A priori mesh grading for distributed optimal control problems

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Abstract. This paper deals with L^2-error estimates for finite element approximations of control constrained distributed optimal control problems governed by linear partial differential equations. First, general assumptions are stated that allow to prove second order convergence in control, state and adjoint state. Afterwards these assumptions are verified for problems where the solution of the state equation has singularities due to corners or edges in the domain or nonsmooth coefficients in the equation. In order to avoid a reduced convergence order, graded finite element meshes are used.

Mathematics Subject Classification (2000). 65N30, 65N15 .

Keywords. linear-quadratic optimal control problems, control constraints, finite element method, error estimates, graded meshes, anisotropic elements.

1. Introduction

In this paper we consider the optimal control problem

$$\min_{(y,u)\in Y\times U} J(y,u) := \frac{1}{2}\|y-y_d\|_Z^2 + \frac{\nu}{2}\|u\|_U^2, \qquad \text{subject to} \quad y = Su, \quad u \in U^{\text{ad}}, \tag{1.1}$$

where $Z,\ U = U^*$ are Hilbert spaces and Y is a Banach space with $Y \hookrightarrow Z \hookrightarrow Y^*$. We introduce a Banach space $X \hookrightarrow Z$ and demand $y_d \in X$. The operator $S : U \to Y \hookrightarrow U$ is the solution operator of a linear elliptic partial differential equation. We assume ν to be a fixed positive number and $U^{\text{ad}} \subset U$ to be non-empty, convex and closed.

A general review of results is given by Hinze and Rösch in this volume [8]; they shall not be repeated here. We focus on results where the solution of the state equation has singularities due to corners or edges in the domain or nonsmooth coefficients in the equation [1, 3, 4, 16]. In Section 2, general assumptions are stated that allow to prove second order convergence in control, state and adjoint state. Afterwards, in Section 3, these assumptions are verified for a scalar elliptic state equation with discontinuous coefficients in a polygonal domain and isotropic graded meshes, for the Poisson equation in a nonconvex prismatic domain and anisotropic graded meshes and for the Stokes equations as state equation in a nonconvex prismatic domain and a nonconforming discretization on anisotropic meshes.

For further use we recall now part of the theory of control constrained optimal control problems.

Remark 1.1. *Problem* (1.1) *is equivalent to the reduced problem*

$$\min_{u\in U^{\mathrm{ad}}} \hat{J}(u) \tag{1.2}$$

with

$$\hat{J}(u) := J(Su,u) = \frac{1}{2}\|Su - y_d\|_Z^2 + \frac{\nu}{2}\|u\|_U^2.$$

The following theorem can be proved with well known arguments, see, e.g., [10].

Theorem 1.1. *The optimal control problem* (1.1) *has a unique optimal solution* $(\bar{y}, \bar{u})$*. Furthermore, for* S^* *being the adjoint of* S*, the optimality conditions*

$$\bar{y} = S\bar{u}, \tag{1.3}$$

$$\bar{p} = S^*(S\bar{u} - y_d) \tag{1.4}$$

$$\bar{u} \in U^{\mathrm{ad}}, \quad (\nu\bar{u} + \bar{p}, u - \bar{u})_U \geq 0 \quad \forall u \in U^{\mathrm{ad}} \tag{1.5}$$

are necessary and sufficient.

Lemma 1.1. *Let* $\Pi_{U^{\mathrm{ad}}} : U \to U^{\mathrm{ad}}$ *be the projection on* U^{ad}*, i.e.,*

$$\Pi_{U^{\mathrm{ad}}}(u) \in U^{\mathrm{ad}}, \quad \|\Pi_{U^{\mathrm{ad}}}(u) - u\|_U = \min_{v\in U^{\mathrm{ad}}} \|v - u\|_U \quad \forall u \in U.$$

Then the projection formula

$$\bar{u} = \Pi_{U^{\mathrm{ad}}}\left(-\frac{1}{\nu}\bar{p}\right) \tag{1.6}$$

is equivalent to the variational inequality (1.5).

