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A Contribution to 3d-Operational Geodesy

Part 4

The Observation Equations of Satellite Geodesy in the Model of Integrated Geodesy

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CONTENTS

1. INTRODUCTION

This work is the fourth part of a series on the development of ***operational*** or ***integrated geodesy***. After having published the observation equations for geodetic measurements of ***terrestrial*** type ***(Hein, 1982a)***, the concept of a solution ***(Hein, 1982b)***, the first operational software OPERA on the processing of terrestrial data in the integrated adjustment model ***(Hein and Landau, 1983)***, we have tried to outline here in detail the observation equations of satellite geodesy including orbit determination in that unified model. Thereby we refer to the main principle outlined already by ***Moritz (1980, p. 225 to 230)*** and other developments of the Stuttgart school on a slightly different approach to operational geodesy, e.g. ***(Grafarend, 1979; 1981; 1982)***, in particular ***(Grafarend and Livieratos, 1978; Grafarend and Heinz, 1978)*** - to mention only some of the publications. The present state-of-the-art in integrated geodesy is summarized in ***Hein (1986)***.

With the detailed study of the satellite observations and the orbit determination in the integrated model we had two things in mind:

(i) With respect to theory the classical textbook of ***Kaula (1966)*** was and is still the fundamental source for all theoretical developments in satellite geodesy. However, the style and approach of how it was presented, seem to separate space geodesy from the other terrestrial parts - at least at the very first sight. Thus, it was our intention to present satellite geodesy in the same context we have discussed the terrestrial measurements. In addition, we tried to fit it into the integrated geodesy adjustment model in order to end up with a consistent approach to geodesy. This, however, can only be the first trial on the way there.

(ii) The present theory offers also a new computational possibility to orbit determination and the processing of satellite observations. There is no doubt that a realization of the theory into an operational software package still requires a lot of efforts. The interested reader will easily recognize that the determination of appropriate covariances in the general collocation algorithm is the crucial point due to a heavy load of time-consuming calculations for it. However, this does not mean that a numerical realization is impossible. Grid-structured and/or equally-spaced data and subsequent use of Toeplitz matrices can overcome these difficulties.

Some other comments related to the source of the above mentioned difficulties. Although we are publishing this report under the main head line: A Contribution to 3D-Operational Geodesy, it is no longer threedimensional! The consideration of satellite geodesy requires the parameter ***time***, thus it is already a ***fourdimensional*** approach.

The reader who is interested in a quick-look how satellite geodesy fits into the integrated model is recommended to start with paragraphs 4.1 to 4.3 with a short look on the structure of the appropriate observation equations in 4.9.

This report can only be a first step on the inclusion of satellite geodesy in an unified approach to geodesy - although we needed much more time to develop it than earlier anticipated. There is also no doubt that in spite of careful typing and proof-reading still some (or more?) errors might be in it. Looking to the many formulas we hope that the reader has some understanding and tries to assist us in a better version.

2. ORBIT DETERMINATION

For the derivation of the observation equations of satellite geodesy it is necessary to get the geocentric position and velocity vector of the considered satellite. Those vectors are a solution of a system of differential equations of second order. Both, position and velocity vector can be expressed as a nonlinear functional of the gravity potential.

2.1 Equation of motion

Let $\underline{x}(t)$ be a vector in an inertial reference system (for its definition see paragraph 3.1.1) describing the orbit of a satellite. Neglecting relativistic effects the motion of a satellite is given in such a system by Newton's second law,

$$\ddot{\underline{x}}(t) = \nabla_x W\big(\underline{x}(t),t\big) + \underline{f}\big(\underline{x}(t),\dot{\underline{x}}(t),t\big) \qquad (2\text{-}1)$$

where

t ...	is the time parameter,
$\underline{x}(t)$ ...	is the position vector of the satellite,
$\underline{x}(t) = d\underline{x}/dt$ ...	is the velocity vector of the satellite,
$\ddot{\underline{x}}(t) = d^2\underline{x}/dt^2$ ...	is the acceleration vector of the satellite,
$\nabla_x W\big(\underline{x}(t),t\big)$ ...	is the gradient of the earth's gravity potential referring to the inertial coordinate system, and
$\underline{f}\big(\underline{x}(t),\dot{\underline{x}}(t),t\big)$...	is the vector of all other resultant accelerations acting on the satellite.

The explicit functional relationship of the gradient $\nabla_x W\big(\underline{x}(t),t\big)$ with time is due to the fact, that the gravity potential W refers to an earth-fixed coordinate system (see paragraph 3.1.2) moving relatively to the inertial system in time.

2.2 Principle of integration of the equation of motion

Eq. (2-1) is a nonlinear vector differential equation of second order. It corresponds to a nonlinear system of three scalar differential equations of second order.

For the solution of the problem the following decomposition is used:

$$W\big(\underline{x},t\big) = W_0\big(\underline{x}\big) + W_1\big(\underline{x},t\big) \qquad (2\text{-}2)$$

Corresponding to (2-2) we get for the gradient of the gravity potential

$$\nabla_x W\Big(\underline{x}(t)\Big) = \nabla_x W_0\big(\underline{x}\big) + \nabla_x W_1\big(\underline{x},t\big) \qquad (2\text{-}3)$$

where

$$W_0(\underline{x}) = kM/|\underline{x}|^{1)} \qquad (2\text{-}4a)$$

and $$\nabla_x W_0(\underline{x}) = -kM \; \underline{x} \; / \; |\underline{x}|^3 \qquad (2\text{-}4b)$$

kM is the product of the gravitational constant k and the mass M of the earth.

Thus, the decomposition (2-2) results in a radial symmetrical part (2-4a) of the gravity potential and a small disturbance $W_1(\underline{x})$. Furthermore, we note that the corresponding resultant acceleration term $\nabla_x W_1(\underline{x},t)$ is small in comparison to (2-4b).

Inserting (2-3) in (2-1) we get

$$\underline{\ddot{x}}(t) = \nabla_x W_0(\underline{x}) + \nabla_x W_1(\underline{x},t) + \underline{f}(\underline{x},\underline{\dot{x}},t) \qquad (2\text{-}5)$$

Adding the two most right terms in (2-5), and expressing the correspon-ing acceleration vector by $\underline{a}$,

$$\underline{a}(\underline{x},\underline{\dot{x}},t) = \nabla_x W_1(\underline{x},t) + \underline{f}(\underline{x},\underline{\dot{x}},t) \qquad (2\text{-}6)$$

we have

$$\underline{\ddot{x}} = \nabla_x W_0(\underline{x}) + \underline{a}(\underline{x},\underline{\dot{x}},t) \qquad (2\text{-}7)$$

Thus, (2-7) can be considered as a perturbed homogeneous problem, where the perturbing part $\underline{a}(\underline{x},\underline{\dot{x}},t)$ is small in comparison to $\nabla_x W_0(\underline{x})$.

The classical solution starts with the solution of the homogeneous problem, which possesses six integration constants. Afterwards the special solution of the perturbed problem can be derived by the method of variation of constants.

2.2.1 The homogeneous problem

According to (2-7) the homogeneous problem is given by

$$\underline{\ddot{x}} = -kM \; \underline{x} / |\underline{x}|^3 \qquad (2\text{-}8)$$

This problem is known as the so-called Kepler problem. Depending on the

---

1) W_0 refers in principle to an earth-fixed coordinate system $\underline{y}$. According to (3-5a) we have $\underline{x} = \underline{R}(t)\,\underline{y}$ where $\underline{R}^T\underline{R} = \underline{R}\,\underline{R}^T = \underline{I}$. $\underline{I}$ is the identity matrix. Due to the invariance of the norm of a matrix with respect to orthogonal transformations,

$$|\underline{x}| = (\underline{x}^T\underline{x})^{0.5} = (\underline{y}\;\underline{R}^T\underline{R}\;\underline{y})^{0.5} = (\underline{y}^T\underline{y})^{0.5} = |\underline{y}|$$

the relation $\underline{W}_0(\underline{y}) = \underline{W}_0(\underline{x})$ holds.

initial values the solution $\underline{x}(t)$ defines an ellipse, parabola or hyperbola. In case of the motion of a satellite only the ellipse is of interest.

The position vector $\underline{x}$ of the so-called Kepler ellipse is of the following form,

$$\underline{x} = \underline{x}(\underline{u},t) \tag{2-9}$$

where $\underline{u}$ is the vector of six integration constants (orbital elements)

$$\underline{u} = [u_1,u_2,u_3,u_4,u_5,u_6]^T \tag{2-10}$$

2.2.2 The inhomogeneous or perturbed problem (method of variation of constants)

In order to get a special solution of the perturbed problem (2-7), the constants u_i in (2-10) are considered as functions of time, e.g. $u_i = u_i(t)$, $i = \{1,...,6\}$.

Following ***Bucerius (19667, p. 171 ff.)*** a differential equation system of first order can be found for the determination of the unknown parameters $u_i(t)$, $i = \{1,...,6\}$.

Replacing the integration constants in (2-9) by the functions $u_i(t)$ we get

$$\underline{x} = \underline{x}(\underline{u}(t),t) \tag{2-11}$$

In order to get the above-mentioned differential equation system for $\underline{u}(t)$, the position vector $\underline{x}$ has to be differentiated twice with respect to time and to be inserted in (2-7).

For the differentiation the following abbreviations are used:

$$\underline{X} := \frac{\partial \underline{x}}{\partial \underline{u}} = \left[\frac{\partial \underline{x}}{\partial u_1},\frac{\partial \underline{x}}{\partial u_2},\frac{\partial \underline{x}}{\partial u_3},\frac{\partial \underline{x}}{\partial u_4},\frac{\partial \underline{x}}{\partial u_5},\frac{\partial \underline{x}}{\partial u_6}\right] \tag{2-12a}$$

$$\dot{\underline{X}} := \frac{\partial \dot{\underline{x}}}{\partial \underline{u}} = \left[\frac{\partial \dot{\underline{x}}}{\partial u_1},\frac{\partial \dot{\underline{x}}}{\partial u_2},\frac{\partial \dot{\underline{x}}}{\partial u_3},\frac{\partial \dot{\underline{x}}}{\partial u_4},\frac{\partial \dot{\underline{x}}}{\partial u_5},\frac{\partial \dot{\underline{x}}}{\partial u_6}\right] \tag{2-12b}$$

Considering $\underline{u}$ as a vector of coordinates, in which the perturbed problem has to be transformed, then (2-12a) is the Jacobi matrix of the position vector $\underline{x}(t)$, and (2-12b) the Jacobi matrix of the velocity vector $\dot{\underline{x}}(t)$ with respect to the coordinates $\underline{u}$.

The dimension of the matrices $\underline{X}$ and $\dot{\underline{X}}$, respectively, is of the order

$$o(\underline{X}) = 3 \times 6 \tag{2-13a}$$

$$o(\dot{\underline{X}}) = 3 \times 6 \tag{2-13b}$$

Thus, the first and second order derivatives of (2-11) are given by

$$\underline{\dot{x}} = \underline{X}\,\underline{\dot{u}} + \partial\underline{x}/\partial t \qquad (2\text{-}14a)$$

$$\underline{\ddot{x}} = \underline{\dot{X}}\,\underline{\dot{u}} + \partial^2\underline{x}/\partial t^2 \qquad (2\text{-}14a)$$

where

$$\underline{X}\,\underline{\dot{u}} = \underline{o} \qquad (2\text{-}14c)$$

and

$$\underline{\dot{u}} = [\dot{u}_1,\dot{u}_2,\dot{u}_3,\dot{u}_4,\dot{u}_5,\dot{u}_6]^T \qquad (2\text{-}14d)$$

The condition (2-14c) may be chosen due to the fact, that the special solution of (2-7) is determined uniquely by three scalar functions (three components of the position vector $\underline{x}$). Thus, only three independent functions $u_i(t)$ can be determined. Condition (2-14c) is also reasonable since $\underline{\ddot{x}}$ takes the simple form (2-14b). The partial derivatives in (2-14a,b) refer to the parameter time t which (2-11) contains.

Inserting (2-14b) in (2-7), results in

$$\underline{\dot{X}}\,\underline{\dot{u}} + \left(\partial^2\underline{x}/\partial t^2\right) = \nabla_x W_0(\underline{X}) + \underline{a}(\underline{x},\dot{x},t) \qquad (2\text{-}15a)$$

where

$$\left(\partial^2\underline{x}/\partial t^2\right) - \nabla_x W_0(\underline{x}) = \underline{o} \qquad (2\text{-}15b)$$

Eq. (2-15b) expresses the fact that $\partial^2\underline{x}/\partial t^2$ just solves homogeneous problem (2-8).

Using (2-14c) and (2-15a) we get a nonlinear differential equation system of first order for the determination of the vector $\underline{u}(t)$.

$$\underline{X}\,\underline{\dot{u}} = \underline{o} \qquad (2\text{-}16a)$$

$$\underline{\dot{X}}\,\underline{\dot{u}} = \underline{a}\big(\underline{x}(u(t),t),\underline{\dot{x}}(u(t),t),t\big) \qquad (2\text{-}16b)$$

Writing (2-16a,b) in matrix form and solving the system by inversion of the resultant block matrix, yields

$$\underline{\dot{u}} = \begin{bmatrix}\underline{X}\\ \underline{\dot{X}}\end{bmatrix}^{-1}\begin{bmatrix}\underline{o}\\ \underline{a}\end{bmatrix} \qquad (2\text{-}17)$$

Since the first three components of the vector on the right hand side are zero-elements, the following simplification might be reasonable,

$$\underline{Y} = \begin{bmatrix}\underline{X}\\ \underline{\dot{X}}\end{bmatrix}^{-1}[0,0,0,1,1,1]^T \qquad (2\text{-}18)$$

Thus, the following differential equation system may be obtained from (2-17) using (2-18).

$$\dot{\underline{u}}(t) = \underline{Y}\big(\underline{u}(t),t\big)\ \underline{a}\big(\underline{u}(t),t\big) \tag{2-19}$$

In the acceleration term $\underline{a}$ the vectors $\underline{x}$, $\dot{\underline{x}}$ are replaced by $\underline{u}$ according to (2-11).

2.3 The solution of the homogeneous problem

In order to solve the differential equation system (2-19) the explicit definition of the matrix $\underline{Y}\big(\underline{u}(t),t\big)$ and of the resultant acceleration term $\underline{a}\big(\underline{u}(t),t\big)$ is necessary. The last is defined by an acceleration model described in paragraph 3 in detail. Here we are concerned with the matrices $\underline{X}$, $\dot{\underline{X}}$, $\underline{Y}$. For that purpose the solution vector $\underline{x}$ of the homogeneous problem (2-8) is needed.

2.3.1 Kepler orbital elements

According to ***Arnold (1970, p. 16 ff.)*** the solution of (2-8) is given by

$$\underline{x}(t) = \underline{R}_3(-\Omega)\,\underline{R}_1(-i)\,\underline{R}_3(-\omega)\ [r\cos v(t), r\sin v(t), 0]^T \tag{2-20}$$

where

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos v} \tag{2-20a}$$

The position vector $\underline{x}(t)$ defines the so-called Kepler ellipse which spans up an orbital plane as shown in Fig. 1 and defined by

- v ... the true anomaly (angle between the perigee vector and the position vector of the orbital ellipse),
- a ... the semimajor axis of the orbital ellipse,
- Ω ... the length of the ascending node of the orbital plane, measured in the (x_1, x_2)-plane,
- ω ... the argument of the perigee in the orbital plane,
- e ... the excentricity of the orbital ellipse, and
- i ... the inclination of the orbital plane with respect to the (x_1, x_2)-plane.

For the definition of the rotation matrices $\underline{R}_i$ in (2-20) see appendix A.

For the sake of simplicity in computations we introduce the excentric anomaly E instead of the true anomaly v as time-dependent parameter of the orbital ellipse.

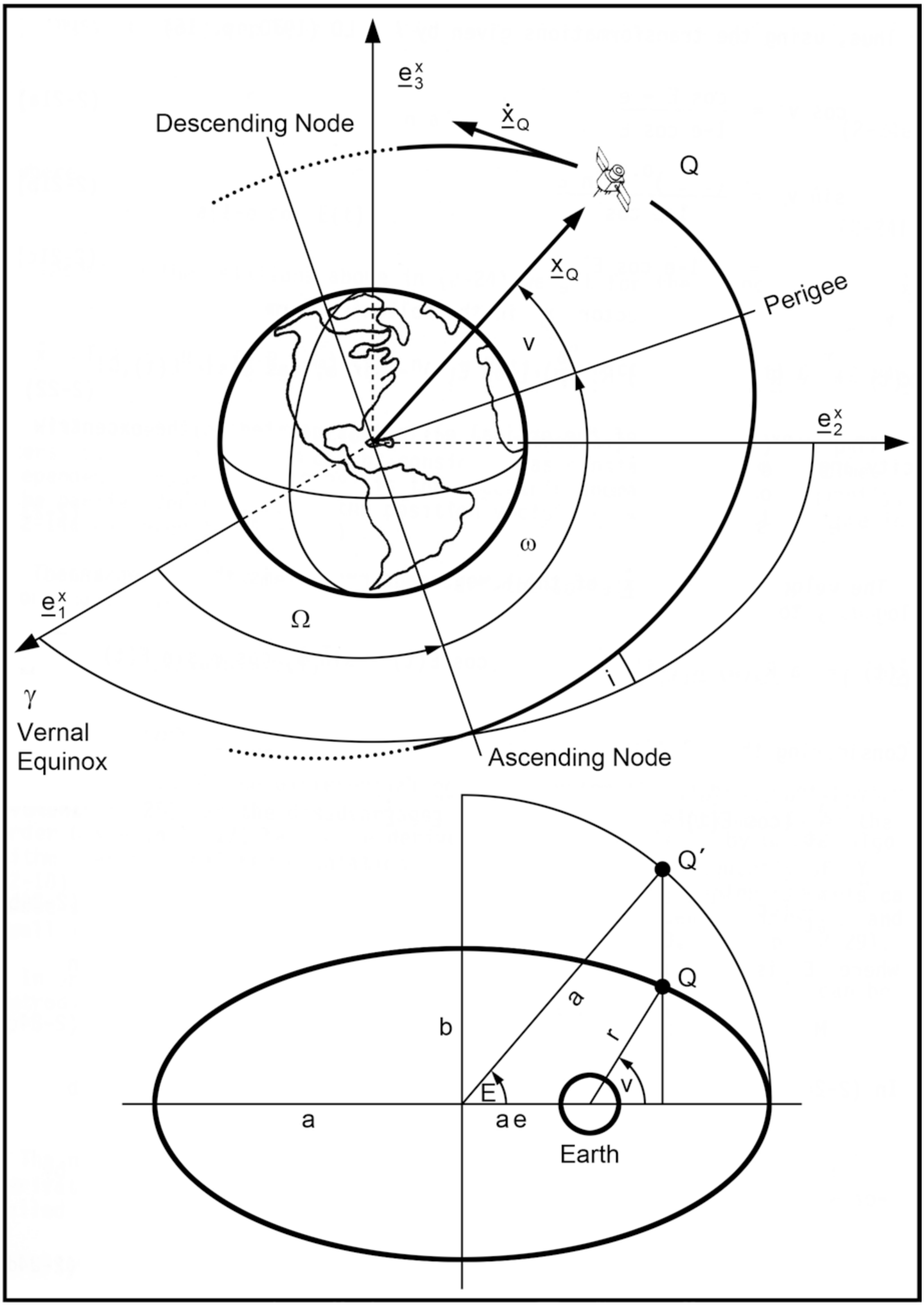

Fig. 1: Kepler orbital elements

Thus, using the transformations given by ***Arnold (1970, p. 15)***

$$\cos v = \frac{\cos E - e}{1 - e \cos E} \tag{2-21a}$$

$$\sin v = \frac{(1-e^2)^{0.5} \sin E}{1 - e \cos E} \tag{2-21b}$$

$$r = a(1 - e \cos E) \tag{2-21c}$$

we get the position vector $\underline{x}$ in the following form

$$\underline{x}(t) = a\, \underline{R}_3(-\Omega)\, R_1(-i)\, R_3(-\omega)\, [\cos E(t) - \sin\tilde{\varphi}, \cos\tilde{\varphi} \sin E(t), 0]^T \tag{2-22}$$

where the excentricity of the orbital ellipse is related to the excentricity angle $\tilde{\varphi}$ by

$$e = \sin\tilde{\varphi} \,. \tag{2-23}$$

The velocity vector $\underline{\dot{x}}$ of the homogeneous problem is then griven analogously to (2-22) by

$$\underline{\dot{x}}(t) = a\, \underline{R}_3(\Omega)\, \underline{R}_1(-i)\, \underline{R}_3(-\omega) \frac{\partial}{\partial t} [\cos E(t) - \sin\tilde{\varphi}, \cos\tilde{\varphi} \sin E(t), 0]^T \tag{2-24}$$

Considering the relations

$$\frac{\partial}{\partial t} [\cos E(t) - \sin\tilde{\varphi}] = -\sin E \cdot \dot{E} \tag{2-24a}$$

$$\frac{\partial}{\partial t} [\cos\tilde{\varphi} \, \sin E(t)] = \cos\tilde{\varphi} \cos E \cdot \dot{E} \tag{2-24b}$$

where E is given as function of the time according to Kepler's equation

$$M = n(t - t_p) = E - e \sin E \tag{2-24c}$$

In (2-24a) we denote by

M ... the mean anomaly, by

t_p ... the time of the perigee passage of the satellite, and by

n ... the mean angular velocity of the satellite,

$$n = \left(\frac{kM}{a^3}\right)^{0.5} . \tag{2-24d}$$

Thus, $\dot{E}$ is given by

$$\dot{E} = \frac{n}{1-e\ \cos E(t)} = n\ a/r \qquad (2\text{-}24e)$$

where

$$r = a\left[1-e\ \cos E(t)\right] . \qquad (2\text{-}24f)$$

Inserting the relations above in (2-24) we get for the velocity vector $\underline{\dot{x}}$

$$\underline{\dot{x}} = \underline{R}_3(\Omega)\ \underline{R}_1(-i)\ \underline{R}_3(-\omega) \left[\frac{-a^2}{r} n\ \sin E(t) , \frac{a^2}{r} n\ \cos\tilde{\varphi}\ \cos E(t) , 0\right]^T \qquad (2\text{-}24g)$$

With respect to the soultion of the homogeneous problem (2-8) the parameters Ω, i, ω, a, e, t_p can be considered as constants. The only time-dependent parameter in (2-22) is the excentric anomaly E. Consequently, the partial derivatives of the position vector $\underline{x}$ with respect to time in (2-14a,b) refer to $E = E(t)$.

The six parameters mentioned above are the so-called ***Kepler elements*** forming the vector $\underline{u}$.

$$\underline{u} = \left[t_p , a , \Omega , \omega , e , i\right]^T \qquad (2\text{-}25)$$

2.3.2 Poincaré orbital elements

The solution of the differential equation system (2-19) by use of Kepler elements (2-25) has the disadvantage, that the inverse block matrix of the order 6 x 6 in (2-17) has to be derived by Cramer's rule or by Gauss' algorithm. An explicit representation of the inverse and consequently of $\underline{Y}$ (2-18) might be extremely difficult. In addition, using Kepler elements can cause singularities of (2-19) when considering small excentricities e and small inclinations i of the satellite (see also ***Arnold, 1970, p. 28-29***).

In order to avoid these disadvantages, ***canonical*** orbital elements can be introduced in the vector $\underline{u}$ (2-25), fulfilling the relation

$$\det \begin{bmatrix} \underline{X} \\ \underline{\dot{X}} \end{bmatrix} = 1$$

The necessary sub-determinants in (2-17) are derived then by partial derivatives. The canonical elements used in this presentation are the so-called ***Poincaré orbital elements***.

Following ***Strumpff (1973, p. 237)*** the Poincaré orbital elements

l, g, h, L, G, H

are given in terms of Kepler elements by

$$l = n\,t + c + \omega + \Omega \qquad \text{where} \quad c = n\,t_p \tag{2-26a}$$

$$g = -2\,L^{0.5}\,\sin\frac{\tilde{\varphi}}{2}\,\sin(\omega+\Omega) \tag{2-26b}$$

$$h = -2\,L^{0.5}\,\cos^{0.5}\tilde{\varphi}\,\sin\frac{i}{2}\,\sin\Omega \tag{2-26c}$$

$$L = \kappa\,a^{0.5} \qquad \text{where} \quad \kappa^2 = kM \tag{2-26d}$$

$$G = 2\,L^{0.5}\,\sin\frac{\tilde{\varphi}}{2}\,\cos(\omega+\Omega) \tag{2-26e}$$

$$H = 2\,L^{0.5}\,\cos^{0.5}\tilde{\varphi}\,\sin\frac{i}{2}\,\cos\Omega \tag{2-26f}$$

and the corresponding inverse relations are

$$M(l,g,G) = n\left(t-t_p\right) = l + \arctan\left(\frac{g}{G}\right) \tag{2-27a}$$

$$a(L) = \frac{L^2}{\kappa^2} \tag{2-27b}$$

$$\Omega(h,H) = -\arctan\left(\frac{h}{H}\right) \tag{2-27c}$$

$$\omega(h,H,g,G) = \arctan\left(\frac{h}{H}\right) - \arctan\left(\frac{g}{G}\right) \tag{2-27d}$$

$$e(g,G,L) = \left[\frac{g^2+G^2}{L} - \frac{\left(g^2+G^2\right)^2}{4L^2}\right]^{0.5} \tag{2-27e}$$

$$i(h,g,H,G,L) = 2\arcsin\left[\frac{h^2+H^2}{4L-2\left(g^2+G^2\right)}\right]^{0.5} \tag{2-27f}$$

The canonical orbital elements (2-26a-f) represent a set of harmonic coordinates. Solving the problem (2-19) in terms of those coordinates implies the advantage of using canonical transformations (see the appendix B; in particular (B-10) and (B-11a to d)).

2.3.3 Modified Poincaré orbital elements

Since the introduced canonical element l (2-26a) is a linear function of time it might not be suited as integration constant in the solution of the homogeneous problem. By modification of l we might be able to construct a set of time-independent non-canonical elements which, however, keep the nice properties of the canonical Poincaré elements as pointed out in appendix B.

In order to derive the necessary modifications we first introduce the so-called ***Hamilton function*** H of the unperturbed Kepler problem (point mass in a radial symmetrical force field)

$$H(\underline{x},\underline{y}) = 0.5(\underline{y}^T\underline{y}) - \frac{\kappa^2}{(\underline{x}^T\underline{x})^{0.5}} \tag{2-28a}$$

where $\underline{y} = \dot{\underline{x}}$ (2-28b)

$$\kappa^2 = kM \tag{2-28c}$$

The Hamilton function of (2-28a) represents the difference between kinetic and potential energy of the Kepler problem.

The equation of motion of that problem is then given by (see also the appendix B, eq. (B-2a,b))

$$\dot{\underline{x}} = \left(\frac{\partial H}{\partial \underline{y}}\right)^T \tag{2-29a}$$

$$\dot{\underline{y}} = -\left(\frac{\partial H}{\partial \underline{x}}\right)^T \tag{2-29b}$$

$$\frac{\partial H}{\partial \underline{y}} = \underline{y}^T \tag{2-29c}$$

$$\frac{\partial H}{\partial \underline{x}} = +\kappa^2 \frac{\underline{x}^T}{(\underline{x}^T\underline{x})^{1.5}} \tag{2-29d}$$

By combining (2-29a-d) we get again the basic equation of motion of (2-8) using $(\dot{\underline{y}} = \ddot{\underline{x}})$

$$\dot{\underline{x}} = \underline{y} \tag{2-30a}$$

$$\dot{\underline{y}} = -\kappa^2 \frac{\underline{x}}{(\underline{x}^T\underline{x})^{1.5}} \tag{2-30b}$$

Expressing H as function of the formally introduced two set of coordinates $\underline{q}$, $\underline{p}$ consisting of the canonical elements (2-26a-f)

$$\underline{q} = [l, g, h]^T \tag{2-31a}$$

$$\underline{p} = [L, G, H]^T \tag{2-31b}$$

yields the differential equations (2-29a,b). The transformation of H with respect to $\underline{q}$ and $\underline{p}$ using $\underline{x} = \underline{x}(\underline{q},\underline{p})$ and $\underline{y} = \underline{y}(\underline{q},\underline{p})$ can be easily derived when inserting the squared velocity of Kepler motion into (2-28a).

