2t) - 2v(t) \exp(-2t) = \exp(-2t) (v'(t) - 2v(t)), \]

\[ y''(t) = -2 \exp(-2t) (v'(t) - 2v(t)) + \exp(-2t) (v''(t) - 2v'(t)) \]

\[ = \exp(-2t) (-4v'(t) + 4v(t) + v''(t)). \]
Introducing these relations into the differential equation (1.48) and collecting terms, we get

\[ \exp(-2t) (v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t)) = 0 \]

respectively

\[ v''(t) = 0. \]

Integrating twice, we get

\[ v(t) = c_1 t + c_2 \]

where \( c_1 \) and \( c_2 \) are constants of integration. Substituting this \( v(t) \) in (1.49) we obtain

\[ y(t) = c_1 t \exp(-2t) + c_2 \exp(-2t) = y_2(t) + y_1(t) \]

The method of d’Alembert can be applied in general for (1.35). We know one solution:

\[ y_1(t) = \exp \left( -\frac{b}{2a} t \right). \]

Assume

\[ y(t) = v(t)y_1(t) = v(t) \exp(-bt/2a) \]

are solutions. Substituting

\[ y'(t) = v'(t) \exp(-bt/2a) - \frac{b}{2a} v(t) \exp(-bt/2a), \]

\[ y''(t) = v''(t) \exp(-bt/2a) - \frac{b}{a} v'(t) \exp(-bt/2a) + \frac{b^2}{4a^2} v(t) \exp(-bt/2a) \]

in (1.35) and collecting terms (exploiting \( b^2 - 4ac = 0 \)), we obtain finally

\[ v''(t) = 0 \quad \Rightarrow \quad v(t) = c_1 t + c_2. \]

Hence,

\[ y(t) = v(t) \exp(-bt/2a) = c_1 t \exp(-bt/2a) + c_2 \exp(-bt/2a) = y_2(t) + y_1(t). \]

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Summary of Sections 1.3.1, 1.3.2 and 1.3.3:
The solution of the linear homogeneous differential equation with constant coefficients

\[ ay''(t) + by'(t) + cy(t) = 0, \quad a, b, c \text{ constant} \]

is given by:

(i) If \( r_1 \neq r_2 \) are real, then the general solution is

\[ y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t). \]

(ii) If \( r_1 = \lambda + i\mu \) and \( r_2 = \lambda - i\mu \) are complex conjugates, then the general solution is

\[ y(t) = c_1 \exp(\lambda t) \cos(\mu t) + c_2 \exp(\lambda t) \sin(\mu t). \]

(iii) If \( r_1 = r_2 \), then the general solution is

\[ y(t) = c_1 \exp(r_1 t) + c_2 t \exp(r_1 t). \]

Herein, \( r_1 \) and \( r_2 \) denote the roots of the characteristic equation

\[ ar^2 + br + c = 0. \]

1.3.4 Reduction of Order
The method of d’Alembert can be used in a more general form. Let a solution \( y_1 \) of

\[ y''(t) + p(t)y'(t) + q(t)y(t) = 0 \]

be given. Ansatz:

\[ y(t) = v(t)y_1(t). \]

Substituting \( y, y', y'' \) in the differential equation and collecting terms, we obtain the following differential equation for \( v \)

\[ y_1(t)v''(t) + (2y_1'(t) + p(t)y_1(t))v'(t) = 0. \]

Substituting \( w(t) := v'(t) \) we get the first order differential equation for \( w \)

\[ y_1(t)w'(t) + (2y_1'(t) + p(t)y_1(t))w(t) = 0. \]

After solving this first order differential equation for \( w \), we can get \( v \) by integration of \( w \). And then, we have \( y = vy_1 \).
Example 1.3.8

Observe, that \( y_1(t) = 1/t \) is a solution of

\[
2t^2y''(t) + 3ty'(t) - y(t) = 0, \quad t > 0.
\]

Ansatz:

\[
\begin{align*}
y(t) &= v(t)/t, \\
y'(t) &= v'(t)/t - v(t)/t^2, \\
y''(t) &= v''(t)/t - v'(t)/t^2 - v'(t)/t^2 + 2v(t)/t^3.
\end{align*}
\]