Proof. The assertion is motivated in [11]. A detailed proof is given, e.g., in [7]. The assertion follows from Lemma 1.11 in that book by setting $\gamma = 1/\nu$. □

2. Discretization and error estimates

The results of this subsection are also published in [17]. Detailed proofs can be found there. We consider a family of triangulations $\mathcal{T}_h = \{T\}$ of Ω, that is admissible in Ciarlet's sense, see [5, Assumptions $(\mathcal{T}_h1) - (\mathcal{T}_h5)$]. The operators $S_h : U \to Y_h$ and $S_h^* : Y^* \to Y_h$ are finite element approximations of S and S^*, respectively where Y_h is a suitable finite element space.

2.1. Variational discrete approach

In [6] Hinze introduced a discretization concept for the optimal control problem (1.2) which is based on the discretization of the state space only. The control space is not discretized. Instead, the discrete optimal control $\bar{u}_h^s$ is defined via the variational inequality

$$(\nu\bar{u}_h^s + S_h^*(S_h\bar{u}_h^s - y_d), u - \bar{u}_h^s)_U \geq 0 \quad \forall u \in U^{\mathrm{ad}}. \tag{2.1}$$

In the following we formulate two assumptions that are are sufficient to prove optimal error estimates.

Assumption VAR1. *The operators S_h and S_h^* are bounded, i.e., the inequalities*

$$\|S_h\|_{U\to Y} \leq c \quad \text{and} \quad \|S_h^*\|_{Y^*\to U^*} \leq c$$

are valid.

As usual in numerical analysis the generic constant c used here and in the sequel does not depend on h.

Assumption VAR2. *The estimates*

$$\|(S - S_h)u\|_U \leq ch^2\|u\|_U \quad \forall u \in U,$$
$$\|(S^* - S_h^*)z\|_U \leq ch^2\|z\|_Z \quad \forall z \in Z$$

hold true.

We introduce the optimal discrete state $\bar{y}_h^s := S_h\bar{u}_h^s$ and optimal adjoint state $\bar{p}_h^s := S_h^*(S_h\bar{u}_h^s - y_d)$ and formulate the error estimates in the following theorem.

Theorem 2.1. *Let Assumptions VAR1 and VAR2 hold. Then the estimates*

$$\|\bar{u} - \bar{u}_h^s\|_U \leq ch^2\left(\|\bar{u}\|_U + \|y_d\|_Z\right), \tag{2.2}$$
$$\|\bar{y} - \bar{y}_h^s\|_U \leq ch^2\left(\|\bar{u}\|_U + \|y_d\|_Z\right), \tag{2.3}$$
$$\|\bar{p} - \bar{p}_h^s\|_U \leq ch^2\left(\|\bar{u}\|_U + \|y_d\|_Z\right) \tag{2.4}$$

hold.

Proof. The first estimate is proved in [6]. For the proof of the second assertion we write

$$\begin{aligned}\|\bar{y} - \bar{y}_h^s\|_U &= \|S\bar{u} - S_h\bar{u}_h^s\|_U \\ &\leq \|(S - S_h)\bar{u}\| + \|S_h(\bar{u} - \bar{u}_h^s)\|_U.\end{aligned}$$

Inequality (2.3) follows then from Assumptions VAR1 and VAR2 and (2.2). For the third assertion we can conclude similarly to above

$$\begin{aligned}\|\bar{p} - \bar{p}_h^s\|_U &= \|S^*(Su - y_d) - S_h^*(S_hu_h^s - y_d)\|_U \\ &= \|S^*(S - S_h)\bar{u} + (S^* - S_h^*)S_h\bar{u} + S_h^*S_h(\bar{u} - \bar{u}_h^s) - (S^* - S_h^*)y_d\|_U.\end{aligned}$$

With the triangle inequality the assertion (2.3) follows from the boundedness of S^*, Assumptions VAR1 and VAR2 and (2.2). □

2.2. Postprocessing approach

We consider the reduced problem (1.2) and choose

$$U = Z = L^2(\Omega)^d, \quad Y = H_0^1(\Omega)^d \text{ or } Y = H^1(\Omega)^d,$$

where $d \in \{1, 2, 3\}$ depending on the problem under consideration. As space of admissible controls we use

$$U^{\mathrm{ad}} := \{u \in U : u_a \le u \le u_b \text{ a.e.}\},$$

where $u_a \le u_b$ are constant vectors from $\mathbb{R}^d$. Then the projection in the admissible set reads for a continuous function f as

$$(\Pi_{U^{\mathrm{ad}}} f)(x) := \max(u_a, \min(u_b, f(x))).$$

This formula is to be understood componentwise for vector-valued functions f. We introduce the discrete control space U_h,

$$U_h = \left\{u_h \in U : u_h|_T \in (\mathcal{P}_0)^d \text{ for all } T \in \mathcal{T}_h\right\} \quad \text{and} \quad U_h^{\mathrm{ad}} = U_h \cap U^{\mathrm{ad}}.$$