According to ***Arnold (1970, p. 13)*** we have

$$\underline{y}^T\underline{y} = \dot{\underline{x}}^T\dot{\underline{x}} = \frac{2\kappa^2}{(\underline{x}^T\underline{x})^{0.5}} - \frac{\kappa^2}{a} \tag{2-32}$$

Inserting (2-32) in (2-28a) yields

$$H\left(\underline{p},\underline{q}\right) = -\frac{\kappa^2}{2a} \tag{2-33}$$

Considering $L = p_1 = \kappa\, a^{0.5}$ (2-26d) and (2-31b) we get the transformed Hamilton function

$$H\left(\underline{p},\underline{q}\right) = -\frac{\kappa^4}{2L^2} = -\frac{\kappa^4}{2p_1^2} \tag{2-34}$$

It can be shown now, that the transformation $\underline{x} = \underline{x}\left(\underline{q},\underline{p}\right)$ and $\underline{y} = \underline{y}\left(\underline{q},\underline{p}\right)$ is canonic, and the Hamilton function does not change.

According to eqs. (B-5a,b) of appendix B the following holds

$$\dot{\underline{q}} = \left(\frac{\partial H}{\partial \underline{p}}\right)^T \tag{2-35a}$$

$$\dot{\underline{p}} = -\left(\frac{\partial H}{\partial \underline{q}}\right)^T \tag{2-35b}$$

The proof can be derived by differencing $\underline{q}$, $\underline{p}$ with respect to time and comparing it with the gradient of the Hamilton function.

$$\dot{\underline{q}} = \left[\dot{l}\,,\,\dot{g}\,,\,\dot{h}\right]^T = [n\,,\,o\,,\,o]^T \tag{2-36a}$$

$$\dot{\underline{p}} = \left[\dot{L}\,,\,\dot{G}\,,\,\dot{H}\right]^T = [o\,,\,o\,,\,o]^T \tag{2-36b}$$

n represents the angular velocity of a satellite moving on a Kepler ellipse

$$n = \frac{\kappa}{a^{1.5}} = \frac{\kappa^4}{L^3} = \frac{\kappa^4}{p_1^3} \tag{2-37}$$

For the gradient of H in (2-35a,b) we find

$$\frac{\partial H}{\partial \underline{p}} = \left[\frac{\partial H}{\partial p_1}\,,\,\frac{\partial H}{\partial p_2}\,,\,\frac{\partial H}{\partial p_3}\right] = \left[\frac{\kappa^4}{p_1^3}\,,\,o\,,\,o\right] \tag{2-38a}$$

$$\frac{\partial H}{\partial \underline{q}} = [o\,,\,o\,,\,o] = \underline{o}^T \tag{2-38b}$$

Comparing (2-38a) and (2-36a) using (2-37), as well as comparing (2-38b) and (2-36b), shows that the Poincaré orbital elements are canonical elements. Again, the first element in (2-36a) shows that $\underline{q}$ is not constant.

A set of constant, time-indepenedent orbital elements $\underline{v}$ can be obtained by subtracting from the coordinate q_1 the time-dependent part $nt = \kappa^4 \cdot t / p_1^3$ corresponding to a shift of the former reference of the potential energy by $\kappa^4 / 2p_1^2$.

$$\underline{v} = \begin{bmatrix} \underline{q} - \frac{\kappa^4}{p_1^3} t \, \underline{e}_1 \\ \underline{p} \end{bmatrix} \tag{2-39a}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} q_1 - \frac{\kappa^4}{p_1^3} t \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{2-39b}$$

where

$$\underline{e}_1 = [1 , 0 , 0]^T \tag{2-39c}$$

Thus, the differentiation of $\underline{v}$ (2-39a) with respect to time yields

$$\dot{\underline{v}} = \begin{bmatrix} \underline{o} \\ \underline{o} \end{bmatrix} \tag{2-39d}$$

proving that $\underline{v}$ are constant coordinates of the homogeneous problem (2-8).

The modification of $\underline{q}$ according to (2-39a) results in a new Hamilton function H^* .

$$H^* = H + \frac{\kappa^4}{2v_4^2} \tag{2-39e}$$

or

$$H^* = H + \frac{\kappa^4}{2p_1^2} \tag{2-39f}$$

The ***modified Poincaré elements*** $\underline{v}$ have the same properties (appendix B) like the original Poincaré elements $\underline{q}$, $\underline{p}$. This should be shown by the following.

(i) ***Proposition:***

The functional determinant $\underline{J}$ (see eq. (8-6) of appendix B) does not change when replacing $\underline{q}$, $\underline{p}$ by $\underline{v}$.

According to (B-6) to (B-10) of appendix B holds

$$\det \underline{J} = \det \underline{J}^T = \det \begin{bmatrix} \frac{\partial \underline{x}^T}{\partial \underline{q}} & \frac{\partial \underline{y}^T}{\partial \underline{q}} \\ \frac{\partial \underline{x}^T}{\partial \underline{p}} & \frac{\partial \underline{y}^T}{\partial \underline{p}} \end{bmatrix} = 1 \qquad (2\text{-}40)$$

Proof:

Let $\partial \underline{q}/\partial \underline{v}$ and $\partial \underline{p}/\partial \underline{v}$ be the corresponding Jacobi matrices of $\underline{p}$ and $\underline{q}$ with respect to (2-39b).

$$\frac{\partial \underline{q}}{\partial \underline{v}} = \left[\frac{\partial \underline{q}}{\partial v_1}, \frac{\partial \underline{q}}{\partial v_2}, \ldots, \frac{\partial \underline{q}}{\partial v_6}\right] = [\underline{I}\ \ \underline{Q}] \qquad (2\text{-}41a)$$

$$\frac{\partial \underline{p}}{\partial \underline{v}} = \left[\frac{\partial \underline{p}}{\partial v_1}, \frac{\partial \underline{p}}{\partial v_2}, \ldots, \frac{\partial \underline{p}}{\partial v_6}\right] = [\underline{0}\ \ \underline{I}] \qquad (2\text{-}41b)$$

where $\underline{I}$ is the identity matrix,

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2\text{-}41c)$$

$$\underline{Q} = \begin{bmatrix} -3\tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2\text{-}41d)$$

$$\underline{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2\text{-}41e)$$

$$\tau = \kappa^4 \, t/v_4^4 = t/a^2 \tag{2-41f}$$

$$\frac{\partial \underline{x}}{\partial \underline{v}} = \left[\frac{\partial \underline{x}}{\partial \underline{q}} \frac{\partial \underline{q}}{\partial \underline{v}} + \frac{\partial \underline{x}}{\partial \underline{p}} \frac{\partial \underline{p}}{\partial \underline{v}} \right] \tag{2-42a}$$

$$\frac{\partial \underline{y}}{\partial \underline{v}} = \left[\frac{\partial \underline{y}}{\partial \underline{q}} \frac{\partial \underline{q}}{\partial \underline{v}} + \frac{\partial \underline{y}}{\partial \underline{p}} \frac{\partial \underline{p}}{\partial \underline{v}} \right] \tag{2-42b}$$

Using (2-41a,b) in (2-42a,b) results in

$$\frac{\partial \underline{x}}{\partial \underline{v}} = \left[\frac{\partial \underline{x}}{\partial \underline{q}} \, , \, \frac{\partial \underline{x}}{\partial \underline{p}} \, , \, \frac{\partial \underline{x}}{\partial \underline{q}} \, \underline{Q} \right] \tag{2-43a}$$

$$\frac{\partial \underline{y}}{\partial \underline{v}} = \left[\frac{\partial \underline{y}}{\partial \underline{q}} \, , \, \frac{\partial \underline{y}}{\partial \underline{p}} \, , \, \frac{\partial \underline{y}}{\partial \underline{q}} \, \underline{Q} \right] \tag{2-43a}$$

The matrix products are explicitly defined by

$$\underline{Q}_1 = \frac{\partial \underline{x}}{\partial \underline{q}} \underline{Q} = -3\tau \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & 0 & 0 \\ \frac{\partial x_2}{\partial q_1} & 0 & 0 \\ \frac{\partial x_3}{\partial q_1} & 0 & 0 \end{bmatrix} = -3\tau \left[\frac{\partial \underline{x}}{\partial q_1} \, , \underline{0} \, , \underline{0} \right] \tag{2-44a}$$

$$\underline{Q}_2 = \frac{\partial \underline{y}}{\partial \underline{q}} \underline{Q} = -3\tau \begin{bmatrix} \frac{\partial y_1}{\partial q_1} & 0 & 0 \\ \frac{\partial y_2}{\partial q_1} & 0 & 0 \\ \frac{\partial y_3}{\partial q_1} & 0 & 0 \end{bmatrix} = -3\tau \left[\frac{\partial \underline{y}}{\partial q_1} \, , \underline{0} \, , \underline{0} \right] \tag{2-44b}$$

Thus, for the determinant of the transformation (2-26a,...,f) we get

$$\det \begin{bmatrix} \dfrac{\partial \underline{x}}{\partial \underline{v}} \\ \\ \dfrac{\partial \underline{y}}{\partial \underline{v}} \end{bmatrix} = \det \begin{bmatrix} \dfrac{\partial \underline{x}}{\partial \underline{q}} , & \dfrac{\partial \underline{x}}{\partial \underline{p}} + \underline{Q}_1 \\ \\ \dfrac{\partial \underline{y}}{\partial \underline{q}} , & \dfrac{\partial \underline{y}}{\partial \underline{p}} + \underline{Q}_2 \end{bmatrix} = \det \begin{bmatrix} \dfrac{\partial \underline{x}}{\partial \underline{q}} , & \dfrac{\partial \underline{x}}{\partial \underline{p}} \\ \\ \dfrac{\partial \underline{y}}{\partial \underline{q}} , & \dfrac{\partial \underline{y}}{\partial \underline{p}} \end{bmatrix}$$

$$= \det \underline{J}^T = \det \underline{J} \qquad (2\text{-}45)$$

The expression after the second equality sign in (2-45) is obtained due to the fact, that by subtraction of $\underline{Q}_1, \underline{Q}_2$ (2-44a,b) just the 3τ - times of the first column is subtracted from the fourth column. Thus, the value of the determinant does not change by this elementary transformation.
q.e.d.

(ii) ***Proposition:***

Let $\underline{J}^T$ be the Jacobi matrix of the position vector $\underline{x}$ and the velocity vector $\underline{y}$ with respect to the coordinates $\underline{v}$.

$$\underline{J}^T = \begin{bmatrix} \dfrac{\partial \underline{x}}{\partial \underline{v}} \\ \\ \dfrac{\partial \underline{y}}{\partial \underline{v}} \end{bmatrix} \qquad (2\text{-}46)$$

The sub-determinants of $\underline{J}^T$ resulting from cancellation of one specific row and column can be derived by partial differentiation according to (B-11a,...,d) of appendix B.

Proof:

Rewriting (2-46) considering (2-45) yields

$$\begin{bmatrix} \frac{\partial x_1}{\partial v_1}, \frac{\partial x_1}{\partial v_2}, \frac{\partial x_1}{\partial v_3}, \frac{\partial x_1}{\partial v_4}, \frac{\partial x_1}{\partial v_5}, \frac{\partial x_1}{\partial v_6} \\ \frac{\partial x_2}{\partial v_1}, \frac{\partial x_2}{\partial v_2}, \frac{\partial x_2}{\partial v_3}, \frac{\partial x_2}{\partial v_4}, \frac{\partial x_2}{\partial v_5}, \frac{\partial x_2}{\partial v_6} \\ \frac{\partial x_3}{\partial v_1}, \frac{\partial x_3}{\partial v_2}, \frac{\partial x_3}{\partial v_3}, \frac{\partial x_3}{\partial v_4}, \frac{\partial x_3}{\partial v_5}, \frac{\partial x_3}{\partial v_6} \\ \frac{\partial y_1}{\partial v_1}, \frac{\partial y_1}{\partial v_2}, \frac{\partial y_1}{\partial v_3}, \frac{\partial y_1}{\partial v_4}, \frac{\partial y_1}{\partial v_5}, \frac{\partial y_1}{\partial v_6} \\ \frac{\partial y_2}{\partial v_1}, \frac{\partial y_2}{\partial v_2}, \frac{\partial y_2}{\partial v_3}, \frac{\partial y_2}{\partial v_4}, \frac{\partial y_2}{\partial v_5}, \frac{\partial y_2}{\partial v_6} \\ \frac{\partial y_3}{\partial v_1}, \frac{\partial y_3}{\partial v_2}, \frac{\partial y_3}{\partial v_3}, \frac{\partial y_3}{\partial v_4}, \frac{\partial y_3}{\partial v_5}, \frac{\partial y_3}{\partial v_6} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1}, \frac{\partial x_1}{\partial q_2}, \frac{\partial x_1}{\partial q_3}, \frac{\partial x_1}{\partial p_1} - 3\tau \frac{\partial x_1}{\partial q_1}, \frac{\partial x_1}{\partial p_2}, \frac{\partial x_1}{\partial p_3} \\ \frac{\partial x_2}{\partial q_1}, \frac{\partial x_2}{\partial q_2}, \frac{\partial x_2}{\partial q_3}, \frac{\partial x_2}{\partial p_1} - 3\tau \frac{\partial x_2}{\partial q_1}, \frac{\partial x_2}{\partial p_2}, \frac{\partial x_2}{\partial p_3} \\ \frac{\partial x_3}{\partial q_1}, \frac{\partial x_3}{\partial q_2}, \frac{\partial x_3}{\partial q_3}, \frac{\partial x_3}{\partial p_1} - 3\tau \frac{\partial x_3}{\partial q_1}, \frac{\partial x_3}{\partial p_2}, \frac{\partial x_3}{\partial p_3} \\ \frac{\partial y_1}{\partial q_1}, \frac{\partial y_1}{\partial q_2}, \frac{\partial y_1}{\partial q_3}, \frac{\partial y_1}{\partial p_1} - 3\tau \frac{\partial y_1}{\partial q_1}, \frac{\partial y_1}{\partial p_2}, \frac{\partial y_1}{\partial p_3} \\ \frac{\partial y_2}{\partial q_1}, \frac{\partial y_2}{\partial q_2}, \frac{\partial y_2}{\partial q_3}, \frac{\partial y_2}{\partial p_1} - 3\tau \frac{\partial y_2}{\partial q_1}, \frac{\partial y_2}{\partial p_2}, \frac{\partial y_2}{\partial p_3} \\ \frac{\partial y_3}{\partial q_1}, \frac{\partial y_3}{\partial q_2}, \frac{\partial y_3}{\partial q_3}, \frac{\partial y_3}{\partial p_1} - 3\tau \frac{\partial y_3}{\partial q_1}, \frac{\partial y_3}{\partial p_2}, \frac{\partial y_3}{\partial p_3} \end{bmatrix} \quad (2\text{-}47)$$

For subsequent considerations we introduce the following schematics:

Let $i = \{1, 2, ..., 6\}$ be the number of a column, and $k = \{1, ..., 3\}$ be the number of a row in (2-47).

For the computation of the sub-determinants

$$\left| \frac{\partial x_k}{\partial v_i} \right| \text{ and } \left| \frac{\partial y_k}{\partial v_i} \right|$$

two cases have to be considered. The matrix element $\partial x_k / \partial v_i$ and $\partial y_k / \partial v_i$ respectively, is a symbol of the place in the matrix $\underline{J}^T$ when cancelling the corresponding row and column for the calculation of the specific sub-determinants.

<u>*Case 1. i ≠ 1*</u>

All columns except the first one can be cancelled. For the sub-determinants for all k we get from (2-47)

$$\left| \frac{\partial x_k}{\partial v_i} \right| = \left| \frac{\partial x_k}{\partial q_i} \right| \quad \text{for} \quad i = 2, 3 \qquad (2\text{-}48a)$$

$$\left| \frac{\partial x_k}{\partial v_i} \right| = \left| \frac{\partial x_k}{\partial p_j} \right| \quad \text{for} \quad \begin{matrix} i = 4, 5, 6 \\ \Leftrightarrow j = 1, 2, 3 \end{matrix} \qquad (2\text{-}48b)$$

$$\left|\frac{\partial y_k}{\partial v_i}\right| = \left|\frac{\partial y_k}{\partial q_i}\right| \quad \text{for} \quad i = 2,3 \tag{2-48c}$$

$$\left|\frac{\partial y_k}{\partial v_i}\right| = \left|\frac{\partial y_k}{\partial p_j}\right| \quad \text{for} \quad \begin{matrix} i = 4,5,6 \\ \Leftrightarrow j = 1,2,3 \end{matrix} \tag{2-48d}$$

The equality of sub-determinants in (2-48a,...,d) is due to the fact, that in case $i \neq 1$ each sub-determinant is determined by the corresponding submatrix containing the first column reduced for the cancelled row-element. Except the trivial case $i \neq 1$ and $i \neq 4$, the fourth column of the sub-determinants is reduced by 3τ - times of the first column. This elementary transformation does not change the value of the determinant.

Example:

$$\left|\frac{\partial x_2}{\partial v_3}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial q_1}, \frac{\partial x_1}{\partial q_2}, \frac{\partial x_1}{\partial p_1} - 3\tau\frac{\partial x_1}{\partial q_1}, \frac{\partial x_1}{\partial p_2}, \frac{\partial x_1}{\partial p_3} \\ \frac{\partial x_3}{\partial q_1}, \frac{\partial x_3}{\partial q_2}, \frac{\partial x_3}{\partial p_1} - 3\tau\frac{\partial x_3}{\partial q_1}, \frac{\partial x_3}{\partial p_2}, \frac{\partial x_3}{\partial p_3} \\ \frac{\partial y_1}{\partial q_1}, \frac{\partial y_1}{\partial q_2}, \frac{\partial y_1}{\partial p_1} - 3\tau\frac{\partial y_1}{\partial q_1}, \frac{\partial y_1}{\partial p_2}, \frac{\partial y_1}{\partial p_3} \\ \frac{\partial y_2}{\partial q_1}, \frac{\partial y_2}{\partial q_2}, \frac{\partial y_2}{\partial p_1} - 3\tau\frac{\partial y_2}{\partial q_1}, \frac{\partial y_2}{\partial p_2}, \frac{\partial y_2}{\partial p_3} \\ \frac{\partial y_3}{\partial q_1}, \frac{\partial y_3}{\partial q_2}, \frac{\partial y_3}{\partial p_1} - 3\tau\frac{\partial y_3}{\partial q_1}, \frac{\partial y_3}{\partial p_2}, \frac{\partial y_3}{\partial p_3} \end{vmatrix} = \left|\frac{\partial x_2}{\partial q_3}\right| \tag{2-49}$$

Case 2. i = 1

The first column is cancelled. For the sub-determinants we get for $k = \{1,2,3\}$ from (2-47)

$$\left|\frac{\partial x_k}{\partial v_1}\right| = \left|\frac{\partial x_k}{\partial q_1}\right| + 3\tau\left|\frac{\partial x_k}{\partial p_1}\right| \tag{2-50a}$$

$$\left|\frac{\partial y_k}{\partial v_1}\right| = \left|\frac{\partial y_k}{\partial q_1}\right| + 3\tau\left|\frac{\partial y_k}{\partial p_1}\right| \tag{2-50b}$$

(2-50a,b) are derived by the following theorem about determinants ($\underline{x}_i$ are here column vectors):

$$\det\left(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k + \underline{c}, \underline{x}_{k+1}, \ldots, \underline{x}_n\right)$$
$$= \det\left(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k, \underline{x}_{k+1}, \ldots, \underline{x}_n\right) + \det\left(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_{k-1}, \underline{c}, \underline{x}_{k+1}, \ldots, \underline{x}_n\right) \tag{2-51}$$

If $i = 1$ the first column is cancelled. The so-derived sub-determinants do not contain a multiple of the first column reduced for the corresponding row element. In the third column of the sub-determinants is the difference of two column vectors. Using (2-51) we get in principle (2-50a,b). Thereby the column vector $-3\tau\left[(\partial\underline{x}/\partial q_1)^T,(\partial\underline{y}/\partial q_1)^T\right]^T$, is situated on the place of the first column by interchanging two columns without change.

Concerning the signs in (2-50a,b) we have to consider the following:

$$\text{sign}\left|\frac{\partial x_k}{\partial q_1}\right| = \text{sign}\left|\frac{\partial x_k}{\partial v_1}\right| \tag{2-51a}$$

$$\text{sign}\left|\frac{\partial x_k}{\partial p_1}\right| = -\text{sign}\left|\frac{\partial x_k}{\partial v_1}\right| \tag{2-51b}$$

$$\text{sign}\left|\frac{\partial y_k}{\partial v_1}\right| = \text{sign}\left|\frac{\partial y_k}{\partial q_1}\right| \tag{2-51c}$$

$$\text{sign}\left|\frac{\partial y_k}{\partial v_1}\right| = -\text{sign}\left|\frac{\partial y_k}{\partial p_1}\right| \tag{2-51d}$$

$j = \{1,2,...,6\}$

Let k be an arbitrary row in (2-47) again. Then the elements

$$\frac{\partial x_k}{\partial v_1},\frac{\partial y_k}{\partial v_1}, \text{ and } \frac{\partial x_k}{\partial q_1},\frac{\partial y_k}{\partial q_1} \qquad k = \{1,2,3\}$$

form the first column, whereas the elements

$$\frac{\partial x_k}{\partial p_1},\frac{\partial y_k}{\partial p_1} \qquad k = \{1,2,3\}$$

form the fourth column. Thus, it holds

$$\begin{aligned}
\text{sign}\left|\frac{\partial x_k}{\partial v_1}\right| &= (-1)^{1+j} = \text{sign}\left|\frac{\partial x_k}{\partial q_1}\right| \\
\text{sign}\left|\frac{\partial y_k}{\partial v_1}\right| &= (-1)^{1+j} = \text{sign}\left|\frac{\partial y_k}{\partial q_1}\right| \\
\text{sign}\left|\frac{\partial x_k}{\partial p_1}\right| &= \text{sign}\left|\frac{\partial y_k}{\partial p_1}\right| = (-1)^{4+j}
\end{aligned} \tag{2-52}$$

By $(-1)^{4+k} = (-1)\,(-1)^{1+k}$ the change of sign of the factor -3τ in (2-50a,b) can be explained. This holds also for (2-51b,d)

Example:

$$\left|\frac{\partial x_1}{\partial v_1}\right| = \begin{vmatrix} \frac{\partial x_2}{\partial q_2}, & \frac{\partial x_2}{\partial q_3}, & \frac{\partial x_2}{\partial p_1} - 3\tau\frac{\partial x_2}{\partial q_1}, & \frac{\partial x_2}{\partial p_2}, & \frac{\partial x_2}{\partial p_3} \\ \frac{\partial x_3}{\partial q_2}, & \frac{\partial x_3}{\partial q_3}, & \frac{\partial x_3}{\partial p_1} - 3\tau\frac{\partial x_3}{\partial q_1}, & \frac{\partial x_3}{\partial p_2}, & \frac{\partial x_3}{\partial p_3} \\ \frac{\partial y_1}{\partial q_2}, & \frac{\partial y_1}{\partial q_3}, & \frac{\partial y_1}{\partial p_1} - 3\tau\frac{\partial y_1}{\partial q_1}, & \frac{\partial y_1}{\partial p_2}, & \frac{\partial y_1}{\partial p_3} \\ \frac{\partial y_2}{\partial q_2}, & \frac{\partial y_2}{\partial q_3}, & \frac{\partial y_2}{\partial p_1} - 3\tau\frac{\partial y_2}{\partial q_1}, & \frac{\partial y_2}{\partial p_2}, & \frac{\partial y_2}{\partial p_3} \\ \frac{\partial y_3}{\partial q_2}, & \frac{\partial y_3}{\partial q_3}, & \frac{\partial y_3}{\partial p_1} - 3\tau\frac{\partial y_3}{\partial q_1}, & \frac{\partial y_3}{\partial p_2}, & \frac{\partial y_3}{\partial p_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x_2}{\partial q_2}, & \frac{\partial x_2}{\partial q_3}, & \frac{\partial x_2}{\partial p_1}, & \frac{\partial x_2}{\partial p_2}, & \frac{\partial x_2}{\partial p_3} \\ \frac{\partial x_3}{\partial q_2}, & \frac{\partial x_3}{\partial q_3}, & \frac{\partial x_3}{\partial p_1}, & \frac{\partial x_3}{\partial p_2}, & \frac{\partial x_3}{\partial p_3} \\ \frac{\partial y_1}{\partial q_2}, & \frac{\partial y_1}{\partial q_3}, & \frac{\partial y_1}{\partial p_1}, & \frac{\partial y_1}{\partial p_2}, & \frac{\partial y_1}{\partial p_3} \\ \frac{\partial y_2}{\partial q_2}, & \frac{\partial y_2}{\partial q_3}, & \frac{\partial y_2}{\partial p_1}, & \frac{\partial y_2}{\partial p_2}, & \frac{\partial y_2}{\partial p_3} \\ \frac{\partial y_3}{\partial q_2}, & \frac{\partial y_3}{\partial q_3}, & \frac{\partial y_3}{\partial p_1}, & \frac{\partial y_3}{\partial p_2}, & \frac{\partial y_3}{\partial p_3} \end{vmatrix} - 3\tau \begin{vmatrix} \frac{\partial x_2}{\partial q_2}, & \frac{\partial x_2}{\partial q_3}, & \frac{\partial x_2}{\partial q_1}, & \frac{\partial x_2}{\partial p_2}, & \frac{\partial x_2}{\partial p_3} \\ \frac{\partial x_3}{\partial q_2}, & \frac{\partial x_3}{\partial q_3}, & \frac{\partial x_3}{\partial q_1}, & \frac{\partial x_3}{\partial p_2}, & \frac{\partial x_3}{\partial p_3} \\ \frac{\partial y_1}{\partial q_2}, & \frac{\partial y_1}{\partial q_3}, & \frac{\partial y_1}{\partial q_1}, & \frac{\partial y_1}{\partial p_2}, & \frac{\partial y_1}{\partial p_3} \\ \frac{\partial y_2}{\partial q_2}, & \frac{\partial y_2}{\partial q_3}, & \frac{\partial y_2}{\partial q_1}, & \frac{\partial y_2}{\partial p_2}, & \frac{\partial y_2}{\partial p_3} \\ \frac{\partial y_3}{\partial q_2}, & \frac{\partial y_3}{\partial q_3}, & \frac{\partial y_3}{\partial q_1}, & \frac{\partial y_3}{\partial p_2}, & \frac{\partial y_3}{\partial p_3} \end{vmatrix}$$

$$
= \begin{vmatrix}
\frac{\partial x_2}{\partial q_2}, \frac{\partial x_2}{\partial q_3}, \frac{\partial x_2}{\partial p_1}, \frac{\partial x_2}{\partial p_2}, \frac{\partial x_2}{\partial p_3} \\
\frac{\partial x_3}{\partial q_2}, \frac{\partial x_3}{\partial q_3}, \frac{\partial x_3}{\partial p_1}, \frac{\partial x_3}{\partial p_2}, \frac{\partial x_3}{\partial p_3} \\
\frac{\partial y_1}{\partial q_2}, \frac{\partial y_1}{\partial q_3}, \frac{\partial y_1}{\partial p_1}, \frac{\partial y_1}{\partial p_2}, \frac{\partial y_1}{\partial p_3} \\
\frac{\partial y_2}{\partial q_2}, \frac{\partial y_2}{\partial q_3}, \frac{\partial y_2}{\partial p_1}, \frac{\partial y_2}{\partial p_2}, \frac{\partial y_2}{\partial p_3} \\
\frac{\partial y_3}{\partial q_2}, \frac{\partial y_3}{\partial q_3}, \frac{\partial y_3}{\partial p_1}, \frac{\partial y_3}{\partial p_2}, \frac{\partial y_3}{\partial p_3}
\end{vmatrix}
- 3\tau
\begin{vmatrix}
\frac{\partial x_2}{\partial q_1}, \frac{\partial x_2}{\partial q_2}, \frac{\partial x_2}{\partial q_3}, \frac{\partial x_2}{\partial p_2}, \frac{\partial x_2}{\partial p_3} \\
\frac{\partial x_3}{\partial q_1}, \frac{\partial x_3}{\partial q_2}, \frac{\partial x_3}{\partial q_3}, \frac{\partial x_3}{\partial p_2}, \frac{\partial x_3}{\partial p_3} \\
\frac{\partial y_1}{\partial q_1}, \frac{\partial y_1}{\partial q_2}, \frac{\partial y_1}{\partial q_3}, \frac{\partial y_1}{\partial p_2}, \frac{\partial y_1}{\partial p_3} \\
\frac{\partial y_2}{\partial q_1}, \frac{\partial y_2}{\partial q_2}, \frac{\partial y_2}{\partial q_3}, \frac{\partial y_2}{\partial p_2}, \frac{\partial y_2}{\partial p_3} \\
\frac{\partial y_3}{\partial q_1}, \frac{\partial y_3}{\partial q_2}, \frac{\partial y_3}{\partial q_3}, \frac{\partial y_3}{\partial p_2}, \frac{\partial y_3}{\partial p_3}
\end{vmatrix}
$$

$$
= \left|\frac{\partial x_1}{\partial q_1}\right| - 3\tau(-1)\left|\frac{\partial x_1}{\partial p_1}\right| = \left|\frac{\partial x_1}{\partial q_1}\right| + 3\tau\left|\frac{\partial x_1}{\partial p_1}\right| \qquad (2\text{-}53)
$$

Using eqs. (B-11a,...,d) of appendix B yields the important formula for the determination of the sub-determinants.