Substituting:

\[
\begin{align*}
2t^2 \left( v''(t)/t - 2v'(t)/t^2 + 2v(t)/t^2 \right) + 3t \left( v'(t)/t - v(t)/t^2 \right) - v(t)/t \\
= 2tv''(t) - 4v'(t) + 4v(t)/t + 3v'(t) - 3v(t)/t - v(t)/t \\
= 2tv''(t) - v'(t) = 0.
\end{align*}
\]

Define \( w(t) := v'(t) - 2tw'(t) - w(t) = 0 \). Separation of variables leads to \( v'(t) = w(t) = c\sqrt{t} \). Integration: \( v(t) = \frac{2}{3}ct^{3/2} + k \). Hence,

\[
y(t) = \frac{v(t)}{t} = \frac{2}{3}ct^{1/2} + k \underbrace{\frac{1}{t}}_{=y_1(t)}.
\]

So, \( y_2(t) = t^{1/2} \) is a new independent solution.

\[\square\]

1.3.5 Nonhomogeneous Equations: Method of Undetermined Coefficients

In the preceding sections we considered the homogeneous linear differential equation (1.35). Now we want to solve the nonhomogeneous differential equation

\[
y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \tag{1.50}
\]

where \( p, q, g \) are given continuous functions on the open interval \( I \). The corresponding homogeneous equation is given by

\[
y''(t) + p(t)y'(t) + q(t)y(t) = 0. \tag{1.51}
\]

The following theorem describes the structure of the general solution of (1.50):

\[\textbf{Theorem 1.3.9}\]

The general solution of (1.50) can be written in the form

\[
y(t) = c_1y_1(t) + c_2y_2(t) + Y(t),
\]

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where \( \{y_1, y_2\} \) is a fundamental set of solutions of (1.51), \( c_1, c_2 \) are arbitrary constants, and \( Y \) is some specific solution of (1.50), a so called \textit{particular solution}.

In other words, to determine the solution of (1.50) we need to

(i) find the \textbf{general solution of the homogeneous equation} (1.51), that is \( y_c(t) = c_1y_1(t) + c_2y_2(t) \); (In the preceding sections we discussed how to obtain the general solution for linear equations with constant coefficients.)

(ii) find some \textbf{solution of the nonhomogeneous equation} (1.50), that is \( Y(t) \);

(iii) add together the solutions found in (i) and (ii), that is \( Y(t) + y_c(t) \).

\[ \text{1.3.6 Method of Undetermined Coefficients} \]

We now focus on the determination of a particular solution \( Y \).

The \textbf{Method of Undetermined Coefficients} requires an initial assumption about the form of the particular solution \( Y \).

**Example 1.3.10**

\textit{Find a particular solution of}

\[ y''(t) - 3y'(t) - 4y(t) = 3 \exp(2t). \]

\textit{Assumption:}

\[ Y(t) = A \exp(2t), \quad A \text{ constant.} \]

\textit{Introducing } \( Y \) \textit{in the differential equation:}

\[ 4A \exp(2t) - 6A \exp(2t) - 4A \exp(2t) = 3 \exp(2t) \iff -6A \exp(2t) = 3 \exp(2t) \]

\textit{Thus, } \( A = -\frac{1}{2} \) \textit{and a particular solution is}

\[ Y(t) = -\frac{1}{2} \exp(2t). \]

But, the following example shows, that the initial assumption may be wrong.

**Example 1.3.11**

\textit{Find a particular solution of}

\[ y''(t) - 3y'(t) - 4y(t) = 2 \sin(t). \]

\textit{Assumption:}

\[ Y(t) = A \sin(t), \quad A \text{ constant.} \]
Introducing $Y$ in the differential equation:

$$-A \sin(t) - 3A \cos(t) - 4 \sin(t) = 2 \sin(t) \quad \Leftrightarrow \quad -5A \sin(t) - 3A \cos(t) = 2 \sin(t)$$

$$\Leftrightarrow \quad (2 + 5A) \sin(t) + 3A \cos(t) = 0$$

Since $\sin$ and $\cos$ are linear independent functions, it follows $2 + 5A = 0$ and $3A = 0$.