Then the discretized optimal control problem can be written as

$$\begin{aligned} J_h(\bar u_h) &= \min_{u_h \in U_h^{\mathrm{ad}}} J_h(u_h), \\ J_h(u_h) &:= \frac{1}{2}\|S_h u_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.5}$$

As in the continuous case, this is a strictly convex and radially unbounded optimal control problem. Consequently, (2.5) admits a unique solution $\bar u_h$, that satisfies the necessary and sufficient optimality conditions

$$\begin{aligned} \bar y_h &= S_h \bar u_h, \\ \bar p_h &= S_h^*(\bar y_h - y_d), \\ (\nu \bar u_h + \bar p_h, u_h - \bar u_h)_U &\ge 0 \quad \forall u_h \in U_h^{\mathrm{ad}}. \end{aligned} \tag{2.6}$$

For later use, we introduce the affine operators $Pu = S^*(Su - y_d)$ and $P_h u = S_h^*(S_h u - y_d)$, that maps a given control u to the adjoint state $p = Pu$ and the approximate adjoint state $p_h = P_h u$, respectively.

The approximate control $\tilde u_h$ is constructed as projection of the approximate adjoint state in the set of admissible controls,

$$\tilde u_h = \Pi_{U^{\mathrm{ad}}}\left(-\frac{1}{\nu}\bar p_h\right). \tag{2.7}$$

This postprocessing technique was originally introduced by Meyer and Rösch [14]. In the following we state four rather general assumptions, that allow to prove optimal discretization error estimates. To this end, we first define two projection operators.

Definition 2.2. *For continuous functions f we define the projection R_h in the space $\mathcal{P}_0$ of piecewise constant functions by*

$$(R_h f)(x) := f(S_T) \text{ if } x \in T \tag{2.8}$$

where S_T denotes the centroid of the element T.

The operator Q_h projects L^2-functions g in the space $\mathcal{P}_0$ of piecewise constant functions,

$$(Q_h g)(x) := \frac{1}{|T|}\int_T g(x)\,\mathrm{d}x \text{ for } x \in T. \tag{2.9}$$

Both operators are defined componentwise for vector valued functions.

Assumption PP1. *The discrete solution operators S_h and S_h^* are bounded,*

$$\|S_h\|_{U\to H_h^1(\Omega)^d} \le c, \qquad \|S_h^*\|_{U\to H_h^1(\Omega)^d} \le c,$$
$$\|S_h\|_{U\to L^\infty(\Omega)^d} \le c, \qquad \|S_h^*\|_{U\to L^\infty(\Omega)^d} \le c,$$

with the space

$$H_h^1(\Omega)^d := \left\{ v : \Omega \to \mathbb{R}^d \ : \ \sum_{T\in\mathcal{T}h} \|v\|^2_{H^1(T)^d} < \infty \right\}.$$

Notice that Assumption PP1 implies

$$\|S_h\|_{U\to U} \le c \quad \text{and} \quad \|S_h^*\|_{U\to U} \le c$$

by the embedding $H_h^1(\Omega) \hookrightarrow U$.

Assumption PP2. *The finite element error estimates*

$$\|(S - S_h)u\|_U \le ch^2\|u\|_U \quad \forall u \in U,$$
$$\|(S^* - S_h^*)z\|_U \le ch^2\|z\|_Z \quad \forall z \in Z$$

hold.

Assumption PP3. *The optimal control $\bar u$ is contained in X and the corresponding adjoint state $\bar p$ satisfy the inequality*

$$\|Q_h\bar p - R_h\bar p\|_U \le ch^2\left(\|\bar u\|_X + \|y_d\|_X\right).$$

for a space $X \hookrightarrow U$. In particular, $\bar p$ is continuous, such that $R_h\bar p$ is well defined.

Assumption PP4. *The optimal control $\bar u$ is contained in X and for all functions $\varphi_h \in Y_h$ the inequality*

$$(Q_h\bar u - R_h\bar u, \varphi_h)_U \le ch^2\|\varphi_h\|_{L^\infty(\Omega)^d}\left(\|\bar u\|_X + \|y_d\|_X\right)$$

holds. In particular, $\bar u$ is continuous, such that $R_h\bar u$ is well defined.