For $i \neq 1$:

$$
\left|\frac{\partial x_k}{\partial v_i}\right| = \left|\frac{\partial x_k}{\partial q_i}\right| = \frac{\partial y_k}{\partial p_i} \quad \text{for} \quad i = 2,3 \qquad (2\text{-}54a)
$$

$$
\left|\frac{\partial x_k}{\partial v_i}\right| = \left|\frac{\partial x_k}{\partial p_j}\right| = \frac{\partial y_k}{\partial q_j} \quad \text{for} \quad i = 4,5,6 \Leftrightarrow j = 1,2,3 \qquad (2\text{-}54b)
$$

$$
\left|\frac{\partial y_k}{\partial v_i}\right| = \left|\frac{\partial y_k}{\partial q_i}\right| = \frac{\partial x_k}{\partial p_i} \quad \text{for} \quad i = 2,3 \qquad (2\text{-}54c)
$$

$$
\left|\frac{\partial y_k}{\partial v_i}\right| = \left|\frac{\partial y_k}{\partial p_j}\right| = \frac{\partial x_k}{\partial q_j} \quad \text{for} \quad i = 4,5,6 \Leftrightarrow j = 1,2,3 \qquad (2\text{-}54d)
$$

For $i = 1$:

$$
\left|\frac{\partial x_k}{\partial v_1}\right| = \left|\frac{\partial x_k}{\partial q_1}\right| + 3\tau\left|\frac{\partial x_k}{\partial p_1}\right| = \frac{\partial y_k}{\partial p_1} - 3\tau\frac{\partial y_k}{\partial q_1} \qquad (2\text{-}54e)
$$

$$
\left|\frac{\partial y_k}{\partial v_1}\right| = \left|\frac{\partial y_k}{\partial q_1}\right| + 3\tau\left|\frac{\partial y_k}{\partial p_1}\right| = \frac{\partial x_k}{\partial p_1} - 3\tau\frac{\partial x_k}{\partial q_1} \qquad (2\text{-}54f)
$$

Corresponding formulas for the plane case can be found in ***Stumpff (1974, p. 247 ff.)***.

2.4 The solution of the inhomogeneous or perturbed problem

2.4.1 Transformation of the inhomogeneous problem into modified Poincaré orbital elements

The inhomogeneous problem is given by (2-16a,b) as a first order differential equation system in $\underline{u}$.

$$X(u)\dot{\underline{u}} = \underline{o} \tag{2-55a}$$

$$\dot{\underline{X}}(\underline{u})\dot{\underline{u}} = \underline{a}(\underline{u},t) \tag{2-55b}$$

$\underline{u}$ is the vector of Kepler elements (2-25). Transforming $\underline{u} = \underline{u}(\underline{v})$ into the modified Poincaré orbital elements $\underline{v}$ by

$$\underline{u} = \underline{u}(\underline{v}) \tag{2-56a}$$

$$\underline{x} = \underline{x}(\underline{u}(\underline{v}),t) \tag{2-56b}$$

$$\dot{\underline{x}} = \dot{\underline{x}}(\underline{u}(\underline{v}),t) \tag{2-56c}$$

$$\dot{\underline{u}} = \frac{\partial \underline{u}}{\partial \underline{v}}\,\dot{\underline{v}} \tag{2-57a}$$

$$\frac{\partial \underline{x}}{\partial \underline{v}} = \frac{\partial \underline{x}}{\partial \underline{u}}\,\frac{\partial \underline{u}}{\partial \underline{v}} = \underline{X}\,\frac{\partial \underline{u}}{\partial \underline{v}} \tag{2-57b}$$

$$\frac{\partial \dot{\underline{x}}}{\partial \underline{v}} = \frac{\partial \underline{x}}{\partial \underline{u}}\,\frac{\partial \underline{u}}{\partial \underline{v}} = \dot{\underline{X}}\,\frac{\partial \underline{u}}{\partial \underline{v}} \tag{2-57c}$$

where $\partial\underline{u}/\partial\underline{v}$ is the Jacobi matrix with respect to the transformation from $\underline{u}$ in $\underline{v}$, results in

$$\underline{X} = \frac{\partial \underline{x}}{\partial \underline{v}}\left(\frac{\partial \underline{u}}{\partial \underline{v}}\right)^{-1} \tag{2-58a}$$

$$\dot{\underline{X}} = \frac{\partial \dot{\underline{x}}}{\partial \underline{v}}\left(\frac{\partial \underline{u}}{\partial \underline{v}}\right)^{-1} \tag{2-58b}$$

Inserting (2-57a) and (2-58a,b) into (2-55a,b) yields

$$\underline{X}\,\dot{\underline{u}} = \frac{\partial \underline{x}}{\partial \underline{v}}\left(\frac{\partial \underline{u}}{\partial \underline{v}}\right)^{-1}\frac{\partial \underline{u}}{\partial \underline{v}}\,\dot{v} = \frac{\partial \underline{x}}{\partial \underline{v}}\,\dot{\underline{v}} = \underline{o} \tag{2-59a}$$

$$\dot{\underline{X}}\,\dot{\underline{u}} = \frac{\partial \underline{x}}{\partial \underline{v}}\left(\frac{\partial \underline{u}}{\partial \underline{v}}\right)^{-1}\frac{\partial \underline{u}}{\partial \underline{v}}\,\dot{\underline{v}} = \frac{\partial \dot{\underline{x}}}{\partial \underline{v}}\,\dot{\underline{v}} = \underline{a}(\underline{v},t) \tag{2-59b}$$

$$\left(\frac{\partial \underline{u}}{\partial \underline{v}}\right)^{-1} \frac{\partial \underline{u}}{\partial \underline{v}} = \underline{I} \qquad \text{where} \quad \det \frac{\partial \underline{u}}{\partial \underline{v}} \neq 0 \tag{2-59c}$$

Note: For the transformed Jacobi matrices $\partial \underline{u}/\partial \underline{v}$, the notations $\underline{X}$, $\dot{\underline{X}}$ are kept in the following.

Thus, the inhomogeneous problem (2-16a,b) changes into the new differential equation system with respect to ccordinates $\underline{v}$.

$$\dot{\underline{v}}(t) = \begin{bmatrix} \underline{X}\big(\underline{v}(t),t\big) \\ \dot{\underline{X}}\big(\underline{v}(t),t\big) \end{bmatrix}^{-1} \begin{bmatrix} \underline{o} \\ \underline{a}\big(\underline{v}(t),t\big) \end{bmatrix} \tag{2-60}$$

or in short form

$$\dot{\underline{v}}(t) = \underline{Y}\big(\underline{v}(t),t\big)\ \underline{a}\big(\underline{v}(t),t\big) \tag{2-61}$$

where

$$\underline{X} = \frac{\partial \underline{x}}{\partial \underline{v}} \tag{2-62a}$$

$$\dot{\underline{X}} = \frac{\partial \dot{\underline{x}}}{\partial \underline{v}} \tag{2-62b}$$

$$\underline{Y} = \begin{bmatrix} \underline{X} \\ \dot{\underline{X}} \end{bmatrix}^{-1} \begin{bmatrix} \underline{0} \\ \underline{1} \end{bmatrix} \tag{2-62c}$$

with

$$\underline{0} = [0, 0, 0]^T$$

$$\underline{1} = [1, 1, 1]^T \ .$$

2.4.2 Determination of the Jacobi matrix of Kepler elements and modified Poincaré orbital elements

Let $\underline{u}$ be the vector of Kepler elements using the mean anomaly M (2-24c) instead of t_p in (2-25).

$$\underline{u} = [M, a, \Omega, \omega, e, i]^T \tag{2-63}$$

Further, $\underline{q}$, $\underline{p}$ are the vectors of Poincaré elements (2-26a,...,f)

$$\underline{q} = [l, g, h]^T \tag{2-64a}$$

$$\underline{p} = [L, G, H]^T \tag{2-64b}$$

By $\underline{v}$ the vector of modified Poincaré elements (2-39b) is denoted.

$$\underline{v} = \begin{bmatrix} \underline{q} \\ \underline{p} \end{bmatrix} + \frac{\kappa^4 \cdot t}{p_1^3} \underline{e}_1 \qquad (2\text{-}65)$$

where

$$\underline{e}_1 = [1, 0, 0, 0, 0, 0]^T \ .$$

The Jacobi matrix $\underline{u}$ with respect to $\underline{v}$ is given by

$$\frac{\partial \underline{u}}{\partial \underline{v}} = \frac{\partial \underline{u}}{\partial \underline{q}} \frac{\partial \underline{q}}{\partial \underline{v}} + \frac{\partial \underline{u}}{\partial \underline{p}} \frac{\partial \underline{p}}{\partial \underline{v}} \qquad (2\text{-}66)$$

Inserting (2-41a,...,f) in (2-66) yields

$$\frac{\partial \underline{u}}{\partial \underline{v}} = \left[\frac{\partial \underline{u}}{\partial \underline{q}} , \frac{\partial \underline{u}}{\partial \underline{p}} \right] \begin{bmatrix} \frac{\partial \underline{q}}{\partial \underline{v}} \\ \frac{\partial \underline{p}}{\partial \underline{v}} \end{bmatrix} = \left[\frac{\partial \underline{u}}{\partial \underline{q}} , \frac{\partial \underline{u}}{\partial \underline{p}} \right] \begin{bmatrix} \underline{I} & \underline{Q} \\ \underline{0} & \underline{I} \end{bmatrix} =$$

$$= \left[\frac{\partial \underline{u}}{\partial \underline{q}} , \frac{\partial \underline{u}}{\partial \underline{p}} + \frac{\partial \underline{u}}{\partial \underline{q}} \underline{Q} \right] = \left[\frac{\partial \underline{u}}{\partial \underline{q}} , \frac{\partial \underline{u}}{\partial \underline{p}} + \underline{Q}_2 \right] \qquad (2\text{-}67)$$

$$\underline{Q}_2 = -3\tau \left[\frac{\partial \underline{u}}{\partial \underline{q}_1} , \underline{0} , \underline{0} \right] \quad \text{where} \quad \tau = \frac{t}{a^2} \qquad (2\text{-}67a)$$

Explicitly,

$$\left[\frac{\partial \underline{u}}{\partial \underline{q}} , \frac{\partial \underline{u}}{d\underline{p}} \right] = \begin{bmatrix} \frac{\partial M}{\partial l} & \frac{\partial M}{\partial g} & \frac{\partial M}{\partial h} & \frac{\partial M}{\partial L} & \frac{\partial M}{\partial G} & \frac{\partial M}{\partial H} \\ \frac{\partial a}{\partial l} & \frac{\partial a}{\partial g} & \frac{\partial a}{\partial h} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial G} & \frac{\partial a}{\partial H} \\ \frac{\partial \Omega}{\partial l} & \frac{\partial \Omega}{\partial g} & \frac{\partial \Omega}{\partial h} & \frac{\partial \Omega}{\partial L} & \frac{\partial \Omega}{\partial G} & \frac{\partial \Omega}{\partial H} \\ \frac{\partial \omega}{\partial l} & \frac{\partial \omega}{\partial g} & \frac{\partial \omega}{\partial h} & \frac{\partial \omega}{\partial L} & \frac{\partial \omega}{\partial G} & \frac{\partial \omega}{\partial H} \\ \frac{\partial e}{\partial l} & \frac{\partial e}{\partial g} & \frac{\partial e}{\partial h} & \frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & \frac{\partial e}{\partial H} \\ \frac{\partial i}{\partial l} & \frac{\partial i}{\partial g} & \frac{\partial i}{\partial h} & \frac{\partial i}{\partial L} & \frac{\partial i}{\partial G} & \frac{\partial i}{\partial H} \end{bmatrix} \qquad (2\text{-}68)$$

The appropriate 36 matrix elements of (2-68) are

$$\frac{\partial M}{\partial l} = 1 \tag{2-69a}$$

$$\frac{\partial M}{\partial g} = \frac{\cos(\omega+\Omega)}{2\,L^{0.5}\,\sin(\tilde{\varphi}/2)} \qquad (e = \sin\tilde{\varphi}) \tag{2-69b}$$

$$\frac{\partial M}{\partial h} = 0 \tag{2-69c}$$

$$\frac{\partial M}{\partial L} = 0 \tag{2-69d}$$

$$\frac{\partial M}{\partial G} = \frac{\sin(\omega+\Omega)}{2\,L^{0.5}\,\sin(\tilde{\varphi}/2)} \tag{2-69e}$$

$$\frac{\partial M}{\partial H} = 0 \tag{2-69f}$$

$$\frac{\partial a}{\partial l} = 0 \tag{2-70a}$$

$$\frac{\partial a}{\partial g} = 0 \tag{2-70b}$$

$$\frac{\partial a}{\partial h} = 0 \tag{2-70c}$$

$$\frac{\partial a}{\partial L} = \frac{2\,L}{\kappa^2} = \frac{2\,a^{0.5}}{\kappa} \tag{2-70d}$$

$$\frac{\partial a}{\partial G} = 0 \tag{2-70e}$$

$$\frac{\partial a}{\partial H} = 0 \tag{2-70f}$$

$$\frac{\partial \Omega}{\partial l} = 0 \tag{2-71a}$$

$$\frac{\partial \Omega}{\partial g} = 0 \tag{2-71b}$$

$$\frac{\partial \Omega}{\partial h} = -\frac{\cos\Omega}{2\,L^{0.5}\,\cos^{0.5}\tilde{\varphi}\,\sin(i/2)} \tag{2-71c}$$

$$\frac{\partial \Omega}{\partial L} = 0 \tag{2-71d}$$

$$\frac{\partial \Omega}{\partial G} = 0 \tag{2-71e}$$

$$\frac{\partial \Omega}{\partial H} = -\frac{\sin \Omega}{2 L^{0.5} \cos^{0.5}\tilde{\varphi} \sin(i/2)} \tag{2-71f}$$

$$\frac{\partial \omega}{\partial l} = 0 \tag{2-72a}$$

$$\frac{\partial \omega}{\partial g} = -\frac{\cos(\omega + \Omega)}{2 L^{0.5} \sin(\tilde{\varphi}/2)} = \frac{\partial M}{\partial g} \tag{2-72b}$$

$$\frac{\partial \omega}{\partial h} = \frac{\cos \Omega}{2 L^{0.5} \cos^{0.5}\tilde{\varphi} \sin(i/2)} = -\frac{\partial \Omega}{\partial h} \tag{2-72c}$$

$$\frac{\partial \omega}{\partial L} = 0 \tag{2-72d}$$

$$\frac{\partial \omega}{\partial G} = -\frac{\sin(\omega + \Omega)}{2 L^{0.5} \sin(\tilde{\varphi}/2)} = -\frac{\partial M}{\partial G} \tag{2-72e}$$

$$\frac{\partial \omega}{\partial H} = \frac{\sin \Omega}{2 L^{0.5} \cos^{0.5}\tilde{\varphi} \sin(i/2)} = -\frac{\partial \Omega}{\partial H} \tag{2-72f}$$

$$\frac{\partial e}{\partial l} = 0 \tag{2-73a}$$

$$\frac{\partial e}{\partial g} = -\frac{\cos \tilde{\varphi} \sin(\omega + \Omega)}{L^{0.5} \cos(\tilde{\varphi}/2)} \tag{2-73b}$$

$$\frac{\partial e}{\partial h} = 0 \tag{2-73c}$$

$$\frac{\partial e}{\partial L} = -\frac{1}{L} \tan(\tilde{\varphi}/2) \sin \tilde{\varphi} \tag{2-73d}$$

$$\frac{\partial e}{\partial G} = \frac{\cos \tilde{\varphi} \cos(\omega + \Omega)}{L^{0.5} \cos(\tilde{\varphi}/2)} \tag{2-73e}$$

$$\frac{\partial e}{\partial h} = 0 \tag{2-73f}$$

$$\frac{\partial i}{\partial l} = 0 \qquad (2\text{-}74a)$$

$$\frac{\partial i}{\partial g} = -\frac{2\sin(\tilde{\varphi}/2)\tan(i/2)\sin(\omega+\Omega)}{L^{0.5}\cos\tilde{\varphi}} \qquad (2\text{-}74b)$$

$$\frac{\partial i}{\partial h} = -\frac{\sin\Omega}{L^{0.5}\cos^{0.5}\tilde{\varphi}\cos(i/2)} \qquad (2\text{-}74c)$$

$$\frac{\partial i}{\partial L} = -\frac{\tan(i/2)}{L\cos\tilde{\varphi}} \qquad (2\text{-}74d)$$

$$\frac{\partial i}{\partial G} = \frac{2\sin(\tilde{\varphi}/2)\tan(i/2)\cos(\omega+\Omega)}{L^{0.5}\cos\tilde{\varphi}} \qquad (2\text{-}74e)$$

$$\frac{\partial i}{\partial H} = \frac{\cos\Omega}{L^{0.5}\cos^{0.5}\tilde{\varphi}\cos(i/2)} \qquad (2\text{-}74f)$$

The matrix $\underline{Q}_2$ (2-67a) is defined using (2-69a,...,f).

$$\underline{Q}_2 = -3\frac{t}{a^2}\begin{bmatrix} \frac{\partial M}{\partial l} & 0 & 0 \\ \frac{\partial a}{\partial l} & 0 & 0 \\ \frac{\partial \Omega}{\partial l} & 0 & 0 \\ \frac{\partial \omega}{\partial l} & 0 & 0 \\ \frac{\partial e}{\partial l} & 0 & 0 \\ \frac{\partial i}{\partial l} & 0 & 0 \end{bmatrix} \qquad (2\text{-}75)$$

2.4.3 Determination of the Jacobi matrix $\underline{X}$

The matrix $\underline{X}$ (2-62a) is given by using the chain rule with respect to the transformation $\underline{u} = \underline{u}(\underline{v})$, see also appendix C.

$$\underline{X} = \frac{\partial \underline{x}}{\partial \underline{v}} = \frac{\partial \underline{x}}{\partial \underline{u}}\frac{\partial \underline{u}}{\partial \underline{v}} \qquad (2\text{-}76)$$

$\partial \underline{u}/\partial \underline{v}$ is defined by (2-69) to (2-75). The Jacobi matrix of the position vector $\underline{x}$ with respect to the Kepler elements (2-63) reads

$$\frac{\partial \underline{x}}{\partial \underline{u}} = \begin{bmatrix} \frac{\partial x_1}{\partial M} & \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial \Omega} & \frac{\partial x_1}{\partial \omega} & \frac{\partial x_1}{\partial e} & \frac{\partial x_1}{\partial i} \\ \frac{\partial x_2}{\partial M} & \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial \Omega} & \frac{\partial x_2}{\partial \omega} & \frac{\partial x_2}{\partial e} & \frac{\partial x_2}{\partial i} \\ \frac{\partial x_3}{\partial M} & \frac{\partial x_3}{\partial a} & \frac{\partial x_3}{\partial \Omega} & \frac{\partial x_3}{\partial \omega} & \frac{\partial x_3}{\partial e} & \frac{\partial x_3}{\partial i} \end{bmatrix} \tag{2-76a}$$

The 18 matrix elements of (2-76a) are derived by partial differentiation of (2-22). Note that for M the relation (2-24c) is used.

$$\frac{\partial x_1}{\partial M} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot$$
$$\cdot \left[-\frac{a^2}{r} \sin E, \frac{a^2}{r} \cos\tilde{\varphi} \cos E, 0\right]^T \tag{2-77a}$$

$$\frac{\partial x_2}{\partial M} = [\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i, \cos\Omega \cos\omega \cos i - \sin\Omega \sin\omega, -\cos\Omega \sin i] \cdot$$
$$\cdot \left[-\frac{a^2}{r} \sin E, \frac{a^2}{r} \cos\tilde{\varphi} \cos E, 0\right]^T \tag{2-77b}$$

$$\frac{\partial x_3}{\partial M} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot$$
$$\cdot \left[-\frac{a^2}{r} \sin E, \frac{a^2}{r} \cos\tilde{\varphi} \cos E, 0\right]^T \tag{2-77c}$$

$$\frac{\partial x_1}{\partial a} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot$$
$$\cdot [\cos E - \sin\tilde{\varphi}, \cos\tilde{\varphi} \sin E, 0]^T \tag{2-78a}$$

$$\frac{\partial x_2}{\partial a} = [\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i, \cos\Omega \cos\omega \cos i - \sin\Omega \sin\omega, -\cos\Omega \sin i] \cdot$$
$$\cdot [\cos E - \sin\tilde{\varphi}, \cos\tilde{\varphi} \sin E, 0]^T \tag{2-78b}$$

$$\frac{\partial x_3}{\partial a} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot$$
$$\cdot [\cos E - \sin\tilde{\varphi}, \cos\tilde{\varphi} \sin E, 0]^T \tag{2-78c}$$

$$\frac{\partial x_1}{\partial \Omega} = [-(\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i), \sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i, \cos\Omega \sin i] \cdot$$
$$\cdot [a(\cos E - \sin\tilde{\varphi}), a \cos\tilde{\varphi} \sin E, 0]^T \tag{2-79a}$$

$$\frac{\partial x_2}{\partial \Omega} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot$$

$$\cdot [a(\cos E - \sin\tilde{\varphi}), a \cos\tilde{\varphi} \sin E, 0]^T \qquad (2\text{-}79b)$$

$$\frac{\partial x_3}{\partial \Omega} = [\quad 0 \quad , \quad 0 \quad , \quad 0 \quad] \cdot$$

$$\cdot [a(\cos E - \sin\tilde{\varphi}), a \cos\tilde{\varphi} \sin E, 0]^T = 0 \qquad (2\text{-}79c)$$

$$\frac{\partial x_1}{\partial \omega} = [-(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin\omega \cos i - \cos\Omega \cos\omega, 0] \cdot$$

$$\cdot [a(\cos E - \sin\tilde{\varphi}), a \cos\tilde{\varphi} \sin E, 0]^T \qquad (2\text{-}80a)$$

$$\frac{\partial x_2}{\partial \omega} = [\cos\Omega \cos\omega \cos i - \sin\Omega \sin\omega, -(\cos\Omega \sin\omega \cos i + \sin\Omega \cos\omega), 0] \cdot$$

$$\cdot [a(\cos E - \sin\tilde{\varphi}), a \cos\tilde{\varphi} \sin E, 0]^T \qquad (2\text{-}80b)$$

$$\frac{\partial x_3}{\partial \omega} = [\cos\omega \sin i, -\sin\omega \sin i, 0] \cdot$$

$$\cdot [a(\cos E - \sin\tilde{\varphi}), a \cos\tilde{\varphi} \sin E, 0]^T \qquad (2\text{-}80c)$$

$$\frac{\partial x_1}{\partial e} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot$$

$$\cdot \left[-\left(a + \frac{a^2}{r} \sin^2 E\right), \frac{a^2 \sin E}{r \cos\varphi} (\cos E - e), 0 \right]^T \qquad (2\text{-}81a)$$

$$\frac{\partial x_2}{\partial e} = [\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i, \cos\Omega \cos\omega \cos i + \sin\Omega \sin\omega, -\cos\Omega \sin i] \cdot$$

$$\cdot \left[-\left(a + \frac{a^2}{r} \sin^2 E\right), \frac{a^2 \sin E}{r \cos\varphi} (\cos E - e), 0 \right]^T \qquad (2\text{-}81b)$$

$$\frac{\partial x_3}{\partial e} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot$$

$$\cdot \left[-\left(a + \frac{a^2}{r} \sin^2 E\right), \frac{a^2 \sin E}{r \cos\varphi} (\cos E - e), 0 \right]^T \qquad (2\text{-}81c)$$

$$\frac{\partial x_1}{\partial i} = [\sin\Omega \sin\omega \sin i, \sin\Omega \cos\omega \sin i, \sin\Omega \cos i] \cdot$$

$$\cdot [a(\cos E - \sin\varphi), a \cos\varphi \sin E, 0]^T \qquad (2\text{-}82a)$$

$$\frac{\partial x_2}{\partial i} = [-\cos\Omega \sin\omega \sin i, -\cos\Omega \cos\omega \sin i, -\cos\Omega \cos i] \cdot$$

$$\cdot [a(\cos E - \sin\varphi), a \cos\varphi \sin E, 0]^T \qquad (2\text{-}82b)$$

$$\frac{\partial x_3}{\partial i} = [\sin\Omega \sin i \,, \cos\omega \cos i \,, -\sin i] \cdot$$

$$\cdot [a(\cos E - \sin\varphi) \,, a \cos\varphi \sin E \,, 0]^T \tag{2-82c}$$

The variable r is given by (2-21c) already

$$r = a(1 - e \cos E) \tag{2-83}$$

2.4.4 Determination of the Jacobi matrix $\underline{\dot{X}}$

Matrix $\underline{\dot{X}}$ (2-62b) is again derived by using the chain rule (see also appendix C) with respect to the transformation $\underline{u} = \underline{u}(\underline{v})$.

$$\underline{\dot{X}} = \frac{\partial \underline{\dot{x}}}{\partial \underline{u}} \frac{\partial \underline{u}}{\partial \underline{v}} \tag{2-84}$$

$\partial\underline{u}/\partial\underline{v}$ is already given by (2-67). The Jacobi matrix of the velocity vector $\underline{\dot{x}}$ with respect to the Kepler elements $\underline{u}$ is explicitely given by

$$\frac{\partial \underline{\dot{x}}}{\partial \underline{u}} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial M} & \frac{\partial \dot{x}_1}{\partial a} & \frac{\partial \dot{x}_1}{\partial \Omega} & \frac{\partial \dot{x}_1}{\partial \omega} & \frac{\partial \dot{x}_1}{\partial e} & \frac{\partial \dot{x}_1}{\partial i} \\ \frac{\partial \dot{x}_2}{\partial M} & \frac{\partial \dot{x}_2}{\partial a} & \frac{\partial \dot{x}_2}{\partial \Omega} & \frac{\partial \dot{x}_2}{\partial \omega} & \frac{\partial \dot{x}_2}{\partial e} & \frac{\partial \dot{x}_2}{\partial i} \\ \frac{\partial \dot{x}_3}{\partial M} & \frac{\partial \dot{x}_3}{\partial a} & \frac{\partial \dot{x}_3}{\partial \Omega} & \frac{\partial \dot{x}_3}{\partial \omega} & \frac{\partial \dot{x}_3}{\partial e} & \frac{\partial \dot{x}_3}{\partial i} \end{bmatrix} \tag{2-85}$$

The 18 matrix elements of (2-85) are derived by partial differentiation of (2-24g).