**Consequence:** Our initial assumption was wrong and there exists no particular solution of the form $A \sin(t)$.

**Next try:** Assume

$$Y(t) = A \sin(t) + B \cos(t), \quad A, B \text{ constant.}$$

Introducing $Y$ into the equation, collecting and comparing coefficients yields the linear equation system

$$-5A + 3B = 2, \quad -3A - 5B = 0 \quad \Rightarrow \quad A = -\frac{5}{17}, B = \frac{3}{17}.$$ 

This time our initial assumption is adequate and a particular solution is

$$Y(t) = -\frac{5}{17} \sin(t) + \frac{3}{17} \cos(t).$$

The method of undetermined coefficients can also be used, if $g$ is a polynomial in $t$ or a product of the above mentioned functions (exp, sin, cos, polynomial). We summarize some ansatz functions for determining a particular solution of

$$ay''(t) + by'(t) + cy(t) = g_i(t):$$

<table>
<thead>
<tr>
<th>$g_i(t)$</th>
<th>$Y_i(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_n$</td>
<td>$t^k (A_0 t^n + A_1 t^{n-1} + \ldots + A_n)$</td>
</tr>
<tr>
<td>$P_n(t) \exp(\alpha t)$</td>
<td>$t^k (A_0 t^n + A_1 t^{n-1} + \ldots + A_n) \exp(\alpha t)$</td>
</tr>
<tr>
<td>$P_n(t) \exp(\alpha t) \sin(\beta t)$</td>
<td>$t^k [(A_0 t^n + A_1 t^{n-1} + \ldots + A_n) \exp(\alpha t) \cos(\beta t) + (B_0 t^n + B_1 t^{n-1} + \ldots + B_n) \exp(\alpha t) \sin(\beta t)]$</td>
</tr>
<tr>
<td>$P_n(t) \exp(\alpha t) \cos(\beta t)$</td>
<td>$t^k [(A_0 t^n + A_1 t^{n-1} + \ldots + A_n) \exp(\alpha t) \cos(\beta t) + (B_0 t^n + B_1 t^{n-1} + \ldots + B_n) \exp(\alpha t) \sin(\beta t)]$</td>
</tr>
</tbody>
</table>
Note:
The number \( s \) is the smallest integer (\( s = 0, 1, \) or \( 2 \)) that will ensure, that no term in \( Y_i(t) \) is solution of the corresponding homogeneous equation.

**Example 1.3.12**

Find a particular solution of

\[
y''(t) + 4y(t) = 3 \cos(2t).
\]

**Assumption:**

\[
Y(t) = A \cos(2t) + B \sin(2t), \quad A, B \text{ constant.}
\]

By introducing \( Y \) in the differential equation we obtain:

\[
(4A - 4A) \cos(2t) + (4B - 4B) \sin(2t) = 3 \cos(2t).
\]

Hence, there is no choice of \( A \) and \( B \) such that this equation is fulfilled.

**Reason:** A fundamental set of solutions for the homogeneous equation is given by \( \{\sin(2t), \cos(2t)\} \). Thus, our initial assumption for the particular solution is actually the general solution of the homogeneous equation and therefore cannot be a solution of the inhomogeneous equation.

If we try

\[
Y(t) = t (A \cos(2t) + B \sin(2t)), \quad A, B \text{ constant}
\]

(that is \( s = 1 \) in the above table) then we obtain

\[
-4A \sin(2t) + 4B \cos(2t) = 3 \cos(2t) \quad \Rightarrow \quad A = 0, B = \frac{3}{4}.
\]

A particular solution is given by

\[
Y(t) = \frac{3}{4} t \sin(2t).
\]

\[
\blacksquare
\]

If the right hand side is a sum of some of the above mentioned functions, then a particular solution can be obtained by splitting the problem. Let \( g(t) = g_1(t) + g_2(t) \) and let \( Y_1 \) be a solution of

\[
ay''(t) + by'(t) + cy(t) = g_1(t)
\]

and \( Y_2 \) a solution of

\[
ay''(t) + by'(t) + cy(t) = g_2(t).
\]
Then $Y_1 + Y_2$ is a solution of

$$ay''(t) + by'(t) + cy(t) = g_1(t) + g_2(t) = g(t).$$

**Example 1.3.13**

Find a particular solution of

$$y''(t) - 3y'(t) - 4y(t) = 3\exp(2t) + 2\sin(t).$$

**Summary:**

To solve the nonhomogeneous equation

$$ay''(t) + by'(t) + cy(t) = g(t), \quad a, b, c \text{ constants}$$

we need to:

(i) find the general solution of the corresponding homogeneous equation $ay''(t) + by'(t) + cy(t) = 0$,

(ii) find a particular solution by the method of undetermined coefficients (if possible),

(iii) add the solutions of (i) and (ii),

(iv) use initial conditions to determine the arbitrary coefficients involved in the general solution.