First, we recall a property of Q_h that is proved in [4].

Lemma 2.1. *For $f,\ g \in H^1(T)$ the inequality*

$$(f - Q_h f, g)_{L^2(T)} \le ch_T^2|f|_{H^1(T)}|g|_{H^1(T)}$$

is valid where h_T denotes the diameter of the element T.

Now we can prove the following properties of the operator R_h.

Lemma 2.2. *Assume that the Assumptions PP1 and PP4 hold. Then the estimates*

$$\|S_h\bar{u} - S_hR_h\bar{u}\|_U \le ch^2\left(\|\bar{u}\|_X + \|y_d\|_X\right) \tag{2.10}$$

$$\|P_h\bar{u} - P_hR_h\bar{u}\|_U \le ch^2\left(\|\bar{u}\|_X + \|y_d\|_X\right) \tag{2.11}$$

are valid.

Proof. The proof of this Lemma is similar to the one given by Apel and Winkler in [4] in the special case of optimal control of the Poisson equation and a discretization with linear finite elements. A proof under assumptions like PP1 and PP4 for the optimal control of the Stokes equation is given in [16]. A detailed proof for the general case can be found in [17]. □

Lemma 2.3. *The inequality*

$$\nu\|R_h\bar{u} - \bar{u}_h\|_U^2 \le (R_h\bar{p} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_U \tag{2.12}$$

holds.

Proof. This lemma was originally proved in [14] and is based on a combination of the variational inequalities (1.5) and (2.6). □

Now we are able to prove the following supercloseness result.

Theorem 2.3. *Assume that Assumptions PP1–PP4 hold. Then the inequality*

$$\|\bar{u}_h - R_h\bar{u}\|_U \le ch^2\left(\|\bar{u}\|_X + \|y_d\|_X\right)$$

is valid.

Proof. The following proof is similar to the one of Theorem 4.21 of [18] which is given in the context of optimal control of the Poisson equation. We give the details here to illustrate the validity under the Assumptions PP1 – PP4. From Lemma 2.3 we have

$$\begin{aligned}\nu\|\bar{u}_h - R_h\bar{u}\|_U^2 &\le (R_h\bar{p} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_U \\ &= (R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_U + (\bar{p} - P_hR_h\bar{u}, \bar{u}_h - R_h\bar{u})_U \\ &\quad + (P_hR_h\bar{u} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_U. \end{aligned} \tag{2.13}$$

We estimate these three terms separately. For the first term, we use that Q_h is an L^2-projection and get

$$\begin{aligned}(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_U &= (R_h\bar{p} - Q_h\bar{p}, \bar{u}_h - R_h\bar{u})_U + (Q_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_U \\ &= (R_h\bar{p} - Q_h\bar{p}, \bar{u}_h - R_h\bar{u})_U.\end{aligned}$$

The Cauchy-Schwarz inequality yields together with Assumption PP3

$$\begin{aligned}(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_U &\le \|R_h\bar{p} - Q_h\bar{p}\|_U\|\bar{u}_h - R_h\bar{u}\|_U \\ &\le ch^2\left(\|\bar{u}\|_X + \|y_d\|_X\right)\|\bar{u}_h - R_h\bar{u}\|_U.\end{aligned} \tag{2.14}$$

For the second term we apply again the Cauchy-Schwarz inequality and use $\bar{p} = P\bar{u}$, so that we arrive at

$$(\bar{p} - P_hR_h\bar{u}, \bar{u}_h - R_h\bar{u})_U \le \|P\bar{u} - P_hR_h\bar{u}\|_U\|\bar{u}_h - R_h\bar{u}\|_U.$$

With Assumptions PP1 and PP2, Lemma 2.2 and the embedding $X \hookrightarrow U$, one can conclude

$$\begin{aligned}\|P\bar{u} - P_h R_h \bar{u}\|_U &\le \|S_h^*\|_{U\to U}\|S\bar{u} - S_h\bar{u}\|_U + \|S^* y_d - S_h^* y_d\|_U \\ &\quad + \|P_h\bar{u} - P_h R_h \bar{u}\|_U \\ &\le ch^2 \left(\|\bar{u}\|_X + \|y_d\|_X\right),\end{aligned}$$

and therefore

$$(\bar{p} - P_h R_h \bar{u}, u_h - R_h\bar{u})_U \le ch^2 \left(\|\bar{u}\|_X + \|y_d\|_X\right) \|\bar{u}_h - R_h\bar{u}\|_U. \tag{2.15}$$