$$\frac{\partial \dot{x}_1}{\partial M} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i \,, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot$$

$$\cdot \left[-\frac{a^4}{r_3} n(\cos E - \sin\tilde{\varphi}) \,, -\frac{a^4}{r_3} n \cos\tilde{\varphi} \sin E \,, 0 \right]^T \tag{2-86a}$$

$$\frac{\partial \dot{x}_2}{\partial M} = [\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i \,, \cos\Omega \cos\omega \cos i - \sin\Omega \sin\omega \,, -\cos\Omega \sin i] \cdot$$

$$\cdot \left[-\frac{a^4}{r_3} n(\cos E - \sin\tilde{\varphi}) \,, -\frac{a^4}{r_3} n \cos\tilde{\varphi} \sin E \,, 0 \right]^T \tag{2-86b}$$

$$\frac{\partial \dot{x}_3}{\partial M} = [\sin\omega \sin i \,, \cos\omega \sin i \,, \cos i] \cdot$$

$$\cdot \left[-\frac{a^4}{r_3} n(\cos E - \sin\tilde{\varphi}) \,, -\frac{a^4}{r_3} n \cos\tilde{\varphi} \sin E \,, 0 \right]^T \tag{2-86c}$$

$$\frac{\partial \dot{x}_1}{\partial a} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i, -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin i] \cdot$$

$$\cdot\left[\frac{a}{2r} n\sin E, -\frac{a}{2r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-87a}$$

$$\frac{\partial \dot{x}_2}{\partial a} = [\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i, \cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega, -\cos\Omega\sin i] \cdot$$

$$\cdot\left[\frac{a}{2r} n\sin E, -\frac{a}{2r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-87b}$$

$$\frac{\partial \dot{x}_3}{\partial a} = [\sin\omega\sin i, \cos\omega\sin i, \cos i] \cdot$$

$$\cdot\left[\frac{a}{2r} n\sin E, -\frac{a}{2r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-87c}$$

$$\frac{\partial \dot{x}_1}{\partial \Omega} = [-(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i), \sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i, \cos\Omega\sin i] \cdot$$

$$\cdot\left[-\frac{a^2}{r} n\sin E, \frac{a^2}{r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-88a}$$

$$\frac{\partial \dot{x}_2}{\partial \Omega} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i, -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin i] \cdot$$

$$\cdot\left[-\frac{a^2}{r} n\sin E, \frac{a^2}{r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-88b}$$

$$\frac{\partial \dot{x}_3}{\partial \Omega} = [\quad 0 \quad , \quad 0 \quad , \quad 0 \quad] \cdot$$

$$\cdot\left[-\frac{a^2}{r} n\sin E, \frac{a^2}{r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-88c}$$

$$\frac{\partial \dot{x}_1}{\partial \omega} = [-(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin\omega\cos i - \cos\Omega\cos\omega, 0] \cdot$$

$$\cdot\left[-\frac{a^2}{r} n\sin E, \frac{a^2}{r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-89a}$$

$$\frac{\partial \dot{x}_2}{\partial \omega} = [\cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega, -(\cos\Omega\sin\omega\cos i + \sin\Omega\cos\omega), 0] \cdot$$

$$\cdot\left[-\frac{a^2}{r} n\sin E, \frac{a^2}{r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-89b}$$

$$\frac{\partial \dot{x}_3}{\partial \omega} = [\cos\omega\sin i, -\sin\omega\sin i, 0] \cdot$$

$$\cdot\left[-\frac{a^2}{r} n\sin E, \frac{a^2}{r} n\cos\tilde{\varphi}\cos E, 0\right]^T \tag{2-89c}$$

$$\frac{\partial \dot{x}_1}{\partial e} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\,, -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin i] \cdot$$

$$\cdot \left[\frac{a^4}{2r^3} n\,(2\sin\tilde{\varphi} + \sin 2E\,(\sin\tilde{\varphi}\cos E - 2))\,,\; -\frac{a^2}{r} n\,(\tan\tilde{\varphi}\cos E + \right.$$

$$\left. + \frac{a}{r}\cos\tilde{\varphi}\left(\frac{a}{r}\sin^2 E - \cos^2 E\right)\right),\; 0 \Big]^T \tag{2-90a}$$

$$\frac{\partial \dot{x}_2}{\partial e} = [\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i\,, \cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega\,, -\cos\Omega\sin i] \cdot$$

$$\cdot \left[\frac{a^4}{2r^3} n\,(2\sin\tilde{\varphi} + \sin 2E\,(\sin\tilde{\varphi}\cos E - 2))\,,\; -\frac{a^2}{r} n\,(\tan\tilde{\varphi}\cos E + \right.$$

$$\left. + \frac{a}{r}\cos\tilde{\varphi}\left(\frac{a}{r}\sin^2 E - \cos^2 E\right)\right),\; 0 \Big]^T \tag{2-90b}$$

$$\frac{\partial \dot{x}_3}{\partial e} = [\sin\omega\sin i\,, \cos\omega\sin i\,, \cos i] \cdot$$

$$\cdot \left[\frac{a^4}{2r^3} n\,(2\sin\tilde{\varphi} + \sin 2E\,(\sin\tilde{\varphi}\cos E - 2))\,,\; -\frac{a^2}{r} n\,(\tan\tilde{\varphi}\cos E + \right.$$

$$\left. + \frac{a}{r}\cos\tilde{\varphi}\left(\frac{a}{r}\sin^2 E - \cos^2 E\right)\right),\; 0 \Big]^T \tag{2-90c}$$

$$\frac{\partial \dot{x}_1}{\partial i} = [\sin\Omega\sin\omega\sin i\,, \sin\Omega\cos\omega\sin i\,, \sin\Omega\cos i] \cdot$$

$$\cdot \left[-\frac{a^2}{r} n\sin E\,,\; \frac{a^2}{r} n\cos\tilde{\varphi}\cos E\,,\; 0 \right]^T \tag{2-91a}$$

$$\frac{\partial \dot{x}_1}{\partial i} = [-\cos\Omega\sin\omega\sin i\,, -\cos\Omega\cos\omega\sin i\,, -\cos\Omega\cos i] \cdot$$

$$\cdot \left[-\frac{a^2}{r} n\sin E\,,\; \frac{a^2}{r} n\cos\tilde{\varphi}\cos E\,,\; 0 \right]^T \tag{2-91b}$$

$$\frac{\partial \dot{x}_3}{\partial i} = [\sin\omega\cos i\,, \cos\omega\cos i\,, -\sin i] \cdot$$

$$\cdot \left[-\frac{a^2}{r} n\sin E\,,\; \frac{a^2}{r} n\cos\tilde{\varphi}\cos E\,,\; 0 \right]^T \tag{2-91c}$$

2.4.5 Determination of the inverse $\underline{Y}$

According to (2-60) has the following linear system of equation of dimension 6 to be solved with respect to $\underline{\dot{v}}$.

$$\begin{bmatrix} \underline{X} \\ \underline{\dot{X}} \end{bmatrix} \underline{\dot{v}} = \begin{bmatrix} \underline{o} \\ \underline{a} \end{bmatrix} \tag{2-92}$$

Using Cramer's rule the element $\dot{v}_i$ of vector $\underline{\dot{v}}$ is determined by inserting the right hand side of (2-92) in column i of the coefficient maxtrix, built-up of the corresponding sub-determinant, and division by the determinant of the coefficient matrix. Thereby we have to note that $\underline{y} = \underline{\dot{x}}$, (2-29a).

$$\dot{v}_i = \frac{1}{\det \underline{J}} \begin{bmatrix} \frac{\partial x_1}{\partial v_1} & \frac{\partial x_1}{\partial v_2} & \cdots\cdots & 0 & \cdots\cdots & \frac{\partial x_1}{\partial v_6} \\ \frac{\partial x_2}{\partial v_1} & \frac{\partial x_2}{\partial v_2} & \cdots\cdots & 0 & \cdots\cdots & \frac{\partial x_2}{\partial v_6} \\ \frac{\partial x_3}{\partial v_1} & \frac{\partial x_3}{\partial v_2} & \cdots\cdots & 0 & \cdots\cdots & \frac{\partial x_3}{\partial v_6} \\ \frac{\partial y_1}{\partial v_1} & \frac{\partial y_1}{\partial v_2} & \cdots\cdots & a_1 & \cdots\cdots & \frac{\partial y_1}{\partial v_6} \\ \frac{\partial y_2}{\partial v_1} & \frac{\partial y_2}{\partial v_2} & \cdots\cdots & a_2 & \cdots\cdots & \frac{\partial y_2}{\partial v_6} \\ \frac{\partial y_3}{\partial v_1} & \frac{\partial y_3}{\partial v_2} & \cdots\cdots & a_3 & \cdots\cdots & \frac{\partial y_3}{\partial v_6} \end{bmatrix} \quad \text{(column } i \text{ marked above the } 0, a \text{ entries)} \tag{2-93}$$

Developing the sub-determinant in (2-93) results in

$$\dot{v}_i = \frac{1}{\det \underline{J}} \left(a_1 \left| \frac{\partial y_1}{\partial v_i} \right| + a_2 \left| \frac{\partial y_2}{\partial v_i} \right| + a_3 \left| \frac{\partial y_3}{\partial v_i} \right| \right) \tag{2-94}$$

Using the special properties of the introduced canonical elements, in particular (appendix B, eq. (B-10)),

$$\det \underline{J} = 1 \tag{2-95}$$

and considering (2-54a,...,f) the explicit sub-determinants (2-94) are easily constructed.

According to (2-61) we have

$$\underline{\dot{v}}(t) = \underline{Y}\left(\underline{v}(t),t\right) \underline{a}\left(\underline{v}(t),t\right) \tag{2-96}$$

where $\underline{Y}$ is given in terms of ***Poincaré elements*** $\underline{p}$, $\underline{q}$ by

$$
\underline{Y} = \begin{bmatrix}
-\frac{\partial x_1}{\partial p_1} + 3\tau \frac{\partial x_1}{\partial q_1} & , & -\frac{\partial x_2}{\partial p_1} + 3\tau \frac{\partial x_2}{\partial q_1} & , & -\frac{\partial x_3}{\partial p_1} + 3\tau \frac{\partial x_3}{\partial q_1} \\
-\frac{\partial x_1}{\partial p_2} & , & -\frac{\partial x_2}{\partial p_2} & , & -\frac{\partial x_3}{\partial p_2} \\
-\frac{\partial x_1}{\partial p_3} & , & -\frac{\partial x_2}{\partial p_3} & , & -\frac{\partial x_3}{\partial p_3} \\
\frac{\partial x_1}{\partial q_1} & , & \frac{\partial x_2}{\partial q_1} & , & \frac{\partial x_3}{\partial q_1} \\
\frac{\partial x_1}{\partial q_2} & , & \frac{\partial x_2}{\partial q_2} & , & \frac{\partial x_3}{\partial q_2} \\
\frac{\partial x_1}{\partial q_3} & , & \frac{\partial x_2}{\partial q_3} & , & \frac{\partial x_3}{\partial q_3}
\end{bmatrix} \qquad (2\text{-}97a)
$$

and in terms of ***modified Poincaré elements*** (2-26a,...,f) by

$$
\underline{Y} = \begin{bmatrix}
-\frac{\partial x_1}{\partial L} + 3\tau \frac{\partial x_1}{\partial l} & , & -\frac{\partial x_2}{\partial L} + 3\tau \frac{\partial x_2}{\partial l} & , & -\frac{\partial x_3}{\partial L} + 3\tau \frac{\partial x_3}{\partial l} \\
-\frac{\partial x_1}{\partial G} & , & -\frac{\partial x_2}{\partial G} & , & -\frac{\partial x_3}{\partial G} \\
-\frac{\partial x_1}{\partial H} & , & -\frac{\partial x_2}{\partial H} & , & -\frac{\partial x_3}{\partial H} \\
\frac{\partial x_1}{\partial l} & , & \frac{\partial x_2}{\partial l} & , & \frac{\partial x_3}{\partial l} \\
\frac{\partial x_1}{\partial g} & , & \frac{\partial x_2}{\partial g} & , & \frac{\partial x_3}{\partial g} \\
\frac{\partial x_1}{\partial h} & , & \frac{\partial x_2}{\partial h} & , & \frac{\partial x_3}{\partial h}
\end{bmatrix} \qquad (2\text{-}97b)
$$

The partial derivatives of $\underline{x}$ can be found in (2-76) to (2-82c).

2.4.6 The solution by the method of successive approximations

In the following we are concerned with the determination of the vector $\underline{v} = \underline{v}(t)$ from (2-96). If $v(t)$ is known as function of time, the position vector $\underline{x}$ and the velocity vector $\dot{\underline{x}}$ of the satellite can be uniquely determined using the transformations (2-26a,...,f), (2-27a,...,f) and the accelerations $\underline{a}$.

We rewrite (2-96) in the form

$$
\dot{\underline{v}}(t) = \underline{Y}\left(\underline{v}(t),t\right)\, \underline{a}\left(\underline{x}(v(t),t)\, ,\, \dot{\underline{x}}\left(\underline{v}(t),t\right) t\right) \qquad (2\text{-}98a)
$$

$$
\dot{v}(t) = \underline{Y}\left(\underline{v}(t),t\right)\, \underline{a}\left(\underline{v}(t),t\right) \qquad (2\text{-}98b)
$$

(2-98b) is derived from (2-98a) by replacing $\underline{x}$ and $\underline{\dot{x}}$ in the acceleration vector by the coordinates $\underline{v}$.

(2-98b) represents a nonlinear 6 x 6 differential equation system for the unknown vector $\underline{v}(t)$. Since $\underline{a}$ (2-6) is considerable small in comparison to $\nabla_x W_o$ (2-4b) (for low satellites we have approximately $|\underline{a}|/|\nabla_x W_o| = 0.002$, ***Picard's iteration method by successive approximations*** is suited for the solution of (2-98b). For that purpose we write (2-98b) in the form of an integral equation

$$\underline{v}(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\big(\underline{v}(t),t\big)\, \underline{a}\big(\underline{v}(t),t\big)\, dt \tag{2-99a}$$

where

$$\underline{v}(t_o) = \underline{v}_o \ . \tag{2-99b}$$

The constant vector v_o is considered as initial condition at time t_o for the solution of (2-98b). The integral on the right hand side of (2-99a) represents the variation of the orbital elements $\underline{v}_o$ in the time interval $(t-t_o)$. Considering further the position vector $\underline{x}(\underline{v}(t),t)$ and the velocity vector $\underline{\dot{x}}(\underline{v}(t),t)$ at $t=t_o$, we recognize that by $\underline{x}(\underline{v}_o,t_o)$ and $\underline{\dot{x}}(v_o,t_o)$ and (2-99a,b) we just obtain the solution (2-20) of the homogeneous problem (2-8). Thus, the space curve $\underline{x}(v(t),t)$ is osculating the Kepler ellipse of the homogeneous problem at $t=t_o$. (Position and velocity vector of the satellite coincide with the corresponding values of the Kepler ellipse at $t=t_o$.)

By Picard's approach successive functions $\underline{v}_k(t)$ of (2-99a) are constructed which are converging under certain prerequisites towards $\underline{v}(t)$. The principle is simple. Starting with $\underline{v}=\underline{v}_o$ in the integral of (2-99a) and solving it for the time interval $(t-t_o)$ yields the first solution $\underline{v}_1$ which again will be used in the integral, $\underline{v}=\underline{v}_1$. Repeating the integration and the whole procedure k-times leads to the following iteration scheme.

$$\underline{v}_{k+1}(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\big(\underline{v}_k(t),t\big)\, \underline{a}\big(\underline{v}_k(t),t\big)\, dt \tag{2-100}$$

where

$$k = \{0,1,\ldots,m\} \quad \text{and} \quad \underline{v}_k(t{=}o) = \underline{v}_o \ .$$

Thus, the successive approximations are determined.

$$\underline{v}_1(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\big(\underline{v}_o,t\big)\, \underline{a}\big(\underline{v}_o,t\big)\, dt \tag{2-101a}$$

$$\underline{v}_2(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\big(\underline{v}_1(t),t\big)\, \underline{a}\big(\underline{v}_1(t),t\big)\, dt \tag{2-101b}$$

$$\underline{v}_3(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\big(\underline{v}_2(t),t\big)\, \underline{a}\big(\underline{v}_2(t),t\big)\, dt \tag{2-101c}$$

....

$$\underline{v}_m(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\left(\underline{v}_{m-1}(t),t\right)\ \underline{a}\left(\underline{v}_{m-1}(t),t\right)\ dt \tag{2-101d}$$

$\underline{v}_k(t)$ is converging to $\underline{v}(t)$ in the interval $[t_1,t_2]$, containing t_o , if Cauchy's theorem of existence is fulfilled. Thus, necessary and sufficient condition for the convergence is the existence of the solution with initial conditions $\underline{v}(t_o) = \underline{v}_o$. This is certified, if

$$\dot{\underline{v}}_1 = \underline{Y}\left(\underline{v}_1(t),t\right)\ \underline{a}\left(\underline{v}_1(t),t\right)$$

is continuous in a domain in the vicinity of $P\left(t_o,\underline{v}_o\right) \in \mathbb{R}^7$, e.g. if

$$|t - t_o| < \varepsilon_a$$

$$\left|\underline{v} - \underline{v}_o\right| < \underline{\varepsilon}_b$$

(Branstein-Semendjajew, 1975, p. 383, p. 375).

In principle, the boundedness of the norm $\left|\underline{v} - \underline{v}_o\right|$ can be interfered by possible singularities in the differential equation system (2-61). It is known ***(Arnold, 1970, p. 28-29)*** that those singularities exist for small excentricities e when using Kepler elements.

Solving the problem by Poincaré orbital elements $\underline{v}$, one can show that no singularities exist for $e = o$, by expanding $v_i(t)$ in a series with respect to the excentricity e and the inclination i . In fact, some of the partial derivatives (2-69a,...,2-74f) containing $\sin(\tilde{\varphi}/2)$ and/or $\sin(i/2)$ in the denominator are singular at the very first sight. Those singularities, however, can be removed by multiplying the corresponding elements of $\partial\underline{x}/\partial\underline{v}$ by the ones of $\partial\underline{x}/\partial\underline{u}$ (2-76) and the subsequent summation.

The method of successive approximations can be applied analytically or numerically. Using an adequate acceleration modell $\underline{a}$ meeting the requirements of modern satellite geodesy does not allow to perform an analytical integration due to the complexity. ***Merson (1961)*** has derived two iteration steps pf (2-100) analytically with considerable effort using only zonal spherical harmonics, which do not yield the necessary geodetic accuracies. Therefore numerical integration might be the only way to solve the integral in (2-100).

A sequential algorithm of successive approximation of (2-100) can easily be derived avoiding that all quantities $\underline{v}_{k-1}(t)$ at step k have to be stored in a computer.

By applying a quadrature formula to (2-100) we get

$$\underline{v}_{k+1}(t_n) = \underline{v}_o + h\sum_{i=o}^{n} \alpha_i\, \underline{Y}\left(\underline{v}_k(t_i),t_i\right)\ \underline{a}\left(\underline{v}_k(t_i),t_i\right) \tag{2-102}$$

By h the integration step size is denoted. α_i are the appropriate constants for the type of quadrature. For the case of the trapezoid rule we have

$$\alpha_i = \begin{cases} 0.5 & \text{for} \quad i = 0 \quad \text{and} \quad i = n \\ 1 & \text{for} \quad i \neq 0 \quad \text{and} \quad i \neq n \end{cases} \tag{2-103a}$$

$$h = (t - t_o)/n \tag{2-103b}$$

Using the expressions above, (2-102) changes into

$$\underline{v}_{k+1}(t_{i+1}) = \underline{v}_{k+1}(t_i) + \\ + \frac{h}{2} \Big\{ \underline{Y}\big(\underline{v}_k(t_i), t_i\big) \, \underline{a}\big(\underline{v}_k(t_i), t_i\big) + \underline{Y}\big(\underline{v}_k(t_{i+1}), t_{i+1}\big) \\ \underline{a}\big(\underline{v}_k(t_{i+1}), t_{i+1}\big) \Big\} \tag{2-104a}$$

For the initial quantities at time $t = t_o$ holds

$$\underline{v}_1(t_o) = \underline{v}_2(t_o) = \ldots = \underline{v}_m(t_o) = \underline{v}_o \tag{2-104b}$$

and the indices i, k are

$$i = \{0, 1, \ldots, n\} \\ k = \{0, 1, \ldots, m\} \tag{2-104c}$$

(2-104a) allows to determine $\underline{v}$ at each time t_i by m iteration steps. By the final vector $\underline{v}_{k+1}(t_i)$ the position vector $\underline{x}\big(\underline{v}_{k+1}(t_i), t_i\big)$ as well as the velocity vector of the satellite $\dot{\underline{x}}\big(\underline{v}_{k+1}(t_i), t_i\big)$ can be computed. Because of the small order of magnitude of $\underline{a}$, m will be considerable smaller than n, $m \ll n$.

In principle, all higher order quadrature formulas can be used to solve (2-102). However, this will be not discussed further in this context.

3. THE ACCELERATION MODELL

In order to determine the position vector $\underline{x}$ and the velocity vector $\dot{\underline{x}}$ of a satellite with sufficient accuracy for nowadays purposes, it is necessary to take into account still other acceleration types besides the gravity acceleration. In this context five different acceleration types are discusses acting on the satellites. These are

(i) $\underline{j}_1$ ... gravity acceleration of the solid earth, [1)]

(ii) $\underline{f}_1$ ... acceleration due to the air-drag of the atmosphere,

(iii) $\underline{f}_2$ ... acceleration due to the solar radiation pressure,

---

[1)] Note, that the vector $\underline{j}_1$ is the gradient of the model potential $(U - U_o)$, and not the one of the actual gravity potential $(W - W_o)$. This is important to know when using the linearization step of integrated geodesy, see also 4.3.

(iv) $\underline{f}_3$ ... acceleration due to the attraction of sun and moon, and,

(v) $\underline{f}_4$ ... acceleration due to the tides of the solid earth.

Depending on the specific purpose of orbit determination further types of accelerations have to be introduced, like e.g. oceanic tides, indirect solar radiation pressure, etc.

3.1 Coordinate reference frames

In order to derive the observation equations of satellite geodesy, determining the earth's gravity field and threedimensional positions on the earth's surface, two reference systems have to ne defined.

(i) For the purpose of satellite orbit determination an inertial reference frame is needed. In such a system Newton's laws of classical mechanics hold in good approximation. The inertial system was already used in paragraph 2., but not yet very well defined.

(ii) In this paper points on the earth's surface as well as the gravity potential are considered as time-independent. They refer to an earth-fixed coordinate system. The International Polar Motion Service (IPMS) and the Bureau International de l'Heure (BIH) determine the instantaneous pole position from quasi-continuous observations of the astronomical latitude and refer it to the Conventional International Origin (CIO) and BIH-pole, respectively.

Between system (i) and (ii) transformations can be defined using time-dependent rotation matrices. For both reference systems the origin is taken to be identical.

3.1.1 The inertial reference system

The inertial reference system (better: quasi-inertial system) used here is the ***instantaneous astronomical system at time*** $t = t_o$.[1)]

The coordinates in this system are denoted by

$$\underline{x} = [x_1, x_2, x_3]^T \tag{3-1}$$

and their orthonormal basis by

$$\underline{e}_1^x, \underline{e}_2^x, \underline{e}_3^x \tag{3-2}$$

1) Time t_o is identical with the lower integration limit in paragraph 2. and 4., e.g. where the orbit integration starts.

where

$\underline{e}_1^x$ ... is the unit vector of the equinox γ at $t = t_0$,

$\underline{e}_2^x$ ... $\underline{e}_2^x = \underline{e}_3^x \times \underline{e}_1^x$, and

$\underline{e}_3^x$ ... is the unit vector parallel to the true rotation axis $\underline{\boldsymbol{\omega}}$ of the earth at $t = t_0$.

The origin is identical with the geocenter, the mass center of the earth.

The so-defined inertial system is free of rotations for $t \geq t_0$. Its basis (3-2) is space-fixed.

In principle, the system (3-1), does not represent a real inertial system which is in rest or is uniformly moving. Neglecting relativistic effects, the earth is moving in space. This movement of the origin is an accelerated movement, contradicting the definition of an inertial system. In order to get Newton's laws fulfilled, an inertial acceleration $\left(-\underline{\ddot{x}}_0\right)$ has to be introduced. As usual in satellite geodesy, $\left(-\underline{\ddot{x}}_0\right)$ is taken into account when determining the acceleration $\underline{f}_3$ due to sun and moon. For that reason only the relative acceleration with respect to the geocenter will be considered in $\underline{f}_3$. The attraction due to other planets can be neglected in that context since the integration interval $[t, t_0]$ is short and only the difference in accelerations is considered. Thus, it is assumed that by the correction $\left(-\underline{\ddot{x}}_0\right)$ the inertial reference system is sufficiently approximated.

3.1.2 The earth-fixed reference system

The earth-fixed reference system is defined by the Conventional International Origin (CIO) of the International Latitude Service (ILS), International Polar Motion Service (IPMS) or by the BIH origin of the Bureau Internationale de l'Heure (BIH), and by the astronomical meridian of Greenwich. Both origins coincide within < 1 m.

The coordinates of the earth-fixed system are denoted by

$$\underline{y} = [y_1, y_2, y_3]^T \qquad (3\text{-}3)$$

and their orthonormal basis by

$$\underline{e}_1^y, \underline{e}_2^y, \underline{e}_3^y \qquad (3\text{-}4)$$

where

$\underline{e}_1^y$ ... is the unit vector in the astronomical meridian plane of Greenwich, $\underline{e}_1^{Ty} \cdot \underline{e}_3^x = 0$,

$\underline{e}_2^y$ ... $\underline{e}_2^y = \underline{e}_3^y \times \underline{e}_1^y$, and

$\underline{e}_3^y$ ... is the unit vector parallel to the line geocenter - CIO.

The origin, again, is identical with the geocenter, the mass center of the earth.

3.1.3 Transformations

The transformation between the coordinate vector $\underline{x}$ in the inertial reference system and the vector $\underline{y}$ in the earth-fixed system is given by (appendix D, eqs. (D-9a.,,,,d)

$$\underline{x} = \underline{R}(t)\,\underline{y} \tag{3-5a}$$

$$\underline{y} = \underline{R}^T(t)\,\underline{x} \tag{3-5b}$$

where the orthogonal rotation matrix $\underline{R}(t)$ is defined by

$$\underline{R}(t) = \underline{N}(t_o)\,\underline{P}(t_o)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\left(-\Theta(t)\right)\,\underline{S}(t) \tag{3-5c}$$

and the following relation holds

$$\underline{R}^{-1}(t) = \underline{R}^T(t) \tag{3-5d}$$

For the explicit definition of the rotation matrices in (3-5c) see appendix D.

For the consideration of the earth's gravity potential we introduce besides the vectors $\underline{x}$, $\underline{y}$ an ***earth-fixed spherical system*** (r, φ, λ) , where

- r ... is the radial distance between the geocenter and the considered point in space,
- φ ... is the latitude of the point in space referenced to the coordinate system $\underline{y}$, and
- λ ... is the longitude of the point in space referenced to the coordinate system $\underline{y}$.

Between the vector $\underline{y}$ and the spherical coordinate the simple relation holds

$$\underline{y} = r \begin{bmatrix} \cos\varphi\,\cos\lambda \\ \cos\varphi\,\sin\lambda \\ \sin\varphi \end{bmatrix} \tag{3-6}$$

The inverse relations are given by

$$r = \left(\underline{y}^T\underline{y}\right)^{0.5} = \left|\,\underline{y}\,\right| \tag{3-7a}$$

$$\sin\varphi = y_3 / |y| \tag{3-7b}$$

$$\cos\varphi = \left(1 - y_3^2 / |y|^2\right)^{0.5} \tag{3-7c}$$

$$\sin\lambda = \frac{y_2}{r\cos\varphi} = \frac{y_2}{\left|\underline{y}\right|\left(1 - y_3^2/\left|\underline{y}\right|^2\right)^{0.5}} \tag{3-7d}$$

$$\cos\lambda = \frac{y_1}{r\cos\varphi} = \frac{y_1}{\left|\underline{y}\right|\left(1 - y_3^2/\left|\underline{y}\right|^2\right)^{0.5}} \tag{3-7e}$$

Partitioning the rotation matrix $\underline{R}^T(t)$ in (3-5c) with respect to row vectors, leads to a simple relation between $\underline{x}$ and (r, φ, λ) using (3-7a,...,e).

$$\underline{R}^T(t) = \begin{bmatrix} \underline{r}_1^T(t) \\ \underline{r}_2^T(t) \\ \underline{r}_3^T(t) \end{bmatrix} \tag{3-8}$$

$$y_1 = \underline{r}_1^T(t)\,\underline{x} \tag{3-9a}$$

$$y_2 = \underline{r}_2^T(t)\,\underline{x} \tag{3-9b}$$

$$y_3 = \underline{r}_3^T(t)\,\underline{x} \tag{3-9c}$$

$$r = \left(\underline{y}^T\,\underline{y}\right)^{0.5} = \left[\underline{x}^T\,\underline{R}(t)\,\underline{R}^T\,\underline{x}\right]^{0.5} = \left(\underline{x}^T\,\underline{x}\right)^{0.5} = |x| \tag{3-10a}$$

$$\sin\varphi = \frac{\underline{r}_3^T(t)\,\underline{x}}{|\underline{x}|} \tag{3-10b}$$

$$\sin\lambda = \frac{\underline{r}_2^T(t)\,\underline{x}}{|\underline{x}|\left[1 - \left(\underline{r}_3^T(t)\,\underline{x}/|\underline{x}|\right)^2\right]^{0.5}} \tag{3-10c}$$

$$\cos\lambda = \frac{\underline{r}_1^T(t)\,\underline{x}}{|\underline{x}|\left[1 - \left(\underline{r}_3^T(t)\,\underline{x}/|\underline{x}|\right)^2\right]^{0.5}} \tag{3-10d}$$

Formulas (3-10a,...,d) are necessary for transforming the position vector $\underline{x}$ of the satellite into the spherical earth-fixed coordinate system (r,φ,λ) .

