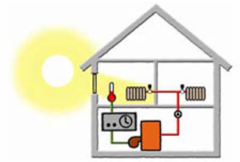
Example: Controlling a Radiator



house/room

radiator

Keep temperature at a given level!

thermostat/controller

Example: Controlling a Radiator

Mathematical model:

$x_1(t)$	:	room temperature at time t
$x_2(t)$	:	temperature of radiator at time t
$u(t)$	:	thermostat/control
		$u > 0$ increases temperature of radiator
		$u < 0$ reduces temperature of radiator
h	:	time period/step size
x_1^*	:	target temperature, $x_1^* = 0$ for simplicity

Rate of change of temperature in the room:

$$x_1'(t) = -x_1(t) + x_2(t)$$

Rate of change of temperature of the radiator:

$$x_2'(t) = u(t)$$

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Example: Controlling a Radiator

Assumption:

It is possible to measure the room temperature x_1.

A first control law:

- If the room temperature is above the target temperature, then reduce temperature of radiator (cool down).
  Mathematically: If $x_1(t) > x_1^*$, then choose $u(t) < 0$.
- If the room temperature is below the target temperature, then increase temperature of radiator (heat up).
  Mathematically: If $x_1(t) < x_1^*$, then choose $u(t) > 0$.

Feedback control law: (proportional controller, P-controller)

$$u(t) = -c \cdot x_1(t), \qquad c \text{ constant}$$

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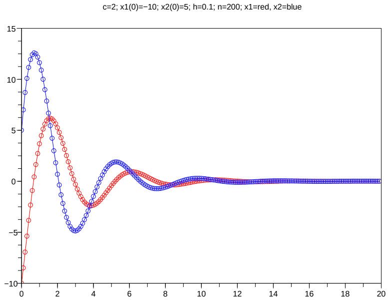
Implementation: (SCILAB, www.scilab.org)

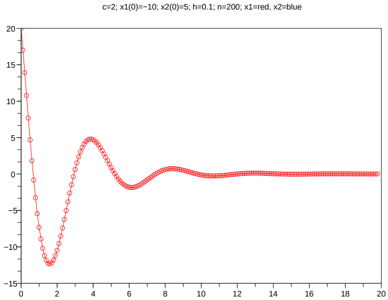
```
function radiator1(x10,x20,h,c,n)
    x1 = zeros(1,n);
    x2 = zeros(1,n);
    u = zeros(1,n-1);
    x1(1) = x10;
    x2(1) = x20;
    for i=1:n-1,
        x1(i+1) = x1(i) + h*(-x1(i)+x2(i));
        u(i) = -c*x1(i);
        x2(i+1) = x2(i) + h*u(i);
    end;
endfunction
```

Example: Controlling a Radiator ($c = 2$)

Target temperature: $x_1^* = 0$

Control: $u(t) = -c \cdot x_1(t)$





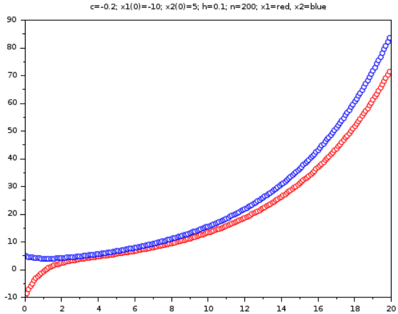
red curve: room temperature
blue curve: radiator temperature

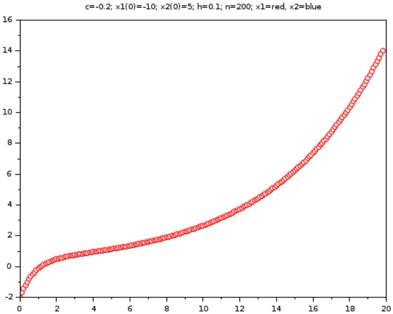
$$
\begin{aligned}
x_1(t+h) &= x_1(t) + h \cdot (-x_1(t) + x_2(t)) \\
x_2(t+h) &= x_2(t) + h \cdot u(t)
\end{aligned}
$$

Example: Controlling a Radiator ($c = -0.2$)

Target temperature: $x_1^* = 0$

Control: $u(t) = -c \cdot x_1(t)$





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blue curve: radiator temperature

$$\begin{aligned} x_1(t+h) &= x_1(t) + h \cdot (-x_1(t) + x_2(t)) \\ x_2(t+h) &= x_2(t) + h \cdot u(t) \end{aligned}$$

Stability and Eigenvalues

Differential equation:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 \\ -c & 0 \end{pmatrix}}_{=A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Ansatz:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \exp(\boldsymbol{\lambda} t) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Eigenvalue problem:

$$\begin{pmatrix} -1 & 1 \\ -c & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \boldsymbol{\lambda} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Solution:

$$\boldsymbol{\lambda} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - c} \qquad \text{(stable, if } Re(\boldsymbol{\lambda}) < 0\text{, i.e. } c > 0\text{)}$$

Special cases:

- $c = 2$: $\quad \boldsymbol{\lambda} = -\frac{1}{2} \pm i\sqrt{\frac{7}{4}}$, $Re(\boldsymbol{\lambda}) = -1/2 < 0$
- $c = -0.2$: $\quad \boldsymbol{\lambda} = -\frac{1}{2} \pm \sqrt{\frac{9}{20}} \approx -\frac{1}{2} \pm 0.671$

Control problems are everywhere...

- autopilot in an airplane
  $\rightarrow$ keep airplane on course
- anti-blocking system (ABS) or electronic stability program (ESP) in a car
  $\rightarrow$ keep car in safe driving conditions
- thermostat in a radiator, heating system or cooling system of a plant
  $\rightarrow$ keep temperature at a certain level
- pressure sensitive valve in a boiler
  $\rightarrow$ keep pressure at a certain level
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