The third term can simply be omitted since

$$\begin{aligned}(P_h R_h \bar{u} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_U &= (P_h R_h \bar{u} - P_h\bar{u}_h, \bar{u}_h - R_h\bar{u})_U \\ &= (S_h(R_h\bar{u} - \bar{u}_h), S_h(\bar{u}_h - R_h\bar{u}))_U \\ &\le 0. \end{aligned} \tag{2.16}$$

Thus, one can conclude from the estimates (2.13)–(2.16)

$$\nu\|\bar{u}_h - R_h\bar{u}\|_U^2 \le ch^2 \left(\|\bar{u}\|_X + \|y_d\|_X\right) \|u_h - R_h\bar{u}\|_U$$

what yields the assertion. □

Now we are able to formulate the main result of this section.

Theorem 2.4. *Assume that the Assumptions PP1–PP4 hold. Then the estimates*

$$\|\bar{y} - \bar{y}_h\|_U \le ch^2 \left(\|\bar{u}\|_X + \|y_d\|_X\right), \tag{2.17}$$

$$\|\bar{p} - \bar{p}_h\|_U \le ch^2 \left(\|\bar{u}\|_X + \|y_d\|_X\right), \tag{2.18}$$

$$\|\bar{u} - \tilde{u}_h\|_U \le ch^2 \left(\|\bar{u}\|_X + \|y_d\|_X\right) \tag{2.19}$$

are valid.

Proof. In order to prove the first assertion we apply the triangle inequality and get

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_U &= \|S\bar{u} - S_h\bar{u}_h\|_U \\ &\le \|Su - S_h u\|_U + \|S_h\bar{u} - S_h R_h\bar{u}\|_U + \|S_h(R_h\bar{u} - \bar{u}_h)\|_U.\end{aligned}$$

The first term is a finite element error and is estimated in the first inequality of Assumption PP2. For the second term an upper bound is given in Lemma 2.2. For the third term we use the supercloseness result of Theorem 2.3 and the boundedness of S_h given in Assumption PP1. These three estimates yield assertion (2.17). In a similar way one can prove inequality (2.18). By using the Lipschitz continuity of the projection operator, we get

$$\|\bar{u} - \tilde{u}_h\|_U = \left\|\Pi_{U^{\mathrm{ad}}}\left(-\frac{1}{\nu}\bar{p}\right) - \Pi_{U^{\mathrm{ad}}}\left(-\frac{1}{\nu}\bar{p}_h\right)\right\|_U \le \frac{1}{\nu}\|\bar{p} - \bar{p}_h\|_U$$

and inequality (2.19) is a direct consequence of estimate (2.18). □

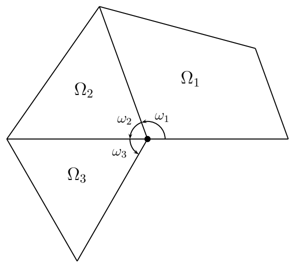

FIGURE 1. Example for subdomains Ω_i in interface problem

3. Examples

3.1. Scalar elliptic state equation with discontinuous coefficients in polygonal domain

We consider the optimal control problem (1.2) with the interface problem for the Laplacian as state equation. We assume that the domain Ω can be partitioned into disjoint, open, polygonal Lipschitz subdomains Ω_i, $i = 1, \dots, n$, on which the diffusion coefficient k has the constant value k_i. Since the singular behaviour is a local phenomenon we restrict our considerations to one corner located at the origin and assume that no singularities occur at the other corners. The interior angle of the subdomains Ω_i at this corner is denoted by ω_i, see Figure 1 for an example. Notice, that for $n = 1$ the state equation reduces to the Poisson equation. This case is treated in [1].