3.2 Gravity acceleration of the solid earth

Using the model potential $U(\underline{x},t)$ approximating the actual potential $W(\underline{x},t)$ in first order, we get the vector of gravity acceleration in analogue to (2-6) by

$$\underline{j}_1 = \text{grad}_x\left(U - U_o\right) = \text{grad}_x\,U_1 \tag{3-11}$$

By U_o the radial-symmetrical part of the gravity potential is denoted which has to be separated for the solution of the homogeneous problem (2-8). The index of the gradient operator indicates differentiation with respect to the coordinate directions of the inertial system $\underline{x}$.

In general, $U_1 = U - U_o$ is given in the form of spherical harmonics refering to an earth-fixed system. Therefore we have to use the transformation

$$\text{grad}_x\,U_1 = \underline{R}(t)\,\text{grad}_y\,U_1 \tag{3-12}$$

where $\underline{R}(t)$ is given by (3-5c). Introducing the earth-fixed spherical coordinate system $\underline{r} = [r,\varphi,\lambda]^T$ we get

$$\text{grad}_y U_1 = \frac{\partial \underline{r}^T}{\partial \underline{y}} \left[\frac{\partial U_1}{\partial r}, \frac{\partial U_1}{\partial \varphi}, \frac{\partial U_1}{\partial \lambda} \right]^T \tag{3-13}$$

Inserting (3-13) into (3-12) yields the desired representation of $\underline{j}_1$ in the inertial system

$$\underline{j}_1 = \underline{R}(t) \frac{\partial \underline{r}^T}{\partial \underline{y}} \left[\frac{\partial U_1}{\partial r}, \frac{\partial U_1}{\partial \varphi}, \frac{\partial U_1}{\partial \lambda} \right]^T \tag{3-14}$$

where $\underline{R}(t)$ is given by (3-5c) and

$$\frac{\partial \underline{r}}{\partial \underline{y}} = \begin{bmatrix} \frac{\partial r}{\partial y_1} & \frac{\partial r}{\partial y_2} & \frac{\partial r}{\partial y_3} \\ \frac{\partial \varphi}{\partial y_1} & \frac{\partial \varphi}{\partial y_2} & \frac{\partial \varphi}{\partial y_3} \\ \frac{\partial \lambda}{\partial y_1} & \frac{\partial \lambda}{\partial y_2} & \frac{\partial \lambda}{\partial y_3} \end{bmatrix} = \begin{bmatrix} \cos\varphi\cos\lambda\,, & \cos\varphi\sin\lambda\,, & \sin\varphi \\ -\frac{\sin\varphi\cos\lambda}{r}\,, & -\frac{\sin\varphi\sin\lambda}{r}\,, & \frac{\cos\varphi}{r} \\ -\frac{\sin\lambda}{r\cos\varphi}\,, & \frac{\cos\lambda}{r\cos\varphi}\,, & 0 \end{bmatrix} \tag{3-15}$$

The spherical harmonics expansion reads

$$U_1(r,\varphi,\lambda) = kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \frac{R^\mu}{r^{\mu+1}} P_{\mu\nu}(\sin\varphi) \left(c_{\mu\nu}\cos\nu\lambda + s_{\mu\nu}\sin\nu\lambda\right) \tag{3-16}$$

and its partial derivatives with respect to r,φ,λ are

$$\frac{\partial U_1}{\partial r} = kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \frac{R^\mu}{r^{\mu+2}} (\mu+1) P_{\mu\nu}(\sin\varphi) \left(c_{\mu\nu}\cos\nu\lambda + s_{\mu\nu}\sin\nu\lambda\right) \tag{3-17a}$$

$$\frac{\partial U_1}{\partial \varphi} = kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \frac{R^\mu}{r^{\mu+2}} \frac{\partial P_{\mu\nu}(\sin\varphi)}{\partial \varphi} \left(c_{\mu\nu}\cos\nu\lambda + s_{\mu\nu}\sin\nu\lambda\right) \tag{3-17b}$$

$$\frac{\partial U_1}{\partial \lambda} = kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \frac{R^\mu}{r^{\mu+2}} \nu P_{\mu\nu}(\sin\varphi) \left(s_{\mu\nu}\cos\nu\lambda - c_{\mu\nu}\sin\nu\lambda\right) \tag{3-17c}$$

$P_{\mu\nu}(\sin\varphi)$ in (3-16) to (3-17a,b,c) represent the so-called associated Legendre polynomicals (see e.g. ***Sigl, 1973, p. 141***).

$$P_{\mu\nu}(\sin\varphi) = \cos^{\nu}\varphi \sum_{\sigma=0}^{\kappa} a_{\mu\nu\sigma} \sin^{\mu-\nu-2\sigma}\varphi \qquad (3\text{-}18)$$

The partial derivative of (3-18) with respect to spherical latitude is given by [1)]

$$\frac{\partial P_{\mu\nu}}{\partial\varphi} = \sum_{\sigma=0}^{\kappa} a_{\mu\nu\sigma} \{(\mu-\nu-2)\cot\varphi - \nu\tan\phi\} \cos^{\nu}\phi \sin^{\mu-\nu-2\sigma}\varphi \qquad (3\text{-}19)$$

where

$$a_{\mu\nu\sigma} = \frac{(-1)^{\sigma}(2\mu-2\sigma)!}{2^{\mu}\sigma!\,(\mu-\sigma)!\,(\mu-\nu-2\sigma)!} \qquad (3\text{-}20a)$$

$$\kappa = \left\{ \begin{array}{ll} \dfrac{\mu-\nu}{2} & \text{if } \mu-\nu \text{ is even,} \\ \dfrac{\mu-\nu-1}{2} & \text{if } \mu-\nu \text{ is odd} \end{array} \right\} \qquad (3\text{-}20b)$$

3.3 Acceleration due to air-drag

On the satellite flying in the atmosphere acts an acceleration opposite to the direction of movement and proportional to the satellite's velocity squared. This leads to a reduction of potential energy of the satellite.

For a spherical-shaped satellite the acceleration due to air-drag is ***(Cappelari et al, 1976, (4-73))***

$$\underline{f}_1 = -0.5\, C_D\, (A/m)\, \rho\, \underline{\dot{x}}_{rel} \left|\underline{\dot{x}}_{rel}\right| \qquad (3\text{-}21)$$

where

C_D ... is the force coefficient,

(A/m) ... is the relation of effective surface to the mass of the satellite,

ρ ... is the density of the atmosphere, and

$\underline{\dot{x}}_{rel}$ ... is the velocity of the satellite relative to the atmosphere in the inertial system. $\underline{\dot{x}}_{rel}$ is approximately the velocity of the satellite relative to the earth when assuming that the earth rotates with the same angle velocity like the atmosphere.

In principle, eq. (3-21) holds also for non-spherical satellites. In that case, the geometry of the satellite, the orientation of its surface

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[1)] For $\varphi = 0 \rightarrow \cot\varphi$ is undefined. The singularity can be removed because either $(\mu-\nu-2) = 0$, or the factor $\sin^{\mu-\nu-2\sigma}\varphi$ can be premultiplied.

with respect to the velocity vector, and the resulting aerodynamic behaviour have to be taken into account when determining C_D .

3.3.1 Velocity of the satellite relative to the atmosphere of the earth

Let $\underline{x}$ be the position vector, $\underline{\dot{x}}$ the velocity vector of the satellite in the inertial system, $\underline{\boldsymbol{\omega}}$ the vector of angular velocity of the earth, and $\underline{\dot{x}}_{rel}$ the relative velocity vector of the satellite with respect to the earth. We assume that the angular velocity $\underline{\boldsymbol{\omega}}_A$ of the earth's atmosphere is defined in first approximation by the angular velocity vector $\underline{\boldsymbol{\omega}}$ of the earth.

The basic relation between the vectors mentioned above is

$$\underline{\dot{x}} = \underline{\dot{x}}_{rel} + \underline{\boldsymbol{\omega}} \times \underline{x} \tag{3-22}$$

Rewriting the vector product on the right hand side of (3-22) in matrix form, yields

$$\underline{\boldsymbol{\omega}} \times \underline{x} = \underline{\boldsymbol{\Omega}}\, \underline{x} \tag{3-23}$$

where

$$\underline{\boldsymbol{\omega}} = [\omega_1, \omega_2, \omega_3]^T , \tag{3-23a}$$

the skew-symmetrical matrix $\underline{\boldsymbol{\Omega}}$ is given by

$$\underline{\boldsymbol{\Omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3-23b}$$

$$\underline{\boldsymbol{\Omega}} = -\underline{\boldsymbol{\Omega}}^T \tag{3-23c}$$

By differentiation of the transformation (3-5a) we get

$$\underline{\dot{x}} = \underline{\dot{R}}\,\underline{y} + \underline{R}\,\underline{\dot{y}} \tag{3-24}$$

where $\underline{y}$ is the position vector, $\underline{\dot{y}}$ is the velocity vector of the satellite, both in the earth-fixed reference system.

Inserting the inverse transformation (3-5b) in (3-24) we get

$$\underline{\dot{x}} = \underline{\dot{R}}\,\underline{R}^T\underline{x} + \underline{R}\,\underline{\dot{y}} \tag{3-25}$$

By comparison of (3-22) with (3-25), and considering (3-23) the following identities hold:

$$\underline{\boldsymbol{\omega}} \times \underline{x} = \underline{\dot{R}}\,\underline{R}^T\underline{x} \tag{3-26a}$$

$$\underline{\dot{x}}_{rel} = \underline{R}\,\underline{\dot{y}} \tag{3-26b}$$

$$\underline{\boldsymbol{\Omega}} = \underline{\dot{R}}\,\underline{R}^T \tag{3-26c}$$

Thus, eq. (3-22) can be rewritten using (3-26a,c).

$$\underline{\dot{x}}_{rel} = \underline{\dot{x}} - \mathbf{\Omega}\, \underline{x} \tag{3-27a}$$

The norm of $\underline{\dot{x}}_{rel}$ is given considering (3-23c) by

$$\left| \underline{\dot{x}}_{rel} \right| = \left(\underline{\dot{x}}^T_{rel}\, \underline{\dot{x}}_{rel} \right)^{0.5} = \left(\underline{\dot{x}}^T \underline{\dot{x}} + 2\, \underline{x}^T \mathbf{\Omega}\, \underline{\dot{x}} - \underline{x}^T \mathbf{\Omega}^2 \right)^{0.5} \tag{3-27b}$$

The time derivative of $\underline{R}$ in (3-26c) is determined by differentiation of (3-5c).

$$\begin{aligned} \underline{\dot{R}}(t) = {} & \underline{N}(t_o)\, \underline{P}(t_o)\, \underline{\dot{P}}^T(t)\, \underline{N}^T(t)\, \underline{R}_3\big(-\Theta(t)\big)\, \underline{S}(t) + \\ & + \underline{N}(t_o)\, \underline{P}(t_o)\, \underline{P}^T(t)\, \underline{\dot{N}}^T(t)\, \underline{R}_3\big(-\Theta(t)\big)\, \underline{S}(t) + \\ & + \underline{N}(t_o)\, \underline{P}(t_o)\, \underline{P}^T(t)\, \underline{N}^T(t)\, \underline{\dot{R}}_3\big(-\Theta(t)\big)\, \underline{S}(t) + \\ & + \underline{N}(t_o)\, \underline{P}(t_o)\, \underline{P}^T(t)\, \underline{N}^T(t)\, \underline{R}_3\big(-\Theta(t)\big)\, \underline{\dot{S}}(t) + \end{aligned} \tag{3-28}$$

Since

(i) the interval $[t, t_o]$ of orbit integration is considerable small and, therefore, the earth-rotation vector $\underline{\boldsymbol{\omega}}$ shows only very small changes due to precession, nutation and polar motion in that intervall, and

(ii) the density models commonly-used are only a rough approximation,

the position and velocity vector in (3-26a,...,c) does not need a very high accuracy. Thus, it is justified to neglect in the air-drag model precession, nutation and polar motion, and to approximate the rotation matrix in (2-28) by

$$\underline{R}(t) \doteq \underline{R}_3\big(-\Theta(t)\big) \tag{3-29a}$$

and consequently, (3-26c) by

$$\mathbf{\Omega} \doteq \underline{\dot{R}}_3\big(-\Theta(t)\big)\, \underline{R}^T_3\big(-\Theta(t)\big) \tag{3-29b}$$

Replacing the angular velocity component $\dot{\Theta}$ by the angular velocity $\dot{\Theta}_A$ of the atmosphere, we get using (3-29,a,b)

$$\underline{\boldsymbol{\omega}}_A = \left[0, 0, \dot{\Theta}_A \right]^T \tag{3-30a}$$

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\dot{\Theta}_A & 0 \\ \dot{\Theta}_A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3-30b}$$

Thus, the vector $\underline{\dot{x}}_{rel}$ is defined by

$$\underline{\dot{x}}_{rel} = \underline{\dot{x}} - \underline{\Omega}_A \underline{x} = \left[\dot{x}_1 + \dot{\Theta}_A x_2 , \dot{x}_2 - \dot{\Theta}_A x_1 , 0 \right]^T \tag{3-31a}$$

3.3.2 The modified Harris-Priester model atmosphere

In order to compute the acceleration $\underline{f}_1$, the density ρ of the atmosphere at position $\underline{x} = \underline{x}(t)$ of the satellite has to be known.

In principle, the density ρ can be considered as a function of position (e.g. the altitude above the earth), of time and of further model parameters. From the various models we will discuss the ***modified Harris-Priester model*** because of its simple structure. It is described in detail in ***Cappelari et al (1976, p. (4-53)ff.).*** Therefore we like to outline it here only shortly.

(i) The model considers the energy flux of the extreme ultra-violet light and other corpuscular heat sources.

(ii) The annual and semi-annual variations of density as functions of latitude are averaged.

(iii) The variation of density due to the variation of the extreme ultra-violet light is not considered.

(iv) The modification of the original Harris-Priester model consists of the consideration of the day/night effect by a cosine function between a density profile of maximum and minimum density.

(v) The models use a table of minimum and maximum density profiles as function of discrete altitudes between those the values are interpolated exponentially.

The density ρ is given according to ***Cappelari et al (1976, (4-132)ff.)*** by

$$\rho(h, t, \overline{\rho}_1, \overline{\mu}, \overline{\lambda}) = \left[1 + \overline{\rho}_1 \cos^{\overline{\mu}}(\psi/2) \right] \left\{ \rho_m(h) + \left[\rho_M(h) - \rho_m(h) \right] \cos^{\overline{\mu}}(\psi/2) \right\} \tag{3-33}$$

where h is the altitude of the satellite above the earth’s surface, e.g. in first approximation above the earth’s ellipsoid,

$$h \doteq |\underline{x}| - R_E \tag{3-34a}$$

$\underline{x}$ is the position vector of the satellite and R_E is the geocentric radius of the sub-satellite point on the ellipsoid,

$$R_E = \frac{R(1-f)}{\left[1-\left(2f-f^2\right)\cos^2\varphi\right]^{0.5}} \tag{3-34b}$$

R is the mean earth radius, f the flattening of the mean earth ellipsoid, and φ is the spherical latitude refering to the earth-fixed reference system.

The angle ψ is given by

$$\cos\psi = \underline{u}^T \underline{x} / |\underline{x}| \tag{3-34c}$$

where the unit vector $\underline{u}$ of the point of the density maximum of the atmosphere is defined by

$$\underline{u} = \begin{bmatrix} \cos\delta_S \ \cos(\alpha_S+\bar{\lambda}) \\ \cos\delta_S \ \sin(\alpha_S+\bar{\lambda}) \\ \sin\delta_S \end{bmatrix} \quad \text{with} \quad |\underline{u}| = 1 \tag{3-34d}$$

$\alpha_S = \alpha_S(t)$, $\delta_S = \delta_S(t)$ are the rectascension and declination, respectively, of the sun in the inertial system at time t . $\bar{\lambda}$ is the phase lag of the density bulk with respect to the movement of the sun in the equator $(\bar{\lambda} \doteq +30°)$.

The minimum and maximum density values $\rho_m(h_i)$, $\rho_M(h_i)$ given in tables as function of altitudes h_i can be interpolated at h , $h_i \le h \le h_{i+1}$, by

$$\rho_m(h) = \rho_m(h_i)\, e^{(h_i-h)/H_m} \tag{3-34e}$$

$$\rho_M(h) = \rho_M(h_i)\, e^{(h_i-h)/H_M} \tag{3-34f}$$

where H_m , H_M are interpolation constants,

$$H_m = \frac{h_i - h_{i+1}}{\ln \rho_m(h_{i+1}) - \ln \rho_m(h_i)} \tag{3-34g}$$

$$H_M = \frac{h_i - h_{i+1}}{\ln \rho_M(h_{i+1}) - \ln \rho_M(h_i)} \tag{3-34h}$$

$\bar{\mu}$, $\bar{\rho}_1$ in (3-33) are additional model parameters.

3.4 Acceleration due to solar radiation pressure

Considering only the direct solar radiation pressure, the corresponding acceleration $\underline{f}_2$ is givven according to ***Cappelari et al (1976, p. (4-60)ff.)*** by

$$\underline{f}_2 = \nu \, k \, a_S^2 \frac{\underline{x} - \underline{x}_S}{\left| \underline{x} - \underline{x}_S \right|^3} \qquad (3\text{-}35)$$

where

$$k = P_S \, (A/m) \, c_R \qquad (3\text{-}36a)$$

$$P_S = S/c \qquad (3\text{-}36b)$$

$$c_R = 1 + \eta \qquad (3\text{-}36c)$$

$$\underline{x}_S = r_S \begin{bmatrix} \cos\delta_S \cos\alpha_S \\ \cos\delta_S \sin\alpha_S \\ \sin\delta_S \end{bmatrix} \qquad (3\text{-}36d)$$

$$r_S = a_S \left(1 - e_S \cos E_S \right) \qquad (3\text{-}36e)$$

The quantities in the formulas presented above are

k ... is a model parameter,

ν ... $\nu = \begin{cases} 1 & \text{if the satellite is outside of the earth's shadow,} \\ 0 & \text{if the satellite is within the earth's shadow,} \end{cases}$

a_S ... is the semi-major axis of the mean orbit of the earth,

S ... is the mean energy flux of the sun in $\left[W m^{-2} \right]$,

c ... is the velocity of light,

η ... is the quantity describing the optical properties of the material of the satellite (e.g. for aluminium $\eta = 0.95$),

A/m ... is the relation of effective surface of the satellite to its mass,

r_S ... is the distance of the sun with respect to the geocenter,

α_S, δ_S ... is the rectascension and declination, respectively, of the sun in the inertial reference system,

e_S ... is the excentricity of the orbit of the earth,

E_S ... is the excentric anomaly of the earth, and

$\underline{x}$ ... is again the position vector of the satellite in the inertial system.

Depending on the accuracy requirements, further effects related to the solar radiation have to be considered:

(1) ***Indirect solar radiation pressure***, caused by reflection of the sun's radiation on the earth's atmosphere ***(Albedo effect)***,

(2) ***Direct radiation pressure of the earth***,

(3) ***Yarkovski/Schach effect***, caused by the one-sited heat-up of the surface of the satellite in direction to the sun,

(4) ***Poynting-Robertson effect***, a relativistic correction to (3-35).

3.5 Acceleration due to the attraction of sun and moon

The acceleration acting on a mass point (satellite) due to the attraction of a planet H , ***relative*** to the geocenter, is given by

$$\underline{f}_{3H} = -k m_H \left[\frac{\underline{x} - \underline{x}_H}{\left| \underline{x} - \underline{x}_H \right|^3} + \frac{\underline{x}_H}{\left| \underline{x}_H \right|^3} \right] \tag{3-37}$$

where

k	...	is the gravitational constant,
m_H	...	mass of the plant H ,
$\underline{x}_H$	...	position vector of the planet H in the inertial system, and
$\underline{x}$	...	is again the position vector of the satellite in the inertial system.

For the question, why (3-37) is treated as ***relative*** acceleration see also the explanation in paragraph 3.1.1.

Considering here only the attraction of sun and moon, the total acceleration on the satellite is

$$\underline{f}_3 = \underline{f}_{3M} + \underline{f}_{3S} \tag{3-38a}$$

$$\underline{f}_3 = -k \left[m_M \left(\frac{\underline{x} - \underline{x}_M}{\left| \underline{x} - \underline{x}_M \right|^3} + \frac{\underline{x}_M}{\left| \underline{x}_M \right|^3} \right) + m_S \left(\frac{\underline{x} - \underline{x}_S}{\left| \underline{x} - \underline{x}_S \right|^3} + \frac{\underline{x}_S}{\left| \underline{x}_S \right|^3} \right) \right] \tag{3-38b}$$

where

m_M	...	is the mass of the moon,
m_S	...	is the mass of the sun, and
$\underline{x}_M, \underline{x}_S$	...	are the position vectors of the moon and sun, respectively, in the inertial system,

$$\underline{x}_M = r_M \begin{bmatrix} \cos\delta_M \cos\alpha_M \\ \cos\delta_M \sin\alpha_M \\ \sin\delta_M \end{bmatrix} \tag{3-38c}$$

$$\underline{x}_S = r_S \begin{bmatrix} \cos\delta_S \cos\alpha_S \\ \cos\delta_S \sin\alpha_S \\ \sin\delta_S \end{bmatrix} \tag{3-38d}$$

$$r_M = a_M \left(1 - e_M \cos E_M\right) \tag{3-38e}$$

$$r_S = a_S \left(1 - e_S \cos E_S\right) \tag{3-38f}$$

α_M, δ_M ... is the rectascension and declination, respectively, of the moon,

α_S, δ_S ... is the rectascension and declination, respectively, of the sun,

r_M, r_S ... are distances geocenter-moon, and geocenter-sun, respectively,

a_M, a_S ... are the semi-major axes of the moon and the earth, respectively,

e_M, e_S ... are the excentricities of the orbit of the moon and the earth, and

E_M, E_S ... are the mean anomalies of the orbit of the moon, and respectively, of the sun.

3.6 Acceleration due to the tides of the solid earth

The elastic earth is deformed by the attraction of sun and moon. The so-caused mass variations produce at the orbit of the satellite an additional gravity potential $\delta U(\underline{x})$, which can be expressed by ***(Lambeck, 1974, p. 44)***

$$\delta U(\underline{x}) = k \frac{m_H}{|\underline{x}_H|} \sum_{\sigma=2}^{\infty} k_\sigma \left(\frac{R}{|\underline{x}_H|} \right)^{\sigma} \left(\frac{R}{|\underline{x}|} \right)^{\sigma+1} P_\sigma (\cos\delta_H) \tag{3-39a}$$

$$\cos\delta_H = \frac{\underline{x}^T \cdot \underline{x}_H}{|\underline{x}| \cdot |\underline{x}_H|} \tag{3-39b}$$

where

k_σ ... are Love numbers. The index σ indicates the frequency dependency of Love numbers,

P_σ ... is the Legendre polynomials of first kind (zonal spherical harmonics), and

R ... is the mean earth's radius.

For all other quantities in (3-39a,b) see the chapters before. The index H at some quantities stands for the specific planet H (sun or moon) under consideration. Tides of the ***solid*** earth means here, that oceanic tides are neglected.

In high altitudes above the earth's surface, (3-39a) can be approximated by the form $\sigma = 2$, e.g.

$$\delta U(\underline{x}) \doteq k m_H \, k_2 \, \frac{R^5}{\left|\underline{x}_H\right|^3 \left|\underline{x}\right|^3} \, P_2 \left(\cos \delta_H\right) \tag{3-40}$$

Since the point of maximum deformation is not in direction to the planet due to inner friction of the earth's body, thus showing up with a certain time delay Δt , the actual position $\underline{x}_H$ of the planet has to be replaced by a fictitious one, $\underline{\tilde{x}}_H$, considering the corresponding time delay Δt .

The equatorial Kepler elements are corrected by

$$\tilde{\Omega}_H = \Omega_H - \left(\dot{\Omega}_H - \dot{\Theta}\right) \Delta t \tag{3-41a}$$

$$\tilde{\omega}_H = \omega_H - \dot{\omega}_H \Delta t \tag{3-41b}$$

$$\tilde{M}_H = M_H - \dot{M}_H \Delta t \tag{3-41b}$$

and thus, $\underline{\tilde{x}}_H$, is given by (2-22):

$$\underline{\tilde{x}}_H = a_H \, \underline{R}_3\left(-\tilde{\Omega}_H\right) \underline{R}_1\left(-i_H\right) \underline{R}_3\left(-\tilde{\omega}_H\right) \begin{bmatrix} \cos \tilde{E}_H - e_H \\ \left(1 - e_H^2\right)^{0.5} \sin \tilde{E}_H \\ 0 \end{bmatrix} \tag{3-42a}$$

where

$$M_H = E_H - e_H \sin E_H \tag{3-42b}$$

$$\cos \tilde{\delta}_H = \frac{\underline{x}^T \cdot \underline{\tilde{x}}_H}{\left|\underline{x}\right| \left|\underline{\tilde{x}}_H\right|} \tag{3-42c}$$

The acceleration of the satellite caused by the elastic deformation of the earth is determined by the gradient of (3-40).

$$\underline{f}_{4H} = \operatorname{grad}_x \delta U(\underline{x})$$

$$= k m_H \, k_2 \, \frac{R^5}{\left|\underline{\tilde{x}}_H\right|^3 \left|\underline{x}\right|^4} \left[P_2' \left(\cos \tilde{\delta}_H\right) \frac{\underline{x}_H}{\left|\underline{\tilde{x}}_H\right|} - P_3' \left(\cos \tilde{\delta}_H\right) \frac{\underline{x}}{\left|\underline{x}\right|} \right] \tag{3-43}$$

where the Legendre polynomials and its derivatives are given by

$$P_2\left(\cos\tilde{\delta}_H\right) = 0.5\left(3\cos^2\tilde{\delta}_H - 1\right) \tag{3-44a}$$

$$P_2'\left(\cos\tilde{\delta}_H\right) = \frac{\partial P_2\left(\cos\tilde{\delta}_H\right)}{\partial\left(\cos\tilde{\delta}_H\right)} = 3\cos\tilde{\delta}_H \tag{3-44b}$$

$$P_3'\left(\cos\tilde{\delta}_H\right) = \frac{\partial P_3\left(\cos\tilde{\delta}_H\right)}{\partial\left(\cos\tilde{\delta}_H\right)} = 1.5\left(5\cos^2\tilde{\delta}_H - 1\right) \tag{3-44c}$$

Thus, the resultant acceleration of sun and moon is derived by adding their specific terms of (3-43), H: = moon M , and H: = sun S .

$$\underline{f}_4 = \underline{f}_{4S} + \underline{f}_{4M} \tag{3-45a}$$

$$\underline{f}_4 = k_2 \frac{R^5}{|\underline{x}|^4}\left\{\frac{km_S}{\left|\underline{\tilde{x}}_S\right|^3}\left[P_2'\left(\cos\tilde{\delta}_S\right)\frac{\underline{\tilde{x}}_S}{\left|\underline{\tilde{x}}_S\right|} - P_3'\left(\cos\tilde{\delta}_S\right)\frac{\underline{x}}{|\underline{x}|}\right] + \right.$$
$$\left. + \frac{km_M}{\left|\underline{\tilde{x}}_M\right|^3}\left[P_2'\left(\cos\tilde{\delta}_M\right)\frac{\underline{\tilde{x}}_M}{\left|\underline{\tilde{x}}_M\right|} - P_3'\left(\cos\tilde{\delta}_M\right)\frac{\underline{x}}{|\underline{x}|}\right]\right\} \tag{3-45b}$$

4. THE OBSERVATION EQUATIONS OF SATELLITE GEODESY

In this paragraph we will derive the observation equations of satellite geodesy in the model of integrated geodesy. In detail, the following types of satellite observations are treated:

Type S1: $\underline{d}$ ... *direction measurements*

S2: S ... *distance measurements from terrestrial ground station to a satellite*

S3: dS/dt ... *Doppler observations of a satellite at a ground station*

S4.1: ds/dt ... *satellite-to-satellite tracking (Doppler type)*

S4.2: s ... *satellite-to-satellite tracking (intersatellite laser distances)*

S5: τ ... *interferometric time delays*

S6: $\dot{\tau}$ ... *interferometric Doppler differences*

S7: h ... *altimetric heights (distances from a satellite to the sea-surface)*

We will assume that above mentioned quantities can be considered as more or less original observations. The introduction of further parameters, like e.g. atmospheric parameters, clock errors, etc., however, is possible in this approach. It can be easily done. Since those parameters might depend on the different types of instruments used, it will be not discussed further on.