1.3.7 **Variation of Parameters**

Another method of finding a particular solution is known as variation of parameters. This method is a general one and is not restricted to a particular class of problems (as the method of undetermined coefficients was).
Example 1.3.14
Find a particular solution of
\[ y''(t) + 4y(t) = 3 \csc(t), \quad \csc(t) = \frac{1}{\sin(t)}. \]

Solution of the corresponding homogeneous equation
\[ y_c(t) = c_1 \cos(2t) + c_2 \sin(2t). \]

Variation of parameters:
Replace \( c_1 \) and \( c_2 \) by functions \( u_1(t) \) and \( u_2(t) \), respectively, and determine \( u_1 \) and \( u_2 \) such that
\[ y(t) = u_1(t) \cos(2t) + u_2(t) \sin(2t) \]
is a solution of the nonhomogeneous equation.

Problem:
If we simply introduce \( y \) into the differential equation, we get a single equation for the two unknown functions \( u_1 \) and \( u_2 \). This usually allows more than one solution.

Idea: Formulate a second equation such that there is a unique solution for \( u_1 \) and \( u_2 \) of both equations.

We have
\[ y'(t) = -2u_1(t) \sin(2t) + 2u_2(t) \cos(2t) + u_1'(t) \cos(2t) + u_2'(t) \sin(2t). \]

We impose the second equation
\[ u_1'(t) \cos(2t) + u_2'(t) \sin(2t) = 0. \]

There are two reasons for this: First, the derivative of \( y' \) is simplified. Second, \( y_1' \) only depends of the function \( u_1 \) and \( u_2 \) and not of their derivatives.

By use of (1.54) Equation (1.53) simplifies to
\[ y'(t) = -2u_1(t) \sin(2t) + 2u_2(t) \cos(2t). \]

Differentiating (1.55), we obtain
\[ y''(t) = -4u_1(t) \cos(2t) - 4u_2(t) \sin(2t) - 2u_1'(t) \sin(2t) + 2u_2'(t) \cos(2t). \]

Introducing (1.52) and (1.56) into the differential equation, we obtain
\[ -2u_1'(t) \sin(2t) + 2u_2'(t) \cos(2t) = 3 \csc(t) \]
Recall, that \( u_1 \) and \( u_2 \) also have to fulfill (1.54). So, we have to solve the following first order linear differential equation system for \( u_1 \) and \( u_2 \):

\[
\begin{align*}
 u_1'(t) \cos(2t) + u_2'(t) \sin(2t) & = 0, \\
 -2u_1'(t) \sin(2t) + 2u_2'(t) \cos(2t) & = 3 \csc(t).
\end{align*}
\]

We can solve the first equation w.r.t. \( u_2' \):

\[
u_2'(t) = -u_1'(t) \frac{\cos(2t)}{\sin(2t)}.
\]

Then we replace \( u_2' \) in the second equation and obtain after some transformations:

\[
u_1'(t) = -\frac{3 \csc(t) \sin(2t)}{2} = -3 \cos(t).
\]

With this expression, we can calculate also \( u_2' \):

\[
u_2'(t) = -u_1'(t) \frac{\cos(2t)}{\sin(2t)} = \frac{3}{2} \csc(t) - 3 \sin(t).
\]

Now, we can integrate \( u_1' \) and \( u_2' \). The result is:

\[
u_1(t) = -3 \sin(t) + c_1, \quad u_2(t) = \frac{3}{2} \ln (|\csc(t) - \cot(t)|) + 3 \cos(t) + c_2.
\]

According to (1.52) we have the general solution

\[
y(t) = u_1(t) \cos(2t) + u_2(t) \sin(2t) \\
= -3 \sin(t) \cos(2t) + \frac{3}{2} \ln (|\csc(t) - \cot(t)|) \sin(2t) + 3 \cos(t) \sin(2t) \\
+ c_1 \cos(2t) + c_2 \sin(2t) \\
= \underbrace{3 \sin(t) + \frac{3}{2} \ln (|\csc(t) - \cot(t)|) \sin(2t)}_{\text{particular solution}} + \underbrace{c_1 \cos(2t) + c_2 \sin(2t)}_{\text{general solution of hom. equation}}.
\]