The variational formulation of the state equation reads as

$$\text{Find } y \in V_0 : \qquad a_I(y, v) = (u, v)_{L^2(\Omega)} \qquad \forall v \in V_0 \tag{3.1}$$

with bilinear form $a_I : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$,

$$a_I(y, v) := \int_\Omega k \nabla y \cdot \nabla v$$

and $V_0 := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. We triangulate the domain Ω by an isotropic graded mesh with element size $h_T := \mathrm{diam} T$, that satisfy

$$\begin{aligned} c_1 h^{1/\mu} \le h_T &\le c_2 h^{1/\mu} \qquad && \text{for } r_T = 0, \\ c_1 h r_T^{1-\mu} \le h_T &\le c_2 h r_T^{1-\mu} \qquad && \text{for } r_T > 0, \end{aligned}$$

where μ is the grading parameter and r_T the distance of the triangle T to the corner. We assume that the triangulation $\mathcal{T}_h$ of Ω is aligned with the partition of Ω, i.e., the boundary $\partial\Omega_i$ is made up of edges of triangles in $\mathcal{T}_h$. The discrete state $y_h = S_h u$ is given as solution of

$$\text{Find } y_h \in V_{0h} : \qquad a_I(y_h, v_h) = (f, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_{0h}.$$

with

$$V_{0h} = \{v_h \in C(\bar{\Omega}) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h \text{ and } v_h = 0 \text{ on } \partial\Omega\} \tag{3.2}$$

We introduce the singular exponent λ_I as smallest positive solution of the *Sturm-Liouville* eigenvalue problem

$$-\Phi_i''(\varphi) = \lambda_I^2 \Phi_i(\varphi), \quad \varphi \in (\omega_{i-1}, \omega_i), \quad i = 1, \ldots, n \tag{3.3}$$

with the boundary and interface conditions

$$\begin{aligned} \Phi_1(0) = \Phi_n(\omega) &= 0, \\ \Phi_i(\omega_i) &= \Phi_{i+1}(\omega_i) \quad i = 1, \ldots, n-1, \\ k_i \Phi_i'(\omega_i) &= k_{i+1} \Phi_{i+1}'(\omega_i) \quad i = 1, \ldots, n-1. \end{aligned}$$

This is derived, e.g., in [15, Example 2.29]. In the following we assume that the mesh grading parameter satisfies $\mu < \lambda_I$. For the Poisson equation Assumptions VAR1 and PP1 are proved in [1] under the condition $\mu > 1/2$, which is a reasonable assumption since $\lambda_I > 1/2$ in that case. For $n > 1$ the singular exponent $\lambda_I > 0$ can become arbitrary small such that a more involved proof is necessary. This is given in [17, Lemma 5.16] for smooth coefficients. The proof can be easiliy adapted for the interface problem (3.1). The finite element error estimates of Assumptions VAR2 and PP2 can be verified using interpolation error estimates in weighted Sobolev spaces and the fact that the triangulation is aligned with the partition (Ω_i) of Ω, see [17, Theorem 4.29]. Assumptions PP3 and PP4 follow with similar arguments to the case of the Poisson equation in a prismatic domain (see Subsection 3.2). The proofs are even less complicated since one has not to exploit an anisotropic behaviour of the solution. Detailed proofs are given in [17, Lemma 5.46 and 5.47].

Altogether this means that the results of Theorems 2.1 and 2.4 hold for this example. In [17, Subsection 5.2.3.3] one can find numerical tests for the postprocessing approach that confirm the theoretical findings.

3.2. Examples in prismatic domains

In this section we consider a scalar elliptic equation and the Stokes equation as state equation in a nonconvex prismatic domain and show that Assumptions VAR1–VAR2 and PP1–PP4 hold on anisotropic graded meshes. Let $\Omega = G \times Z$ be a domain with boundary $\partial\Omega$, where $G \subset \mathbb{R}^2$ is a bounded polygonal domain and $Z := (0, z_0) \subset \mathbb{R}$ is an interval. Since situations with more than one singular edge can be reduced to the case of only one reentrant edge by a localization argument, see, e.g., [9], we assume that the cross-section G has only one corner with interior angle $\omega > \pi$ located at the origin.