The basic reference system used in the derivation of observation equations, is the ***inertial reference frame*** $\underline{x}$ defined in paragraph 3.1.1.

4.1 General form of observation equations

Every satellite observation l_i can be considered as nonlinear functional depending on the three-dimensional position vector $\underline{x}$ and on the velocity vector $\underline{\dot{x}}$ of the ground station G and of the satellite Q, respectively.

$$l_i = l_i\left(\underline{x}_Q, \underline{\dot{x}}_Q, \underline{x}_G, \underline{\dot{x}}_G\right) \tag{4-1}$$

The position vector $\underline{x}_G$ of the ground station can also be expressed in the earth-fixed coordinate system, e.g., $\underline{y}_G$.

Introducing the approximate coordinate vectors $\underline{x}_Q^o, \underline{\dot{x}}_Q^o, \underline{x}_G^o, \underline{\dot{x}}_G^o$ in the decompositions,

$$\underline{x}_Q = \underline{x}_Q^o + \delta\underline{x}_Q \tag{4-2a}$$

$$\underline{\dot{x}}_Q = \underline{\dot{x}}_Q^o + \delta\underline{\dot{x}}_Q \tag{4-2b}$$

$$\underline{x}_G = \underline{x}_G^o + \delta\underline{x}_G \tag{4-2c}$$

$$\underline{\dot{x}}_G = \underline{\dot{x}}_G^o + \delta\underline{\dot{x}}_G \tag{4-2d}$$

as well as considering the corresponding approximate observations l_i^o ,

$$l_i = l_i^o + \delta l_i \tag{4-2e}$$

and expanding δl_i in a Taylor series neglecting higher order terms, leads to

$$\delta l_i = \frac{\partial l_i}{\partial \underline{x}_Q}\Bigg|^o \delta\underline{x}_Q + \frac{\partial l_i}{\partial \underline{\dot{x}}_Q}\Bigg|^o \delta\underline{\dot{x}}_Q + \frac{\partial l_i}{\partial \underline{x}_G}\Bigg|^o \delta\underline{x}_G + \frac{\partial l_i}{\partial \underline{\dot{x}}_G}\Bigg|^o \delta\underline{\dot{x}}_G \tag{4-3}$$

The partial derivatives with respect to the vectors $\underline{x}_Q^o, \underline{\dot{x}}_Q^o, \underline{x}_G^o, \underline{\dot{x}}_G^o$ are nothing else than the gradients

$$\frac{\partial l_i}{\partial \underline{x}_Q} = \left[\frac{\partial l_i}{\partial \underline{x}_{1Q}}, \frac{\partial l_i}{\partial \underline{x}_{2Q}}, \frac{\partial l_i}{\partial \underline{x}_{3Q}}\right] \tag{4-4a}$$

$$\frac{\partial l_i}{\partial \underline{\dot{x}}_Q} = \left[\frac{\partial l_i}{\partial \underline{\dot{x}}_{1Q}} , \frac{\partial l_i}{\partial \underline{\dot{x}}_{2Q}} , \frac{\partial l_i}{\partial \underline{\dot{x}}_{3Q}} \right] \tag{4-4b}$$

$$\frac{\partial l_i}{\partial \underline{x}_G} = \left[\frac{\partial l_i}{\partial \underline{x}_{1G}} , \frac{\partial l_i}{\partial \underline{x}_{2G}} , \frac{\partial l_i}{\partial \underline{x}_{3G}} \right] \tag{4-4c}$$

$$\frac{\partial l_i}{\partial \underline{\dot{x}}_G} = \left[\frac{\partial l_i}{\partial \underline{\dot{x}}_{1G}} , \frac{\partial l_i}{\partial \underline{\dot{x}}_{2G}} , \frac{\partial l_i}{\partial \underline{\dot{x}}_{3G}} \right] \tag{4-4d}$$

Since the position vector $\underline{x}$ and the velocity vector $\underline{\dot{x}}$ of the satellite is a function of dynamical parameters, e.g., parameters of the acceleration model (see chapter 3) and of the gravity potential, the linearization process should result in a linear functional of the form (see also ***Moritz (1980, p. 226 ff.)***)

$$\delta l_i = \underline{b}_i^T \delta \underline{y}_G + \underline{a}_i^T \delta \underline{p} + L_i(T) \tag{4-5}$$

where

$\delta \underline{y}_G$ ... is the vector of residual coordinate unknowns of the ground station in the earth-fixed reference frame (see 3.1.2),

$\delta \underline{p}$ ... is a vector of residual dynamical parameters, e.g., parameters of the acceleration model,

$L_i(T)$ ... is a linear operator of the residual (or disturbing) potential, and

$\underline{a}_i$, $\underline{b}_i$ ... are coefficient vectors of the unknowns $\delta \underline{y}_G$, $\delta \underline{p}$

The linear relation between $\delta \underline{x}_Q$, $\delta \underline{\dot{x}}_Q$ in (4-3) and $\delta \underline{p}$, $L(T)$ in (4-5) can only be derived by considering the coordinate vector $\underline{v}$ resulting from the successive approximation (2-100).

$$\underline{x}_Q = \underline{x}_Q\big(\underline{v}(t), t\big) \tag{4-6a}$$

$$\underline{\dot{x}}_Q = \underline{\dot{x}}_Q\big(\underline{v}(t), t\big) \tag{4-6b}$$

Decomposing $\underline{v}(t)$ into an approximate vector $\underline{v}^o(t)$ and a residual (linear) vector $\delta \underline{v}(t)$,

$$\underline{v}(t) = \underline{v}^o(t) + \delta \underline{v}(t) \tag{4-7}$$

it follows from (4-6a,b)

$$\underline{x}_Q = \underline{x}_Q\big(\underline{v}^o(t), t\big) + \frac{\partial \underline{x}_Q}{\partial \underline{v}}\big(\underline{v}^o(t), t\big)\, \delta \underline{v}(t) \tag{4-8a}$$

$$\dot{\underline{x}}_Q = \dot{\underline{x}}_Q\big(\underline{v}^o(t),t\big) + \frac{\partial \dot{\underline{x}}_Q}{\partial \underline{v}}\big(\underline{v}^o(t),t\big)\ \delta\underline{v}(t) \tag{4-8b}$$

The Jacobi matrices $\partial\underline{x}_Q/\partial\underline{v}$, $\partial\dot{\underline{x}}_Q/\partial\underline{v}$ are already explicitly given by (2-76) and (2-84) in connection with (2-68). The meaning of the approximate vector $\underline{v}^o(t)$ is discussed later.

Usinf the abbreviations $\underline{X}$, $\dot{\underline{X}}$ for the mentioned Jacobi matrices (see also 2.4.3 and 2.4.4)

$$\underline{x}_Q(t) = \underline{x}_Q\big(\underline{v}^o(t),t\big) + \underline{X}\big(\underline{v}^o(t),t\big)\ \delta\underline{v}(t) \tag{4-9a}$$

$$\dot{\underline{x}}_Q(t) = \dot{\underline{x}}_Q\big(\underline{v}^o(t),t\big) + \dot{\underline{X}}\big(\underline{v}^o(t),t\big)\ \delta\underline{v}(t) \tag{4-9b}$$

Thus, we find for $\delta\underline{x}_Q$, $\delta\dot{\underline{x}}_Q$ in (4-2a,b) the relation

$$\delta\underline{x}_Q = \underline{X}\ \delta\underline{v} \tag{4-9c}$$

$$\delta\dot{\underline{x}}_Q = \dot{\underline{X}}\ \delta\underline{v} \tag{4-9d}$$

4.2 The vector $\underline{p}$ of model parameters

The vector $\underline{p}$ (or its residual vector $\delta\underline{p}$) forms together with the ground station coordinates the ***deterministic unknown*** part of model (4-5). It is defined by the introduced acceleration model. Here we restrict ourselves to the general force modelling given in paragraph 3. It is obvious, that further parametrization is possible.

For the following considerations we partition $\underline{p}$ in five sub-vectors,

$$\underline{p} = \left[\underline{p}_1^T, \underline{p}_2^T, \underline{p}_3^T, \underline{p}_4^T, \dot{\underline{p}}_4^T\right]^T \tag{4-10}$$

where

$\underline{p}_1$ is the vector of the initial state of the satellite at $t = t_o$, here defined by the modified Poincaré elements,

$$\underline{p}_1 = \left[l_o, g_o, h_o, L_o, G_o, H_o\right]^T \tag{4-11a}$$

$\underline{p}_2$ is the vector of gravity potential parameters (spherical harmonic coefficients of an earth model of order n),

$$\underline{p}_2 = \left[c_{20}, c_{21}, \ldots, c_{nn}, s_{20}, s_{21}, \ldots, s_{nn}\right]^T \tag{4-11b}$$

$\underline{p}_3$ is the vector of all other remaining parameters of the different parts of the introduced acceleration model, e.g.,

$$\underline{p}_3 = \left[\dot{\Theta}, C_D, \overline{\rho}_1, \overline{\mu}, \overline{\lambda}, k, k_2, \Delta t\right]^T \tag{4-11c}$$

$\underline{p}_4$ is the vector of earth rotation parameters (pole coordinates ξ , η , and sidereal time Θ),

$$\underline{p}_4 = [\xi, \eta, \Theta]^T \tag{4-11d}$$

and the vector $\underline{\dot{p}}_4$ consists of their variations with time

$$\underline{\dot{p}}_4 = [\dot{\xi}, \dot{\eta}, \dot{\Theta}]^T \tag{4-11e}$$

The vector $\underline{p}$ and its sub-vectors have the following order:

$$0(\underline{p}) = [11 + (n+1) \cdot (n+2)] \times 1 = m \times 1 \tag{4-12a}$$

$$0(\underline{p}_1) = 6 \times 1 \tag{4-12b}$$

$$0(\underline{p}_2) = [(n+1) \cdot (n+2) - 6] \times 1 \tag{4-12c}$$

$$0(\underline{p}_3) = 8 \times 1 \tag{4-12d}$$

$$0(\underline{p}_4) = 3 \times 1 \tag{4-12e}$$

$$0(\underline{\dot{p}}_4) = 3 \times 1 \tag{4-12f}$$

4.3 Determination of $\delta\underline{v}(t)$ by the linearization principle of integrated geodesy

According to (2-7) the resulting acceleration vector $\underline{a}$ is given by

$$\underline{a} = \text{grad}_x W_1 + \underline{f}_1$$

where $W_1 = W - W_o$ and $W_o = kM / |\underline{x}_Q|$.

The Linearization principle of integrated geodesy consists of two steps:

(i) $W_1 = U_1 + T$ (4-13)

(ii) $\underline{p} = \underline{p}^o + \delta\underline{p}$ (4-14)

U_1 in (4-13) is the so-called ***normal (or model) potential U*** reduced for the radial-symmetrical part $U_o = kM/r$, consequently $U_1 = U - U_o$, and T is the gravity ***disturbing potential*** of the earth. $\underline{p}^o$ is the vector of approximate parameters of $\underline{p}$, see (4-11a,...,e).

Thus, the gradient of the gravity potential W_1 (4-13) is

$$\text{grad}_x W_1 = \text{grad}_x U_1 + \text{grad}_x T \tag{4-15}$$

or using the formerly introduced notations of ***Hein (1981, p. 43)***,

$$\underline{g}_1 = \underline{j}_1 + \delta\underline{g} \tag{4-16}$$

where

$$\underline{g}_1 = \text{grad}_x W_1 \tag{4-17a}$$

$$\underline{j}_1 = \text{grad}_x U_1 \tag{4-17b}$$

$$\delta\underline{g} = \text{grad}_x T \tag{4-17c}$$

Introducing (4-16) in (2-7) we get

$$\underline{a} = \underline{g}_1 + \underline{f} = \underline{j}_1 + \underline{f} + \delta\underline{g} \tag{4-18}$$

Note, that the vector $\delta\underline{g}$ in (4-18) is already of first order.

Next, we insert (4-18) in the successive approximation series (2-100), resulting in

$$\underline{v}_{k+1}(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k(t),t\right) \left[\underline{j}_1\left(\underline{v}_k(t),t\right) + \underline{f}\left(\underline{v}_k(t),t\right) + \delta\underline{g}\left(\underline{v}_k(t),t\right)\right] dt \tag{4-19}$$

or

$$\underline{v}_{k+1}(t) = \underline{v}_o + \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k(t),t\right) \left[\underline{j}_1\left(\underline{v}_k(t),t\right) + \underline{f}\left(\underline{v}_k(t),t\right)\right] dt + \\ + \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k(t),t\right) \delta\underline{g}\left(\underline{v}_k(t),t\right) dt \tag{4-20}$$

Linearizing (4-20) by (4-14) having in mind the functional relationship

$$\underline{v}_o = \underline{v}_o\left(\underline{p}\right) \tag{4-21a}$$

$$\underline{j}_1 = \underline{j}_1\left(\underline{v}_k(t),t,\underline{p}\right) \tag{4-21b}$$

$$\delta\underline{g} = \delta\underline{g}\left(\underline{v}_k(t),t,\underline{p}\right) \tag{4-21c}$$

$$\underline{f} = \underline{f}\left(\underline{v}_k(t),t,\underline{p}\right) \tag{4-21d}$$

$$\underline{v}_k(t) = \underline{v}_k\left(t,\underline{p}\right) \tag{4-21e}$$

We get neglecting higher order terms,

$$\begin{aligned}
\underline{v}_{k+1}\left(t,\underline{p}\right) &= \underline{v}_o\left(t,\underline{p}^o\right) + \\
&+ \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k\left(t,\underline{p}^o\right),t\right) \left[\underline{j}_1\left(\underline{v}_k\left(t,\underline{p}^o\right),\underline{p}^o,t\right) + \underline{f}\left(\underline{v}_k\left(t,\underline{p}^o\right),\underline{p}^o,t\right)\right] dt + \\
&+ \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k\left(t,\underline{p}^o\right),t\right) \delta\underline{g}\left(\underline{v}_k\left(t,\underline{p}^o\right),\underline{p}^o,t\right) dt + \left(\frac{\partial \underline{v}_o}{\partial \underline{p}}\right)_{\underline{p}=\underline{p}^o} \delta\underline{p} + \\
&+ \frac{\partial}{\partial \underline{p}} \left\{ \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k\left(t,\underline{p}^o\right),t\right) \left[\underline{j}_1\left(\underline{v}_k\left(t,\underline{p}^o\right),\underline{p}^o,t\right) + \underline{f}\left(\underline{v}_k\left(t,\underline{p}^o\right),\underline{p}^o,t\right)\right] dt \right\}_{\underline{p}=\underline{p}^o} \delta\underline{p} + \\
&+ \frac{\partial}{\partial \underline{p}} \left\{ \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k\left(t,\underline{p}^o\right),t\right) \delta\underline{g}\left(\underline{v}_k\left(t,\underline{p}^o\right),\underline{p}^o,t\right) dt \right\}_{\underline{p}=\underline{p}^o} \delta\underline{p}
\end{aligned} \tag{4-22}$$

The integrals in (4-22) depend not only on the integration parameter t, but also on $\underline{p}$. The question whether the last two integrals (4-22) may be partially differentiated with respect to $\underline{p}$ can be answered by the following.

Partial differentiation under the integral sign is always possible, if the corresponding expression is continuous in the domain $G: \{t_1 \le t \le t_2\ ,\ \underline{p}_1 \le \underline{p} \le \underline{p}_2\}$, and continuously differentiable with respect to p_i ***(Bronstein, Semendjajew, 1975, p. 348)***. Without proof we will assume that these conditions are fulfilled.

Considering the last integral in (4-22), which is already of second order due to the first-order term $\delta\underline{g}$ in the product, and using eq. (C-12) of appendix C, we get

$$\frac{\partial}{\partial \underline{p}} \int_{t_o}^{t} \underline{Y}\, \delta\underline{g}\, dt\, \delta\underline{p} = \int_{t_o}^{t} \frac{\partial}{\partial \underline{p}} \left(\underline{Y}\, \delta\underline{g}\right) dt\, \delta\underline{p} \tag{4-23a}$$

$$\frac{\partial}{\partial \underline{p}} \left(\underline{Y}\, \delta\underline{g}\right) = \left[\frac{\partial \underline{Y}}{\partial p_1}\delta\underline{g},\ \frac{\partial \underline{Y}}{\partial p_2}\delta\underline{g},\ \ldots,\ \frac{\partial \underline{Y}}{\partial p_n}\delta\underline{g}\right] + \underline{Y}\,\frac{\partial \delta\underline{g}}{\partial \underline{p}} \tag{4-23b}$$

According to appendix C eq. (4-23b) is the Jacobi matrix of the product $\underline{Y}\,\delta\underline{g}$ with respect to $\underline{p}$, which is also of first order. By post-multiplying (4-23b) with $\delta\underline{p}$ (see (4-23a)) results in second-order terms which can be neglected if the approximate values were sufficient for the linearization process. Consequently, the integral offers the possibility of proving this fact after estimation of $\delta\underline{p}$ and $\delta\underline{g}$.

Recording of (4-22) using (4-23a,b) results in

$$\underline{v}_{k+1}\left(t, \underline{p}\right) = \underline{v}_{k+1}\left(t, \underline{p}^o\right) + \delta\underline{v}_{k+1}\left(t, \underline{p}^o\right) \tag{4-24}$$

$$\underline{v}_{k+1}\left(t, \underline{p}^o\right) = \underline{v}_o\left(t, \underline{p}^o\right) + \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k\left(t, \underline{p}^o\right), t\right) \left[\underline{j}_1\left(\underline{v}_k\left(t, \underline{p}^o\right), t\right) + \underline{f}\left(\underline{v}_k\left(t, \underline{p}^o\right), \underline{p}^o, t\right)\right] dt \tag{4-24a}$$

$$\begin{aligned}\delta\underline{v}_{k+1}\left(t, \underline{p}^o\right) = \frac{\partial}{\partial \underline{p}} \Big\{ \underline{v}_o\left(t, \underline{p}\right) &+ \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k\left(t, \underline{p}\right), t\right) \left[\underline{j}_1\left(\underline{v}_k\left(t, \underline{p}\right), \underline{p}, t\right) + \underline{f}\left(\underline{v}_k\left(t, \underline{p}\right), \underline{p}, t\right)\right] dt \Big\}_{(\underline{p}^o)} \delta\underline{p} + \\ &+ \int_{t_o}^{t} \underline{Y}\left(\underline{v}_k\left(t, \underline{p}^o\right), t\right)\, \delta\underline{g}\left(\underline{v}_k\left(t, \underline{p}^o\right), \underline{p}^o, t\right) dt\end{aligned} \tag{4-24b}$$

With the above mentioned expressions we are able now to define the linear variations $\delta\underline{x}_Q, \delta\dot{\underline{x}}_Q$ (4-9a,b) as functions of $\delta\underline{p}$ and $\delta\underline{g}$. Eq. (4-24a) in connection with the normal (or model) gravity vector $\underline{j}_1$ and the approximate vector $\underline{p}^o$ form the approximate vector $\underline{v}^o(t)$ in (4-7).

$$\underline{v}^o(t) \doteq \underline{v}_{k+1}\left(t, \underline{p}^o\right) \qquad (4\text{-}25a)$$

$$\delta\underline{v}(t) \doteq \delta\underline{v}_{k+1}\left(t, \underline{p}^o\right) \qquad (4\text{-}25b)$$

4.4 The determination of the necessary Jacobi matrices

The Jacobi matrix $\partial\underline{v}/\partial\underline{p}$

The matrix standing left to the vector $\delta\underline{p}$ in (4-24b) is nothing else than an approximation of the Jacobi matrix $\partial\underline{v}/\partial\underline{p}$ in the iteration step $k+1$.

Thus,

$$\frac{\partial\underline{v}}{\partial\underline{p}} \doteq \frac{\partial\underline{v}_{k+1}}{\partial\underline{p}} = \frac{\partial}{\partial\underline{p}}\left[\underline{v}_o + \int_{t_o}^{t} \underline{Y}\left(\underline{j}_1 + \underline{f}\right) dt\right] \qquad (4\text{-}26)$$

In order to determine $\partial\underline{v}/\partial\underline{p}$ we use the partitioning of $\underline{p}$ introduced already by (4-10). This is reasonable since the vectors $\underline{v}_o$, $\underline{j}_1$, $\underline{f}$ in (4-26) depend on different systems of parameters.

$$\underline{v}_o = \underline{v}_o\left(\underline{p}_1\right) \qquad (4\text{-}27a)$$

$$\underline{v}_k = \underline{v}_k\left(t, \underline{p}_1, \underline{p}_2, \underline{p}_3, \underline{p}_4\right) \qquad (4\text{-}27b)$$

$$\underline{j}_1 = \underline{j}_1\left(\underline{v}_k(t), t, \underline{p}_2, \underline{p}_4\right) \qquad (4\text{-}27c)$$

$$\underline{f} = \underline{f}\left(\underline{v}_k(t), t, \underline{p}_3\right) \qquad (4\text{-}27d)$$

$\underline{v}_o$ and $\underline{v}_k$ do not depend on the sub-vector $\dot{\underline{p}}_4$ (4-11e). The sub-vector $\dot{\underline{p}}_4$ was introduced in order to get a consistent representation later on. It is needed in the transformation of the velocity vector of the ground station between inertial and earth-fixed reference frame (see (4-132b)).

Thus, $\partial\underline{v}/\partial\underline{p}$ consists of the following block matrices:

$$\frac{\partial\underline{v}_{k+1}}{\partial\underline{p}} = \left[\ \frac{\partial\underline{v}_{k+1}}{\partial\underline{p}_1} \quad \frac{\partial\underline{v}_{k+1}}{\partial\underline{p}_2} \quad \frac{\partial\underline{v}_{k+1}}{\partial\underline{p}_3} \quad \frac{\partial\underline{v}_{k+1}}{\partial\underline{p}_4} \quad \frac{\partial\underline{v}_{k+1}}{\partial\dot{\underline{p}}_4}\ \right] \qquad (4\text{-}28)$$

The dimensions of the matrices are

$$0\left(\frac{\partial\underline{v}_{k+1}}{\partial\underline{p}_1}\right) = 6 \times 6 \qquad (4\text{-}29a)$$

$$O\left(\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_2}\right) = 6 \times [(n+1)(n+2)-6] \tag{4-29b}$$

$$O\left(\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_3}\right) = 6 \times 8 \tag{4-29c}$$

$$O\left(\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_4}\right) = 6 \times 3 \tag{4-29d}$$

$$O\left(\frac{\partial \underline{v}_{k+1}}{\partial \underline{\dot{p}}_4}\right) = 6 \times 3 \tag{4-29e}$$

$$O\left(\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}}\right) = 6 \times 14 + (n+1)(n+2) = (6 \times m) \tag{4-29f}$$

Considering (4-26) and the functional relationship (4-27a,...d) the block matrices in (4-28) can be defined as a successive series of Jacobi matrices, $j = \{1,2,3\}$.

$$\begin{aligned} \frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_j}(t) = {} & \frac{\partial \underline{v}_o}{\partial \underline{p}_j}\left(\underline{p}_1\right) + \\ & + \frac{\partial}{\partial \underline{p}_j} \int_{t_o}^{o} \underline{Y}\left(\underline{v}_k\left(\underline{p}_1,\underline{p}_2,\underline{p}_3,t\right),t\right)\ \underline{j}_1\left(\underline{v}_k\left(\underline{p}_1,\underline{p}_2,\underline{p}_3,t\right),\underline{p}_2,t\right) dt + \\ & + \frac{\partial}{\partial \underline{p}_j} \int_{t_o}^{o} \underline{Y}\left(\underline{v}_k\left(\underline{p}_1,\underline{p}_2,\underline{p}_3,t\right),t\right)\ \underline{f}\left(\underline{v}_k\left(\underline{p}_1,\underline{p}_2,\underline{p}_3,t\right),\underline{p}_3,t\right) dt \end{aligned} \tag{4-30}$$

After partial differentiation under the integral sign, and using (C-16), we get with $\partial\cdot/\partial\underline{v}_k = \partial\cdot/\partial\underline{v}$ according to (2-100)

$$\begin{aligned} \frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_1} = {} & \frac{\partial \underline{v}_o}{\partial \underline{p}_1} + \\ & + \int_{t_o}^{o} \left(\underline{Y}\,\frac{\partial \underline{j}_1}{\partial \underline{v}} + \underline{Y}\,\frac{\partial \underline{\underline{f}}}{\partial \underline{v}}\right) \frac{\partial \underline{v}_k}{\partial \underline{p}_1}\, dt + \\ & + \int_{t_o}^{o} \left[\frac{\partial \underline{Y}}{\partial v_1}\left(\underline{j}_1+\underline{f}\right),\ \frac{\partial \underline{Y}}{\partial v_2}\left(\underline{j}_1+\underline{f}\right),\ \ldots,\ \frac{\partial \underline{Y}}{\partial v_6}\left(\underline{j}_1+\underline{f}\right)\right] \frac{\partial \underline{v}_k}{\partial \underline{p}_1}\, dt + \\ & + \int_{t_o}^{o} \left(\underline{Y}\,\frac{\partial \underline{j}_1}{\partial \underline{p}_1} + \underline{Y}\,\frac{\partial \underline{f}}{\partial \underline{p}_1}\right) dt \end{aligned} \tag{4-31}$$

where

$$\frac{\partial \underline{j}_1}{\partial \underline{p}_1} = \underline{0} \tag{4-31a}$$

$$\frac{\partial \underline{f}}{\partial \underline{p}_1} = \underline{0} \tag{4-31b}$$

$$\begin{aligned}\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_2} = \frac{\partial \underline{v}_o}{\partial \underline{p}_2} &+ \\ &+ \int_{t_o}^{o} \left(\underline{Y} \frac{\partial \underline{j}_1}{\partial \underline{v}} + \underline{Y} \frac{\partial \underline{f}}{\partial \underline{v}} \right) \frac{\partial \underline{v}_k}{\partial \underline{p}_2} dt + \\ &+ \int_{t_o}^{o} \left[\frac{\partial \underline{Y}}{\partial v_1} \left(\underline{j}_1 + \underline{f} \right) , \frac{\partial \underline{Y}}{\partial v_2} \left(\underline{j}_1 + \underline{f} \right) , \ldots , \frac{\partial \underline{Y}}{\partial v_6} \left(\underline{j}_1 + \underline{f} \right) \right] \frac{\partial \underline{v}_k}{\partial \underline{p}_2} dt + \\ &+ \int_{t_o}^{o} \left(\underline{Y} \frac{\partial \underline{j}_1}{\partial \underline{p}_2} + \underline{Y} \frac{\partial \underline{f}}{\partial \underline{p}_2} \right) dt \end{aligned} \tag{4-32}$$

where

$$\frac{\partial \underline{v}_o}{\partial \underline{p}_2} = \underline{0} \tag{4-32a}$$

$$\frac{\partial \underline{f}}{\partial \underline{p}_2} = \underline{0} \tag{4-32b}$$

$$\begin{aligned}\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_3} = \frac{\partial \underline{v}_o}{\partial \underline{p}_3} &+ \\ &+ \int_{t_o}^{o} \left(\underline{Y} \frac{\partial \underline{j}_1}{\partial \underline{v}} + \underline{Y} \frac{\partial \underline{f}}{\partial \underline{v}} \right) \frac{\partial \underline{v}_k}{\partial \underline{p}_3} dt + \\ &+ \int_{t_o}^{o} \left[\frac{\partial \underline{Y}}{\partial v_1} \left(\underline{j}_1 + \underline{f} \right) , \frac{\partial \underline{Y}}{\partial v_2} \left(\underline{j}_1 + \underline{f} \right) , \ldots , \frac{\partial \underline{Y}}{\partial v_6} \left(\underline{j}_1 + \underline{f} \right) \right] \frac{\partial \underline{v}_k}{\partial \underline{p}_3} dt + \\ &+ \int_{t_o}^{o} \left(\underline{Y} \frac{\partial \underline{j}_1}{\partial \underline{p}_3} + \underline{Y} \frac{\partial \underline{f}}{\partial \underline{p}_3} \right) dt \end{aligned} \tag{4-33}$$

where

$$\frac{\partial \underline{v}_o}{\partial \underline{p}_3} = \underline{0} \tag{4-33a}$$

$$\frac{\partial \underline{j}_1}{\partial \underline{p}_3} = \underline{0} \tag{4-33b}$$

$$\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_4} = \frac{\partial \underline{v}_o}{\partial \underline{p}_4} + $$

$$+ \int_{t_o}^{o} \left(\underline{Y} \frac{\partial \underline{j}_1}{\partial \underline{v}} + \underline{Y} \frac{\partial \underline{f}}{\partial \underline{v}} \right) \frac{\partial \underline{v}_k}{\partial \underline{p}_4} dt +$$

$$+ \int_{t_o}^{o} \left[\frac{\partial \underline{Y}}{\partial v_1} \left(\underline{j}_1 + \underline{f} \right), \frac{\partial \underline{Y}}{\partial v_2} \left(\underline{j}_1 + \underline{f} \right), \ldots, \frac{\partial \underline{Y}}{\partial v_6} \left(\underline{j}_1 + \underline{f} \right) \right] \frac{\partial \underline{v}_k}{\partial \underline{p}_4} dt +$$

$$+ \int_{t_o}^{o} \left(\underline{Y} \frac{\partial \underline{j}_1}{\partial \underline{p}_4} + \underline{Y} \frac{\partial \underline{f}}{\partial \underline{p}_4} \right) dt \tag{4-34}$$

where

$$\frac{\partial \underline{v}_o}{\partial \underline{p}_4} = \underline{0} \tag{4-34a}$$

$$\frac{\partial \underline{f}}{\partial \underline{p}_4} = \underline{0} \tag{4-34b}$$

$$\frac{\partial \underline{v}_{k+1}}{\partial \dot{\underline{p}}_4} = \underline{0} \tag{4-35}$$

Eq. (4-35) holds, since

$$\frac{\partial \underline{v}_o}{\partial \dot{\underline{p}}_4} = \underline{0} \tag{4-35a}$$

$$\frac{\partial \underline{j}_1}{\partial \dot{\underline{p}}_4} = \underline{0} \tag{4-35b}$$

$$\frac{\partial \underline{f}}{\partial \dot{\underline{p}}_4} = \underline{0} \tag{4-35c}$$

The matrix equations (4-31) to (4-34) are given in a series of successive approximations similar to those for $\underline{v}_{k+1}(t)$. Since (4-31) to (4-34) are Jacobi matrices building up the coefficient vector for $\delta\underline{p}$, some few iteration steps might be sufficient. As already known from adjustment theory coefficient or design matrices have not to show a high accuracy.

The Jacobi matrix $\partial\underline{v}_o/\partial\underline{p}_1$ as starting sequence in (4-31)

According to (4-11a) the vector $\underline{p}_1$ consists of the six ***modified Poincaré elements*** at $t = t_o$.

$$\underline{p}_1 = \left[l_o, g_o, h_o, L_o, G_o, H_o \right]^T$$

The vector $\underline{v}_o$ is then given by (2-36a,...,f) in connection with (2-28a, ...,f) as function of $\underline{p}_1$.

$$\underline{v}_o = \begin{bmatrix} l_o - \kappa^4 t_o/L_o^3 \\ g_o \\ h_o \\ L_o \\ G_o \\ H_o \end{bmatrix} \tag{4-36}$$

The Jacobi matrix of (4-36) with respect to $\underline{p}_1$ is then

$$\frac{\partial\underline{v}_o}{\partial\underline{p}_1} = \left[\frac{\partial\underline{v}_o}{\partial l_o} \quad \frac{\partial\underline{v}_o}{\partial g_o} \quad \frac{\partial\underline{v}_o}{\partial h_o} \quad \frac{\partial\underline{v}_o}{\partial L_o} \quad \frac{\partial\underline{v}_o}{\partial G_o} \quad \frac{\partial\underline{v}_o}{\partial H_o} \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3\,\dfrac{\kappa^4 t_o}{L_o^4} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4-37}$$

Eq. (4-37) can be rewritten in matrix form by

$$\frac{\partial\underline{v}_o}{\partial\underline{p}_1} = \underline{I} + \underline{Q}_3(t_o) \tag{4-37a}$$

where $\underline{I}$ is the identity matrix and

$$\underline{Q}_3 = \begin{bmatrix} \underline{0} & -\underline{Q}(t_o) \\ \underline{0} & \underline{0} \end{bmatrix} \tag{4-37b}$$

and

$$\underline{Q}(t_o) = \begin{bmatrix} +3\tau_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4-37c}$$

Matrix $\underline{Q}$ is according to (2-41d) defined for $t = t_o$ by $\tau = t/a^2$.

The Jacobi matrix $\partial\underline{j}_1/\partial\underline{v}$

Using the chain rule for Jacobi matrices in Appendix C, we can define $\partial\underline{j}_1/\partial\underline{v}$ by

$$\frac{\partial\underline{j}_1}{\partial\underline{v}} = \frac{\partial\underline{j}_1}{\partial\underline{x}}\,\frac{\partial\underline{x}}{\partial\underline{v}} \tag{4-38}$$

where $\partial\underline{x}/\partial\underline{v}$ is already given by (2-76) in connection with (2-66f).

$$\frac{\partial\underline{x}}{\partial\underline{v}} = \frac{\partial\underline{x}}{\partial\underline{u}}\,\frac{\partial\underline{u}}{\partial\underline{v}} \tag{4-38a}$$

In defining (4-38) the transformation between inertial and earth-fixed reference frame and additionally, the transformation between spherical and cartesian coordinates have to be considered.

According to (3-11) we have

$$\underline{j}_1 = \text{grad}_x\, U_1 = \left[\frac{\partial U_1}{\partial x_1} \quad \frac{\partial U_1}{\partial x_2} \quad \frac{\partial U_1}{\partial x_3} \right]^T \tag{4-39}$$

Thus,

$$\frac{\partial\underline{j}_1}{\partial\underline{x}} = \text{grad}_x\, \text{grad}_x^T\, U_1 = \left[\frac{\partial\underline{j}_1}{\partial x_1} \quad \frac{\partial\underline{j}_1}{\partial x_2} \quad \frac{\partial\underline{j}_1}{\partial x_3} \right] \tag{4-40a}$$

$$\frac{\partial \underline{j}_1}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial^2 U_1}{\partial x_1^2} & \frac{\partial^2 U_1}{\partial x_2 \partial x_1} & \frac{\partial^2 U_1}{\partial x_3 \partial x_1} \\ \frac{\partial^2 U_1}{\partial x_1 \partial x_2} & \frac{\partial^2 U_1}{\partial x_2^2} & \frac{\partial^2 U_1}{\partial x_3 \partial x_2} \\ \frac{\partial^2 U_1}{\partial x_1 \partial x_3} & \frac{\partial^2 U_1}{\partial x_2 \partial x_3} & \frac{\partial^2 U_1}{\partial x_3^2} \end{bmatrix} \tag{4-40b}$$

For the transformation from the inertial into the earth-fixed reference frame we use the rotation matrix $\underline{R}$ (3-5c),

$$\frac{\partial \underline{j}_1}{\partial \underline{x}} = \underline{R} \, \mathrm{grad}_y \, \mathrm{grad}_x^T \, U_1 \, \underline{R}^T = \underline{R} \, \frac{\partial \underline{j}_1}{\partial \underline{y}} \, \underline{R}^T \tag{4-41}$$

and expressing (4-41) in spherical coordinates $\underline{r} = [r, \varphi, \lambda]^T$ using

$$\underline{j}_1 = \mathrm{grad}_y \, U_1 = \frac{\partial \underline{r}^T}{\partial \underline{y}} \, \mathrm{grad}_r \, U_1 \tag{4-42a}$$

$$\mathrm{grad}_y = \left[\frac{\partial}{\partial y_1} \quad \frac{\partial}{\partial y_2} \quad \frac{\partial}{\partial y_3} \right]^T \tag{4-42b}$$

$$\mathrm{grad}_r = \left[\frac{\partial}{\partial r} \quad \frac{\partial}{\partial \varphi} \quad \frac{\partial}{\partial \lambda} \right]^T \tag{4-42b}$$

results in

$$\frac{\partial \underline{j}_1}{\partial \underline{y}} = \left[\frac{\partial \underline{r}^T}{\partial y_1 \partial \underline{y}} \, \mathrm{grad}_r \, U_1 \, , \, \frac{\partial \underline{r}^T}{\partial y_2 \partial \underline{y}} \, \mathrm{grad}_r \, U_1 \, , \, \frac{\partial \underline{r}^T}{\partial y_3 \partial \underline{y}} \, \mathrm{grad}_r \, U_1 \right] +$$

$$+ \frac{\partial \underline{r}^T}{\partial \underline{y}} \begin{bmatrix} \frac{\partial^2 U_1}{\partial r^2} & \frac{\partial^2 U_1}{\partial \varphi \partial r} & \frac{\partial^2 U_1}{\partial \lambda \partial r} \\ \frac{\partial^2 U_1}{\partial r \partial \varphi} & \frac{\partial^2 U_1}{\partial \varphi^2} & \frac{\partial^2 U_1}{\partial \lambda \partial \varphi} \\ \frac{\partial^2 U_1}{\partial r \partial \lambda} & \frac{\partial^2 U_1}{\partial \varphi \partial \lambda} & \frac{\partial^2 U_1}{\partial \lambda^2} \end{bmatrix} \frac{\partial \underline{r}}{\partial \underline{y}} \tag{4-43}$$

where the matrices of second derivatives of spherical coordinates are given by

$$\frac{\partial \underline{r}^T}{\partial y_1 \partial \underline{y}} = \begin{bmatrix} \frac{\partial^2 r}{\partial y_1^2} & \frac{\partial^2 \varphi}{\partial y_1^2} & \frac{\partial^2 \lambda}{\partial y_1^2} \\ \frac{\partial^2 r}{\partial y_1 \partial y_2} & \frac{\partial^2 \varphi}{\partial y_1 \partial y_2} & \frac{\partial^2 \lambda}{\partial y_1 \partial y_2} \\ \frac{\partial^2 r}{\partial y_1 \partial y_3} & \frac{\partial^2 \varphi}{\partial y_1 \partial y_3} & \frac{\partial^2 \lambda}{\partial y_1 \partial y_3} \end{bmatrix} \qquad (4\text{-}44a)$$

$$\frac{\partial \underline{r}^T}{\partial y_2 \partial \underline{y}} = \begin{bmatrix} \frac{\partial^2 r}{\partial y_2 \partial y_1} & \frac{\partial^2 \varphi}{\partial y_2 \partial y_1} & \frac{\partial^2 \lambda}{\partial y_2 \partial y_1} \\ \frac{\partial^2 r}{\partial y_2^2} & \frac{\partial^2 \varphi}{\partial y_2^2} & \frac{\partial^2 \lambda}{\partial y_2^2} \\ \frac{\partial^2 r}{\partial y_2 \partial y_3} & \frac{\partial^2 \varphi}{\partial y_2 \partial y_3} & \frac{\partial^2 \lambda}{\partial y_2 \partial y_3} \end{bmatrix} \qquad (4\text{-}44b)$$

$$\frac{\partial \underline{r}^T}{\partial y_3 \partial \underline{y}} = \begin{bmatrix} \frac{\partial^2 r}{\partial y_3 \partial y_1} & \frac{\partial^2 \varphi}{\partial y_3 \partial y_1} & \frac{\partial^2 \lambda}{\partial y_3 \partial y_1} \\ \frac{\partial^2 r}{\partial y_3 \partial y_2} & \frac{\partial^2 \varphi}{\partial y_3 \partial y_2} & \frac{\partial^2 \lambda}{\partial y_3 \partial y_2} \\ \frac{\partial^2 r}{\partial y_3^2} & \frac{\partial^2 \varphi}{\partial y_3^2} & \frac{\partial^2 \lambda}{\partial y_3^2} \end{bmatrix} \qquad (4\text{-}44c)$$

The scalar values in the matrices (4-44a,b,c) are obtained by differentiation of (3-15).

$$\frac{\partial^2 r}{\partial y_1^2} = \frac{\sin^2\varphi \cos^2\lambda + \sin^2\lambda}{r} \qquad (4\text{-}45a)$$

$$\frac{\partial^2 r}{\partial y_2^2} = \frac{\cos^2\lambda + \sin^2\lambda \sin^2\varphi}{r} \tag{4-45b}$$

$$\frac{\partial^2 r}{\partial y_3^2} = \frac{\cos^2\varphi}{r} \tag{4-45c}$$

$$\frac{\partial^2 r}{\partial y_1 \partial y_2} = -\frac{\cos^2\varphi \sin 2\lambda}{2r} \tag{4-45d}$$

$$\frac{\partial^2 r}{\partial y_1 \partial y_3} = -\frac{\sin 2\varphi \cos\lambda}{2r} \tag{4-45e}$$

$$\frac{\partial^2 r}{\partial y_2 \partial y_3} = -\frac{\sin 2\varphi \sin\lambda}{2r} \tag{4-45f}$$

$$\frac{\partial^2 \varphi}{\partial y_1^2} = \frac{\tan\varphi}{r^2}\left(2\cos^2\lambda \cos^2\varphi - \sin^2\lambda\right) \tag{4-46a}$$

$$\frac{\partial^2 \varphi}{\partial y_2^2} = \frac{\tan\varphi}{r^2}\left(2\cos^2\varphi \sin^2\lambda - \cos^2\lambda\right) \tag{4-46b}$$

$$\frac{\partial^2 \varphi}{\partial y_3^2} = -\frac{\sin 2\varphi}{r^2} \tag{4-46c}$$

$$\frac{\partial^2 \varphi}{\partial y_1 \partial y_2} = \frac{\tan\varphi \sin 2\lambda}{2r^2}\left(2\cos^2\varphi + 1\right) \tag{4-46d}$$

$$\frac{\partial^2 \varphi}{\partial y_1 \partial y_3} = -\frac{\cos\lambda \cos 2\varphi}{r^2} \tag{4-46e}$$

$$\frac{\partial^2 \varphi}{\partial y_2 \partial y_3} = -\frac{\sin\lambda \cos 2\varphi}{r^2} \tag{4-46f}$$

$$\frac{\partial^2 \lambda}{\partial y_1^2} = \frac{\sin 2\lambda}{r^2 \cos^2 \varphi} \qquad (4\text{-}47a)$$

$$\frac{\partial^2 \lambda}{\partial y_2^2} = -\frac{\sin 2\lambda}{r^2 \cos^2 \varphi} \qquad (4\text{-}47b)$$

$$\frac{\partial^2 \lambda}{\partial y_3^2} = 0 \qquad (4\text{-}47c)$$

$$\frac{\partial^2 \lambda}{\partial y_1 \, \partial y_2} = -\frac{\cos 2\lambda}{r^2 \cos^2 \varphi} \qquad (4\text{-}47d)$$

$$\frac{\partial^2 \lambda}{\partial y_1 \, \partial y_3} = 0 \qquad (4\text{-}47e)$$

$$\frac{\partial^2 \lambda}{\partial y_2 \, \partial y_3} = 0 \qquad (4\text{-}47f)$$

The six independent partial second order derivatives of the model or reference potential $\underline{U}_1$ with respect to spherical coordinates are defined by differentiation of (3-17a,b,c).

$$\frac{\partial^2 U_1}{\partial r^2} = kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} (\mu+1)(\mu+2) \frac{R^\mu}{r^{\mu+2}} \frac{\partial P_{\mu\nu}(\sin\varphi)}{\partial \varphi} \left[c_{\mu\nu} \cos \nu\lambda + s_{\mu\nu} \sin \nu\lambda \right] \qquad (4\text{-}48a)$$

$$\frac{\partial^2 U_1}{\partial \varphi \, \partial r} = -kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} (\mu+1) \frac{R^\mu}{r^{\mu+2}} \frac{\partial P_{\mu\nu}(\sin\varphi)}{\partial \varphi} \left[c_{\mu\nu} \cos \nu\lambda + s_{\mu\nu} \sin \nu\lambda \right] \qquad (4\text{-}48b)$$

$$\frac{\partial^2 U_1}{\partial \lambda \, \partial r} = -kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \nu (\mu+1) \frac{R^\mu}{r^{\mu+2}} P_{\mu\nu}(\sin\varphi) \left[s_{\mu\nu} \cos \nu\lambda - c_{\mu\nu} \sin \nu\lambda \right] \qquad (4\text{-}48c)$$

$$\frac{\partial^2 U_1}{\partial \varphi^2} = kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \frac{R^\mu}{r^{\mu+1}} \frac{\partial^2 P_{\mu\nu}(\sin\varphi)}{\partial \varphi^2} \left[c_{\mu\nu} \cos \nu\lambda + s_{\mu\nu} \sin \nu\lambda \right] \qquad (4\text{-}48d)$$

$$\frac{\partial^2 U_1}{\partial \lambda \, \partial r} = kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \nu \frac{R^\mu}{r^{\mu+1}} \frac{\partial P_{\mu\nu}(\sin\varphi)}{\partial \varphi} \left[s_{\mu\nu} \cos \nu\lambda - c_{\mu\nu} \sin \nu\lambda \right] \qquad (4\text{-}48e)$$

$$\frac{\partial^2 U_1}{\partial \lambda^2} = -kM \sum_{\mu=2}^{n} \sum_{\nu=0}^{\mu} \nu^2 \frac{R^\mu}{r^{\mu+1}} P_{\mu\nu}(\sin\varphi) \left[s_{\mu\nu} \cos \nu\lambda - c_{\mu\nu} \sin \nu\lambda \right] \qquad (4\text{-}48f)$$

The associated Legendre polynomials $P_{\mu\nu}(\sin\varphi)$ as well as their first derivatives are already given by (3-18), (3-19) and (3-20a,b). The necessary second order derivatives are

$$\frac{\partial^2 P_{\mu\nu}(\sin\varphi)}{\partial\varphi^2} =$$

$$= \sum_{\sigma=0}^{\kappa} a_{\mu\nu\sigma} \left[\frac{\nu^2 \sin^2\varphi - \nu}{\cos^2\varphi} + (\mu-\nu-2\sigma) \frac{(\mu-\nu-2\sigma)\cos^2\varphi - 2\nu\sin^2\varphi - 1}{\sin^2\varphi} \right] \cdot$$

$$\cdot \cos^{\nu} \sin^{\mu-\nu-2\sigma}\varphi \tag{4-49}$$

where the coefficients $a_{\mu\nu\sigma}$ are already given by (3-20a,b).

The Jacobi matrix $\partial \underline{f}/\partial \underline{v}$

In the following the necessary matrix $\partial \underline{f}/\partial \underline{v}$ for the determination of $\partial \underline{v}/\partial \underline{p}$ (4-31) will be derived. The accelerations $\underline{f}_1$ are primarily function of the position vector $\underline{x} = \underline{x}(\underline{v})$ and of the velocity $\underline{\dot{x}} = \underline{\dot{x}}(v)$, see chapter 3.

Using the chain-rule for differentiation (see appendix C) we get

$$\frac{\partial \underline{f}}{\partial \underline{v}} = \frac{\partial \underline{f}}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \underline{v}} + \frac{\partial \underline{f}}{\partial \underline{\dot{x}}} \frac{\partial \underline{\dot{x}}}{\partial \underline{v}} \tag{4-50}$$

where

$$\frac{\partial \underline{f}}{\partial \underline{x}} = \frac{\partial \underline{f}_1}{\partial \underline{x}} + \frac{\partial \underline{f}_2}{\partial \underline{x}} + \frac{\partial \underline{f}_3}{\partial \underline{x}} + \frac{\partial \underline{f}_4}{\partial \underline{x}} \tag{4-51a}$$

$$\frac{\partial \underline{f}}{\partial \underline{\dot{x}}} = \frac{\partial \underline{f}_1}{\partial \underline{\dot{x}}} + \frac{\partial \underline{f}_2}{\partial \underline{\dot{x}}} + \frac{\partial \underline{f}_3}{\partial \underline{\dot{x}}} + \frac{\partial \underline{f}_4}{\partial \underline{\dot{x}}} \tag{4-51b}$$

Matrix $\partial \underline{x}/\partial \underline{v}$ is given by (2-76) in connection with (2-68), whereas $\partial \underline{\dot{x}}/\partial \underline{v}$ is given by (2-85) together with (2-68).

The detailed matrices $\partial \underline{f}_i/\partial \underline{x}$ and $\partial \underline{f}_i/\partial \underline{\dot{x}}$ have still to be defined.

Using $\underline{f}_1$ (3-21) we are looking for

$$\frac{\partial \underline{f}_1}{\partial \underline{x}} = \left[\frac{\partial \underline{f}_1}{\partial x_1}, \frac{\partial \underline{f}_1}{\partial x_2}, \frac{\partial \underline{f}_1}{\partial x_3} \right] \tag{4-52a}$$

$$\frac{\partial \underline{f}_1}{\partial \underline{\dot{x}}} = \left[\frac{\partial \underline{f}_1}{\partial \dot{x}_1} , \frac{\partial \underline{f}_1}{\partial \dot{x}_2} , \frac{\partial \underline{f}_1}{\partial \dot{x}_3} \right] \tag{4-52b}$$

Matrix (4-52a) can be given explicitly using outer products.

$$\frac{\partial \underline{f}_1}{\partial \underline{x}} = -\frac{1}{2} c_D \frac{A}{m} \left(\left| \underline{\dot{x}}_{rel} \right| \underline{\dot{x}}_{rel} \frac{\partial \rho}{\partial \underline{x}} + \rho \frac{\partial \underline{\dot{x}}_{rel}}{\partial \underline{x}} \left| \underline{\dot{x}}_{rel} \right| + \rho \, \underline{\dot{x}}_{rel} \frac{\partial \underline{\dot{x}}_{rel}}{\partial \underline{x}} \right) \tag{4-53}$$

where the density derivative with respect to $\underline{x}$, $\partial\rho/\partial\underline{x}$ is

$$\frac{\partial \rho}{\partial \underline{x}} = \text{grad}_x^T \rho \tag{4-54}$$

Considering further the density ρ as functional of $\underline{x}$ according to (3-33), we find

$$\frac{\partial \rho}{\partial \underline{x}} = \frac{\partial \rho}{\partial h} \frac{\partial h}{\partial \underline{x}} + \frac{\partial \rho}{\partial \psi} \frac{\partial \psi}{\partial \underline{x}} \tag{4-54a}$$

where the scalars and finctions are derived by partial differentiation, e.g. by the gradients of (3-34a,...,h).

$$\frac{\partial \rho}{\partial h} = \left(1 + \bar{\rho}_1 \cos^{\bar{\mu}} \frac{\psi}{2} \right) \left[\frac{\partial \rho_m}{\partial h} + \left(\frac{\partial \rho_M}{\partial h} - \frac{\partial \rho_m}{\partial h} \right) \cos^{\bar{\mu}} \frac{\psi}{2} \right] \tag{4-54b}$$

$$\frac{\partial \rho_m}{\partial h} = -\frac{\rho_m}{H_m} \tag{4-54c}$$

$$\frac{\partial \rho_M}{\partial h} = -\frac{\rho_M}{H_M} \tag{4-54d}$$

$$\frac{\partial h}{\partial \underline{x}} = \frac{\partial |\underline{x}|}{\partial \underline{x}} - \frac{\partial R_E}{\partial \underline{x}} \tag{4-54e}$$

$$\frac{\partial |\underline{x}|}{\partial \underline{x}} = \frac{\underline{x}^T}{|\underline{x}|} \tag{4-54f}$$

$$\frac{\partial R_E}{\partial \underline{x}} = \frac{R_E^3 \left(2f - f^2 \right)}{R^2 (1-f)^2} \frac{x_3}{|\underline{x}|^2} \left(\frac{x_3}{|\underline{x}|^2} \underline{x}^T - \underline{e}_3^{xT} \right) \; ; \; \underline{e}_3^{xT} = [0 , 0 , 1] \tag{4-54g}$$

Inserting (4-54f) and (4-54g) in (4-54e) results in

$$\frac{\partial h}{\partial \underline{x}} = \frac{\underline{x}^T}{|\underline{x}|} + \frac{R_E^3 \left(2f - f^2\right)}{R^2 (1-f)^2} \frac{x_3}{|\underline{x}|^2} \left(\underline{e}_3^{xT} - \frac{x_3}{|\underline{x}|^2} \underline{x}^T \right) \tag{4-54h}$$

Further,

$$\frac{\partial \rho}{\partial \psi} = -\frac{\bar{\mu}}{2} \sin\frac{\psi}{2} \cos^{\bar{\mu}-1}\frac{\psi}{2} \left[\rho_M - \rho_m + \bar{\rho}_1 \rho_m + 2\bar{\rho}_1 (\rho_M - \rho_m) \cos^{\bar{\mu}} \frac{\psi}{2} \right] \tag{4-54i}$$

$$\frac{\partial \psi}{\partial \underline{x}} = -\frac{1}{\sin\psi} \frac{\underline{u}^T}{|\underline{x}|} \left(\underline{I} - \frac{\underline{x}\, \underline{x}^T}{|\underline{x}|^2} \right) \tag{4-54j}$$

$\partial \dot{\underline{x}}_{rel} / \partial \underline{x}$ in (4-53) is derived by partial differentiation of (3-31a).

$$\frac{\partial \dot{\underline{x}}_{rel}}{\partial \underline{x}} = -\underline{\mathbf{\Omega}}_A \tag{4-55}$$

The derivative of the norm of $\left|\dot{\underline{x}}_{rel}\right|$ with respect to $\underline{x}$ is

$$\frac{\partial \left|\dot{\underline{x}}_{rel}\right|}{\partial \underline{x}} = \frac{\underline{x}^T \underline{\mathbf{\Omega}}_A - \dot{\underline{x}}^T}{\left|\dot{\underline{x}}_{rel}\right|} \underline{\mathbf{\Omega}}_A \tag{4-56}$$

where the matrix $\underline{\mathbf{\Omega}}_A$ of the components of the earth rotation vector $\underline{\boldsymbol{\omega}}$ is given by (3-26c) or, in approximation, by (3-30b).