To construct such an anisotropic graded mesh we first triangulate the two-dimensional domain G by an isotropic graded mesh, then extrude this mesh in x_3-direction with uniform mesh size h and finally divide each of the resulting pentahedra into tetrahedra. If one denotes by $h_{T,i}$ the length of the projection of an element T on the x_i-axis, these element sizes satisfy

$$\begin{aligned} c_1 h^{1/\mu} &\le h_{T,i} \le c_2 h^{1/\mu} && \text{for } r_T = 0,\\ c_1 h r_T^{1-\mu} &\le h_{T,i} \le c_2 h r_T^{1-\mu} && \text{for } r_T > 0,\\ c_1 h &\le h_{T,3} \le c_2 h, \end{aligned} \tag{3.4}$$

for $i = 1, 2$, where r_T is the distance of the element T to the edge,

$$r_T := \inf_{x \in T} \sqrt{x_1^2 + x_2^2},$$

and μ is the grading parameter. Note that these meshes are anisotropic, i.e. their elements are not shape regular. Anisotropic refinement near edges is more effective than grading with isotropic (shape-regular) elements. Nevertheless, the latter is also investigated, see [4].

3.2.1. Scalar elliptic state equation. We consider the optimal control problem (1.2) with operator S that associates the state $y = Su$ to the control u as the solution of the Dirichlet problem

$$\text{Find } y \in V_0 := \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \partial\Omega\} : \quad a_D(y, v) = (f, v)_{L^2(\Omega)} \quad \forall v \in V_0, \tag{3.5}$$

where the bilinear form $a_D : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is defined as

$$a_D(y, v) = \int_\Omega \nabla y \cdot \nabla v,$$

or as the solution of the Neumann problem

$$\text{Find } y \in H^1(\Omega) : \qquad a_N(y, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega) \tag{3.6}$$

where the bilinear form $a_N : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is defined as

$$a_N(y, v) = \int_\Omega \nabla y \cdot \nabla v + \int_\Omega y \cdot v.$$

The corresponding finite element approximation $y_h = S_h u$ is given as the unique solution of

$$a_D(y_h, v_h) = (u, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_{0h}$$

or

$$a_N(y_h, v_h) = (u, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h$$

respectively, where the spaces V_{0h} and V_h are defined as

$$\begin{aligned} V_{0h} &= \left\{ v \in C(\bar\Omega) : v|_T \in \mathcal{P}_1 \ \forall T \in T_h \text{ and } v_h = 0 \text{ on } \partial\Omega \right\},\\ V_h &= \left\{ v \in C(\bar\Omega) : v|_T \in \mathcal{P}_1 \ \forall T \in T_h \right\}. \end{aligned}$$

In the following we assume that the domain Ω is discretized according to (3.4) with grading parameter $\mu < \pi/\omega$. The boundedness of the operators S_h and S_h^* is proved in [18, Subsection 3.6] by using Green function techniques. This means Assumptions VAR1 and PP1 are true. For the proof of the finite element error estimates of Assumptions VAR2 and PP2 one had to deal with quasi-interpolation operators and exploit the regularity properties of the solution. For details we refer to [2]. Assumption PP3 is proved in [17, Lemma 5.38]. The main idea is to split Ω into the sets $K_s = \bigcup_{\{T\in\mathcal{T}_h : r_T=0\}} T$ and $K_r = \Omega\backslash\bar{K}_s$ and to utilize the different regularity properties of the optimal adjoint state $\bar{p}$ in these subdomains. Additionally, one has to take account of the fact that the number of elements in K_s is bounded by ch^{-1}. To prove Assumption PP4 we have to assume that the boundary of the active set has finite two dimensional measure. Furthermore we utilize the boundedness of $r^\beta\nabla\bar{p}$ for $\beta > 1-\lambda$, which can be proved with the help of a regularity result by Maz'ya and Rossmann [13]. For a detailed proof we refer to [17].

In summary this means that Theorems 2.1 and 2.4 hold true in this setting. Numerical tests for the postprocessing approach that confirm the theoretical results are given in [17, Subsection 5.2.2.3].

3.2.2. Stokes equation as state equation. For the Stokes equation the state has actually two components, namely velocity and pressure. Therefore we slightly change the notation. We denote by v the velocity field and by q the pressure. The velocity field v plays the role of the state y in Section 2. Consequently, we substitute y_d by v_d, such that the optimal control problem (1.1) reads as

$$\begin{aligned} J(\bar{u}) &= \min_{u\in U^{\mathrm{ad}}} J(u) \\ J(u) &:= \frac{1}{2}\|Su - v_d\|^2_{L^2(\Omega)^3} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)^3}. \end{aligned} \tag{3.7}$$

The operator S maps the control u to the velocity v as solution of the Stokes equations,

$$\begin{aligned} &\text{Find } (v,q)\in X\times M: \\ &\qquad a(v,\varphi)+b(\varphi,q) = (u,\varphi) \qquad \forall\varphi\in X \\ &\qquad\qquad\qquad b(v,\psi) = 0 \qquad \forall\psi\in M \end{aligned}$$

with the bilinear forms $a : X\times X \to \mathbb{R}$ and $b : X\times M \to \mathbb{R}$ defined as

$$a(v,\varphi) := \sum_{i=1}^{3}\int_\Omega \nabla v_i\cdot\nabla\varphi_i \quad\text{ and }\quad b(\varphi,q) := -\int_\Omega q\nabla\cdot\varphi,$$

and the spaces

$$X = \left\{v\in (H^1(\Omega))^3 : v|_{\partial\Omega} = 0\right\} \quad\text{ and }\quad M = \left\{q\in L^2(\Omega) : \int_\Omega q = 0\right\}.$$

The finite element solution $v_h = S_h u$ is given as the unique solution of

$$\begin{aligned} &\text{Find } (v_h,q_h)\in X_h\times M_h \text{ such that} \\ &\qquad a_h(v_h,\varphi_h) + b_h(\varphi_h,q_h) = (u,\varphi_h) \qquad \forall\varphi_h\in X_h \\ &\qquad\qquad\qquad b_h(v_h,\psi_h) = 0 \qquad \forall\psi_h\in M_h. \end{aligned}$$

with the weaker bilinear forms $a_h : X_h \times X_h \to \mathbb{R}$ and $b_h : X_h \times M_h \to \mathbb{R}$,

$$a_h(v_h, \varphi_h) := \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{3} \int_T \nabla v_{h,i} \cdot \nabla \varphi_{h,i} \quad \text{and} \quad b_h(\varphi_h, p_h) := -\sum_{T \in \mathcal{T}_h} \int_T p_h \nabla \cdot \varphi_h.$$

Here, the i-th component of the vectors v_h and φ_h is denoted by $v_{h,i}$ and $\varphi_{h,i}$, respectively. We approximate the velocity by Crouzeix-Raviart elements,

$$X_h := \left\{ v_h \in L^2(\Omega)^3 : v_h|_T \in (\mathcal{P}_1)^3 \ \forall T, \int_F [v_h]_F = 0 \ \forall F \right\}$$

where F denotes a face of an element and $[v_h]_F$ means the jump of v_h on the face F,

$$[v_h(x)]_F := \begin{cases} \lim_{\alpha \to 0} (v_h(x + \alpha n_F) - v_h(x - \alpha n_F)) & \text{for an interior face F,} \\ v_h(x) & \text{for a boundary face F.} \end{cases}$$

Here n_F is a fixed normal of F. For the approximation of the pressure we use piecewise constant functions, this means

$$M_h := \left\{ q_h \in L^2(\Omega) : q_h|_T \in \mathcal{P}_0 \ \forall T, \int_\Omega q_h = 0 \right\}.$$

For our further considerations we assume that the mesh is graded according to (3.4) with parameter $\mu < \lambda_s$, where λ_s is the smallest positive solution of

$$\sin(\lambda_s \omega) = -\lambda_s \sin \omega.$$

This eigenvalue equation is given in [12, Theorem 6.1]. With the help of a discrete Poincaré inequality in X_h one can prove the boundedness of the operators S_h and S_h^*, comp. [16], such that Assumptions VAR1 and PP1 hold true. The finite element error estimates of Assumptions VAR2 and PP2 are shown in [17, Lemma 4.38]. For the proof of Assumption PP3 one cannot apply the arguments of the scalar elliptic case componentwise since there are only regularity results for the derivatives of the solution in edge direction in $L^2(\Omega)$, but not in $L^p(\Omega)$ for general p. For the proof of this assumption we refer to [17, Lemma 5.57]. The proof of Assumption PP4 is similiar to the one for the scalar elliptic case. The weaker regularity result in edge direction, however, results in the condition, that the number of elements in the set K_1 is bounded by ch^{-2}, which is slightly stronger than the condition in Subsection 3.2.1, where the active set is assumed to have bounded two-dimensional measure.

Altogether one can conclude that Theorems 2.1 and 2.4 are valid. Corresponding numerical tests for the postprocessing approach that confirm the theoretical findings can be found in [16, 17].

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