Matrix (4-52b) is given by

$$\frac{\partial \underline{f}_1}{\partial \dot{\underline{x}}} = -\frac{1}{2} c_D \frac{A}{m} \rho \left(\frac{\partial \dot{\underline{x}}_{rel}}{\partial \dot{\underline{x}}} \left|\dot{\underline{x}}_{rel}\right| + \dot{\underline{x}}_{rel} \frac{\partial \left|\dot{\underline{x}}_{rel}\right|}{\partial \dot{\underline{x}}} \right) \tag{4-57}$$

where (using (3-31a,b))

$$\frac{\partial \dot{\underline{x}}_{rel}}{\partial \dot{\underline{x}}} = \underline{I} \tag{4-57a}$$

$$\frac{\partial \left|\dot{\underline{x}}_{rel}\right|}{\partial \dot{\underline{x}}} = \frac{\dot{\underline{x}}^T + \dot{\underline{x}}^T \underline{\mathbf{\Omega}}_A}{\left|\dot{\underline{x}}_{rel}\right|} = \frac{\dot{\underline{x}}_{rel}^T}{\left|\dot{\underline{x}}_{rel}\right|} \tag{4-57b}$$

Thus,

$$\frac{\partial \underline{f}_1}{\partial \dot{\underline{x}}} = -\frac{1}{2} c_D \frac{A}{m} \rho \left(\left|\dot{\underline{x}}_{rel}\right| \underline{I} + \dot{\underline{x}}_{rel} \frac{\dot{\underline{x}}_{rel} \dot{\underline{x}}_{rel}^T}{\left|\dot{\underline{x}}_{rel}\right|} \right) \tag{4-57c}$$

The acceleration $\underline{f}_2$ was given by (3-35). Its derivatives are

$$\frac{\partial \underline{f}_2}{\partial \underline{x}} = \left[\frac{\partial \underline{f}_2}{\partial \underline{x}_1}, \frac{\partial \underline{f}_2}{\partial \underline{x}_2}, \frac{\partial \underline{f}_2}{\partial \underline{x}_3} \right] \tag{4-58a}$$

$$\frac{\partial \underline{f}_2}{\partial \underline{\dot{x}}} = \left[\frac{\partial \underline{f}_2}{\partial \underline{\dot{x}}_1}, \frac{\partial \underline{f}_2}{\partial \underline{\dot{x}}_2}, \frac{\partial \underline{f}_2}{\partial \underline{\dot{x}}_3} \right] = \underline{0} \tag{4-58b}$$

In detail,

$$\frac{\partial \underline{f}_2}{\partial \underline{x}} = \frac{\nu\, k\, a_S^2}{\left|\underline{x} - \underline{x}_S\right|^3} \left[\underline{I} - 3\, \frac{\left(\underline{x} - \underline{x}_S\right)\left(\underline{x} - \underline{x}_S\right)^T}{\left|\underline{x} - \underline{x}_S\right|^2} \right] \tag{4-59}$$

The acceleration $\underline{f}_3$ was given by (3-38b). Its derivatives are

$$\frac{\partial \underline{f}_3}{\partial \underline{x}} = \left[\frac{\partial \underline{f}_3}{\partial \underline{x}_1}, \frac{\partial \underline{f}_3}{\partial \underline{x}_2}, \frac{\partial \underline{f}_3}{\partial \underline{x}_3} \right] \tag{4-60a}$$

$$\frac{\partial \underline{f}_3}{\partial \underline{\dot{x}}} = \left[\frac{\partial \underline{f}_3}{\partial \underline{\dot{x}}_1}, \frac{\partial \underline{f}_3}{\partial \underline{\dot{x}}_2}, \frac{\partial \underline{f}_3}{\partial \underline{\dot{x}}_3} \right] = \underline{0} \tag{4-60b}$$

In detail,

$$\frac{\partial \underline{f}_3}{\partial \underline{x}} = \frac{k m_M}{\left|\underline{x} - \underline{x}_M\right|^3} \left[3\, \frac{\left(\underline{x} - \underline{x}_M\right)\left(\underline{x} - \underline{x}_M\right)^T}{\left|\underline{x} - \underline{x}_M\right|^2} - \underline{I} \right] + \frac{k m_S}{\left|\underline{x} - \underline{x}_S\right|^3} \left[3\, \frac{\left(\underline{x} - \underline{x}_S\right)\left(\underline{x} - \underline{x}_S\right)^T}{\left|\underline{x} - \underline{x}_S\right|^2} - \underline{I} \right] \tag{4-61}$$

The acceleration $\underline{f}_4$ was given by (3-54b). Its derivatives are

$$\frac{\partial \underline{f}_4}{\partial \underline{x}} = \left[\frac{\partial \underline{f}_4}{\partial \underline{x}_1}, \frac{\partial \underline{f}_4}{\partial \underline{x}_2}, \frac{\partial \underline{f}_4}{\partial \underline{x}_3} \right] \tag{4-62a}$$

$$\frac{\partial \underline{f}_4}{\partial \underline{\dot{x}}} = \left[\frac{\partial \underline{f}_4}{\partial \underline{\dot{x}}_1}, \frac{\partial \underline{f}_4}{\partial \underline{\dot{x}}_2}, \frac{\partial \underline{f}_4}{\partial \underline{\dot{x}}_3} \right] \tag{4-62b}$$

In detail,

$$\frac{\partial \underline{f}_4}{\partial \underline{x}} = 3\, k m_S\, k_2 \frac{R^5}{\left|\underline{\hat{x}}_S\right|^3 \left|\underline{x}\right|^5} \left[\frac{1}{2}\left(1 - 5\,\hat{u}_S^2\right) \underline{I} - 5\,\hat{u}_S \left(\frac{\underline{\hat{x}}_S\, \underline{x}^T}{\left|\underline{\hat{x}}_S\right| \left|\underline{x}\right|} + \frac{\underline{x}\, \underline{\hat{x}}_S^T}{\left|\underline{x}\right| \left|\underline{\hat{x}}_S\right|} \right) + \right.$$

$$\left. + \frac{5}{2}\left(7\,\hat{u}_S^2 - 1\right) \frac{\underline{x}\, \underline{x}^T}{\left|\underline{x}\right|^2} + \frac{\underline{\hat{x}}_S\, \underline{\hat{x}}_S^T}{\left|\underline{\hat{x}}_S\right|^2} \right] +$$

$$+ 3\, k m_M\, k_2 \frac{R^5}{\left|\underline{\hat{x}}_M\right|^3 \left|\underline{x}\right|^5} \left[\frac{1}{2}\left(1 - 5\,\hat{u}_M^2\right) \underline{I} - 5\,\hat{u}_M \left(\frac{\underline{\hat{x}}_M\, \underline{x}^T}{\left|\underline{\hat{x}}_M\right| \left|\underline{x}\right|} + \frac{\underline{x}\, \underline{\hat{x}}_M^T}{\left|\underline{x}\right| \left|\underline{\hat{x}}_M\right|} \right) + \right.$$

$$\left. + \frac{5}{2}\left(7\,\hat{u}_M^2 - 1\right) \frac{\underline{x}\, \underline{x}^T}{\left|\underline{x}\right|^2} + \frac{\underline{\hat{x}}_M\, \underline{\hat{x}}_M^T}{\left|\underline{\hat{x}}_M\right|^2} \right] \qquad (4\text{-}63a)$$

where

$$\hat{u}_S = \cos \hat{\delta}_S = \frac{\underline{x}^T\, \underline{\hat{x}}_S}{\left|\underline{x}\right| \left|\underline{\hat{x}}_S\right|} \qquad (4\text{-}63b)$$

$$\hat{u}_M = \cos \hat{\delta}_M = \frac{\underline{x}^T\, \underline{\hat{x}}_M}{\left|\underline{x}\right| \left|\underline{\hat{x}}_M\right|} \qquad (4\text{-}63c)$$

The Jacobi matrix $\partial \underline{Y} / \partial \underline{v}$ in (4-31)

Using the modified Poincaré orbital elements in the form (2-36a,...,f)

$$l = l(v_1, v_4) = v_1 + \frac{\kappa^4}{v_4^3}\, t \qquad (4\text{-}66a)$$

$$g = g(v_2) = v_2 \qquad (4\text{-}66b)$$

$$h = h(v_3) = v_3 \qquad (4\text{-}66c)$$

$$L = L(v_4) = v_4 \qquad (4\text{-}66d)$$

$$G = G(v_5) = v_5 \qquad (4\text{-}66e)$$

$$H = H(v_6) = v_6 \qquad (4\text{-}66f)$$

we get

$$\frac{\partial \underline{Y}}{\partial v_1} = \frac{\partial \underline{Y}}{\partial l}\frac{\partial l}{\partial v_1} \tag{4-67a}$$

$$\frac{\partial \underline{Y}}{\partial v_2} = \frac{\partial \underline{Y}}{\partial g}\frac{\partial g}{\partial v_2} \tag{4-67b}$$

$$\frac{\partial \underline{Y}}{\partial v_3} = \frac{\partial \underline{Y}}{\partial h}\frac{\partial h}{\partial v_3} \tag{4-67c}$$

$$\frac{\partial \underline{Y}}{\partial v_4} = \frac{\partial \underline{Y}}{\partial l}\frac{\partial l}{\partial v_4} + \frac{\partial \underline{Y}}{\partial L}\frac{\partial L}{\partial v_4} \tag{4-67d}$$

$$\frac{\partial \underline{Y}}{\partial v_5} = \frac{\partial \underline{Y}}{\partial G}\frac{\partial G}{\partial v_5} \tag{4-67e}$$

$$\frac{\partial \underline{Y}}{\partial v_6} = \frac{\partial \underline{Y}}{\partial H}\frac{\partial H}{\partial v_6} \tag{4-67f}$$

where

$$\frac{\partial l}{\partial v_1} = 1 \tag{4-68a}$$

$$\frac{\partial l}{\partial v_4} = -3\,\frac{\kappa^4 t}{v_4^4} = -3\,\frac{\kappa^4 t}{L^4} = -3\,\frac{t}{a^2} = -3\,\tau \tag{4-68b}$$

$$\frac{\partial g}{\partial v_2} = 1 \tag{4-68c}$$

$$\frac{\partial h}{\partial v_3} = 1 \tag{4-68d}$$

$$\frac{\partial L}{\partial v_4} = 1 \tag{4-68e}$$

$$\frac{\partial G}{\partial v_5} = 1 \tag{4-68f}$$

$$\frac{\partial H}{\partial v_6} = 1 \tag{4-68g}$$

Using the abbreviations (2-64a,b) and (2-63)

$$\underline{q} = [q_1, q_2, q_3]^T = [l, g, h]^T \tag{4-69a}$$

$$\underline{p} = [p_1, p_2, p_3]^T = [L, G, H]^T \tag{4-69b}$$

$$\underline{u} = [u_1, u_2, u_3, u_4, u_5, u_6]^T = [M, a, \Omega, \omega, e, i]^T \tag{4-69c}$$

we can systematize the derivatives. For the first-order derivatives of the components x_k of the vektor $\underline{x}_k$, $k = \{1,2,3\}$, see (4-72b), with respect to the Kepler elements u_j, $j = \{1,....,6\}$ we get

$$\frac{\partial x_k}{\partial p_i} = \underline{x}_k^T \frac{\partial \underline{u}}{\partial p_i} \tag{4-70a}$$

$$\frac{\partial x_k}{\partial q_i} = \underline{x}_k^T \frac{\partial \underline{u}}{\partial q_i} \tag{4-70b}$$

and for the second-order derivatives of $\underline{x}_k$, $k = \{1,2,3\}$ with respect to the modified Poincaré elements we have

$$\frac{\partial^2 x_k}{\partial p_j \partial p_i} = \frac{\partial \underline{u}^T}{\partial p_j} \underline{X}_k \frac{\partial \underline{u}}{\partial p_i} + \underline{x}_k^T \frac{\partial^2 \underline{u}}{\partial p_j \partial p_i} \tag{4-71a}$$

$$\frac{\partial^2 x_k}{\partial q_j \partial p_i} = \frac{\partial \underline{u}^T}{\partial q_j} \underline{X}_k \frac{\partial \underline{u}}{\partial p_i} + \underline{x}_k^T \frac{\partial^2 \underline{u}}{\partial q_j \partial p_i} \tag{4-71b}$$

$$\frac{\partial^2 x_k}{\partial p_j \partial q_i} = \frac{\partial \underline{u}^T}{\partial p_j} \underline{X}_k \frac{\partial \underline{u}}{\partial q_i} + \underline{x}_k^T \frac{\partial^2 \underline{u}}{\partial p_j \partial q_i} \tag{4-71c}$$

$$\frac{\partial^2 x_k}{\partial q_j \partial q_i} = \frac{\partial \underline{u}^T}{\partial q_j} \underline{X}_k \frac{\partial \underline{u}}{\partial q_i} + \underline{x}_k^T \frac{\partial^2 \underline{u}}{\partial q_j \partial q_i} \tag{4-71d}$$

where

$$\underline{X}_k = \begin{bmatrix} \frac{\partial^2 x_k}{\partial u_1^2} & \frac{\partial^2 x_k}{\partial u_1 \partial u_2} & \frac{\partial^2 x_k}{\partial u_1 \partial u_3} & \frac{\partial^2 x_k}{\partial u_1 \partial u_4} & \frac{\partial^2 x_k}{\partial u_1 \partial u_5} & \frac{\partial^2 x_k}{\partial u_1 \partial u_6} \\ \frac{\partial^2 x_k}{\partial u_2 \partial u_1} & \frac{\partial^2 x_k}{\partial u_2^2} & \frac{\partial^2 x_k}{\partial u_2 \partial u_3} & \frac{\partial^2 x_k}{\partial u_2 \partial u_4} & \frac{\partial^2 x_k}{\partial u_2 \partial u_5} & \frac{\partial^2 x_k}{\partial u_2 \partial u_6} \\ \frac{\partial^2 x_k}{\partial u_3 \partial u_1} & \frac{\partial^2 x_k}{\partial u_3 \partial u_2} & \frac{\partial^2 x_k}{\partial u_3^2} & \frac{\partial^2 x_k}{\partial u_3 \partial u_4} & \frac{\partial^2 x_k}{\partial u_3 \partial u_5} & \frac{\partial^2 x_k}{\partial u_3 \partial u_6} \\ \frac{\partial^2 x_k}{\partial u_4 \partial u_1} & \frac{\partial^2 x_k}{\partial u_4 \partial u_2} & \frac{\partial^2 x_k}{\partial u_4 \partial u_3} & \frac{\partial^2 x_k}{\partial u_4^2} & \frac{\partial^2 x_k}{\partial u_4 \partial u_5} & \frac{\partial^2 x_k}{\partial u_4 \partial u_6} \\ \frac{\partial^2 x_k}{\partial u_5 \partial u_1} & \frac{\partial^2 x_k}{\partial u_5 \partial u_2} & \frac{\partial^2 x_k}{\partial u_5 \partial u_3} & \frac{\partial^2 x_k}{\partial u_5 \partial u_4} & \frac{\partial^2 x_k}{\partial u_5^2} & \frac{\partial^2 x_k}{\partial u_5 \partial u_6} \\ \frac{\partial^2 x_k}{\partial u_6 \partial u_1} & \frac{\partial^2 x_k}{\partial u_6 \partial u_2} & \frac{\partial^2 x_k}{\partial u_6 \partial u_3} & \frac{\partial^2 x_k}{\partial u_6 \partial u_4} & \frac{\partial^2 x_k}{\partial u_6 \partial u_5} & \frac{\partial^2 x_k}{\partial u_6^2} \end{bmatrix} \quad (4\text{-}72a)$$

$$\underline{x}_k^T = \begin{bmatrix} \frac{\partial x_k}{\partial u_1} & \frac{\partial x_k}{\partial u_2} & \frac{\partial x_k}{\partial u_3} & \frac{\partial x_k}{\partial u_4} & \frac{\partial x_k}{\partial u_5} & \frac{\partial x_k}{\partial u_6} \end{bmatrix} \quad (4\text{-}72b)$$

$$\frac{\partial \underline{u}}{\partial p_j} = \begin{bmatrix} \frac{\partial u_1}{\partial p_j} & \frac{\partial u_2}{\partial p_j} & \frac{\partial u_3}{\partial p_j} & \frac{\partial u_4}{\partial p_j} & \frac{\partial u_5}{\partial p_j} & \frac{\partial u_6}{\partial p_j} \end{bmatrix}^T \quad (4\text{-}72c)$$

$$\frac{\partial \underline{u}}{\partial p_i} = \begin{bmatrix} \frac{\partial u_1}{\partial p_i} & \frac{\partial u_2}{\partial p_i} & \frac{\partial u_3}{\partial p_i} & \frac{\partial u_4}{\partial p_i} & \frac{\partial u_5}{\partial p_i} & \frac{\partial u_6}{\partial p_i} \end{bmatrix}^T \quad (4\text{-}72d)$$

$$\frac{\partial \underline{u}}{\partial q_j} = \begin{bmatrix} \frac{\partial u_1}{\partial q_j} & \frac{\partial u_2}{\partial q_j} & \frac{\partial u_3}{\partial q_j} & \frac{\partial u_4}{\partial q_j} & \frac{\partial u_5}{\partial q_j} & \frac{\partial u_6}{\partial q_j} \end{bmatrix}^T \quad (4\text{-}72e)$$

$$\frac{\partial \underline{u}}{\partial q_i} = \left[\frac{\partial u_1}{\partial q_i} \quad \frac{\partial u_2}{\partial q_i} \quad \frac{\partial u_3}{\partial q_i} \quad \frac{\partial u_4}{\partial q_i} \quad \frac{\partial u_5}{\partial q_i} \quad \frac{\partial u_6}{\partial q_i} \right]^T \tag{4-72f}$$

The second-order derivatives of the Kepler elements $\underline{u}$ with respect to the Poincaré orbital elements $\underline{p}$, $\underline{q}$ in (4-71) can be found by the following way: Rewriting (2-68) in the abbreviated form (4-73) and differentiating with respect to the Poincaré elements, yields six matrices of second-order derivatives.

$$\underline{U} = \left[\frac{\partial \underline{u}}{\partial \underline{q}} \quad \frac{\partial \underline{u}}{\partial \underline{p}} \right] = \left[\frac{\partial \underline{u}}{\partial l} \quad \frac{\partial \underline{u}}{\partial g} \quad \frac{\partial \underline{u}}{\partial h} \quad \frac{\partial \underline{u}}{\partial L} \quad \frac{\partial \underline{u}}{\partial G} \quad \frac{\partial \underline{u}}{\partial H} \right] \tag{4-73}$$

$$\underline{U}_l = \frac{\partial \underline{U}}{\partial l} = \left[\frac{\partial^2 \underline{u}}{\partial l^2} \quad \frac{\partial^2 \underline{u}}{\partial l\, \partial g} \quad \frac{\partial^2 \underline{u}}{\partial l\, \partial h} \quad \frac{\partial^2 \underline{u}}{\partial l\, \partial L} \quad \frac{\partial^2 \underline{u}}{\partial l\, \partial G} \quad \frac{\partial^2 \underline{u}}{\partial l\, \partial H} \right] \tag{4-74a}$$

$$\underline{U}_g = \frac{\partial \underline{U}}{\partial g} = \left[\frac{\partial^2 \underline{u}}{\partial g\, \partial l} \quad \frac{\partial^2 \underline{u}}{\partial g^2} \quad \frac{\partial^2 \underline{u}}{\partial g\, \partial h} \quad \frac{\partial^2 \underline{u}}{\partial g\, \partial L} \quad \frac{\partial^2 \underline{u}}{\partial g\, \partial G} \quad \frac{\partial^2 \underline{u}}{\partial g\, \partial H} \right] \tag{4-74b}$$

$$\underline{U}_h = \frac{\partial \underline{U}}{\partial h} = \left[\frac{\partial^2 \underline{u}}{\partial h\, \partial l} \quad \frac{\partial^2 \underline{u}}{\partial h\, \partial g} \quad \frac{\partial^2 \underline{u}}{\partial h^2} \quad \frac{\partial^2 \underline{u}}{\partial h\, \partial L} \quad \frac{\partial^2 \underline{u}}{\partial h\, \partial G} \quad \frac{\partial^2 \underline{u}}{\partial h\, \partial H} \right] \tag{4-74c}$$

$$\underline{U}_L = \frac{\partial \underline{U}}{\partial L} = \left[\frac{\partial^2 \underline{u}}{\partial L\, \partial l} \quad \frac{\partial^2 \underline{u}}{\partial L\, \partial g} \quad \frac{\partial^2 \underline{u}}{\partial L\, \partial h} \quad \frac{\partial^2 \underline{u}}{\partial L^2} \quad \frac{\partial^2 \underline{u}}{\partial L\, \partial G} \quad \frac{\partial^2 \underline{u}}{\partial L\, \partial H} \right] \tag{4-74d}$$

$$\underline{U}_G = \frac{\partial \underline{U}}{\partial G} = \left[\frac{\partial^2 \underline{u}}{\partial G\, \partial l} \quad \frac{\partial^2 \underline{u}}{\partial G\, \partial g} \quad \frac{\partial^2 \underline{u}}{\partial G\, \partial h} \quad \frac{\partial^2 \underline{u}}{\partial G\, \partial L} \quad \frac{\partial^2 \underline{u}}{\partial G^2} \quad \frac{\partial^2 \underline{u}}{\partial G\, \partial H} \right] \tag{4-74e}$$

$$\underline{U}_H = \frac{\partial \underline{U}}{\partial H} = \left[\frac{\partial^2 \underline{u}}{\partial H\, \partial l} \quad \frac{\partial^2 \underline{u}}{\partial H\, \partial g} \quad \frac{\partial^2 \underline{u}}{\partial H\, \partial h} \quad \frac{\partial^2 \underline{u}}{\partial H\, \partial L} \quad \frac{\partial^2 \underline{u}}{\partial H\, \partial G} \quad \frac{\partial^2 \underline{u}}{\partial H^2} \right] \tag{4-74f}$$

The second-order derivatives of the Kepler elements with respect to $\underline{p}$, $\underline{q}$ can be found by taking the corresponding (column) vectors from (4-74a,...,f).

Assuming that the second-order partial derivatives of the Kepler elements with respect to the Poincaré elements are continuous in the specific domain of interest, we can interchange the sequence of differentiation. Thus, it holds for (4-74a,...,f)

$$\frac{\partial^2 \underline{u}}{\partial p_i\, \partial p_j} = \frac{\partial^2 \underline{u}}{\partial p_j\, \partial p_i} \tag{4-75a}$$

$$\frac{\partial^2 \underline{u}}{\partial q_i \, \partial p_j} = \frac{\partial^2 \underline{u}}{\partial p_j \, \partial q_i} \qquad (4\text{-}75b)$$

$$\frac{\partial^2 \underline{u}}{\partial q_i \, \partial q_j} = \frac{\partial^2 \underline{u}}{\partial q_j \, \partial q_i} \qquad (4\text{-}75c)$$

$$\frac{\partial^2 \underline{u}}{\partial p_i \, \partial q_j} = \frac{\partial^2 \underline{u}}{\partial q_j \, \partial p_i} \qquad (4\text{-}75d)$$

and for (4-72a)

$$\underline{X}_K = \underline{X}_K^T \qquad (4\text{-}76)$$

As a consequence, not all of the vectors (4-75a,...,d) are independent of each other. (4-76) is uniquely determined by the upper or lower triangle of $\underline{X}_K$.

Matrix $\underline{X}_1$

Using (4-72a) and the vector $\underline{u}$ (4-69c) matrix $\underline{X}_1$ is given by

$$\underline{X}_1 = \begin{bmatrix} \frac{\partial^2 x_1}{\partial M^2} & \frac{\partial^2 x_1}{\partial M \, \partial a} & \frac{\partial^2 x_1}{\partial M \, \partial \Omega} & \frac{\partial^2 x_1}{\partial M \, \partial \omega} & \frac{\partial^2 x_1}{\partial M \, \partial e} & \frac{\partial^2 x_1}{\partial M \, \partial i} \\ \frac{\partial^2 x_1}{\partial a \, \partial M} & \frac{\partial^2 x_1}{\partial a^2} & \frac{\partial^2 x_1}{\partial a \, \partial \Omega} & \frac{\partial^2 x_1}{\partial a \, \partial \omega} & \frac{\partial^2 x_1}{\partial a \, \partial e} & \frac{\partial^2 x_1}{\partial a \, \partial i} \\ \frac{\partial^2 x_1}{\partial \Omega \, \partial M} & \frac{\partial^2 x_1}{\partial \Omega \, \partial a} & \frac{\partial^2 x_1}{\partial \Omega^2} & \frac{\partial^2 x_1}{\partial \Omega \, \partial \omega} & \frac{\partial^2 x_1}{\partial \Omega \, \partial e} & \frac{\partial^2 x_1}{\partial \Omega \, \partial i} \\ \frac{\partial^2 x_1}{\partial \omega \, \partial M} & \frac{\partial^2 x_1}{\partial \omega \, \partial a} & \frac{\partial^2 x_1}{\partial \omega \, \partial \Omega} & \frac{\partial^2 x_1}{\partial \omega^2} & \frac{\partial^2 x_1}{\partial \omega \, \partial e} & \frac{\partial^2 x_1}{\partial \omega \, \partial i} \\ \frac{\partial^2 x_1}{\partial e \, \partial M} & \frac{\partial^2 x_1}{\partial e \, \partial a} & \frac{\partial^2 x_1}{\partial e \, \partial \Omega} & \frac{\partial^2 x_1}{\partial e \, \partial \omega} & \frac{\partial^2 x_1}{\partial e^2} & \frac{\partial^2 x_1}{\partial e \, \partial i} \\ \frac{\partial^2 x_1}{\partial i \, \partial M} & \frac{\partial^2 x_1}{\partial i \, \partial a} & \frac{\partial^2 x_1}{\partial i \, \partial \Omega} & \frac{\partial^2 x_1}{\partial i \, \partial \omega} & \frac{\partial^2 x_1}{\partial i \, \partial e} & \frac{\partial^2 x_1}{\partial i^2} \end{bmatrix} \qquad (4\text{-}77)$$

The elements of (4-77) are derived by partial differentiation of (2-77a), (2-78a), (2-79a), (2-80a), (2-81a) and (2-82a). $\underline{X}_1$ is uniquely determined by elements of its upper triangle. This is due to the possibility of interchanging the sequence of differentiation.

$$\frac{\partial^2 x_1}{\partial M^2} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\,, -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin i]\cdot$$
$$\cdot\left[-\frac{a^4}{r^3}(\cos E - e), -\frac{a^4}{r^3}\cos\varphi\sin E\,, 0\right]^T \tag{4-78a}$$

$$\frac{\partial^2 x_1}{\partial M\,\partial a} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\,, -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin i]\cdot$$
$$\cdot\left[-\frac{a}{r}\sin E\,, \frac{a}{r}\cos\varphi\cos E\,, 0\right]^T \tag{4-78b}$$

$$\frac{\partial^2 x_1}{\partial M\,\partial\Omega} = [-(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i), \sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i\,, \cos\Omega\sin i]\cdot$$
$$\cdot\left[-\frac{a^2}{r}\sin E\,, \frac{a^2}{r}\cos\varphi\cos E\,, 0\right]^T \tag{4-78c}$$

$$\frac{\partial^2 x_1}{\partial M\,\partial\omega} = [-(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin\omega\cos i - \cos\Omega\cos\omega\,, 0]\cdot$$
$$\cdot\left[-\frac{a^2}{r}\sin E\,, \frac{a^2}{r}\cos\varphi\cos E\,, 0\right]^T \tag{4-78d}$$

$$\frac{\partial^2 x_1}{\partial M\,\partial e} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\,, -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin i]\cdot$$
$$\cdot\left[-\frac{a^4}{r^3}\sin E\left(2\cos E - e\left(1+\cos^2 E\right)\right), \frac{a^4}{r^3\cos\varphi}\left(e^2 + \cos E - e\cos E\left(1+\cos^2 E\right)\right), 0\right]^T \tag{4-78e}$$

$$\frac{\partial^2 x_1}{\partial M\,\partial i} = [\sin\Omega\sin\omega\sin i\,, \sin\Omega\cos\omega\sin i\,, \sin\Omega\cos i]\cdot$$
$$\cdot\left[-\frac{a^2}{r}\sin E\,, \frac{a^2}{r}\cos\varphi\cos E\,, 0\right]^T \tag{4-78f}$$

$$\frac{\partial^2 x_1}{\partial a^2} = 0 \tag{4-78g}$$

$$\frac{\partial^2 x_1}{\partial a\,\partial\Omega} = [-(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i), \sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i\,, \cos\Omega\sin i]\cdot$$
$$\cdot[\cos E - e, \cos\varphi\sin E\,, 0]^T \tag{4-78h}$$

$$\frac{\partial^2 x_1}{\partial a\,\partial\omega} = [-(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin\omega\cos i - \cos\Omega\cos\omega\,, 0]\cdot$$
$$\cdot[\cos E - e\,, \cos\varphi\sin E\,, 0]^T \tag{4-78i}$$

$$\frac{\partial^2 x_1}{\partial a\,\partial e} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\,, -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i), \sin\Omega\sin i]\cdot$$
$$\cdot\left[-\left(1+\frac{a}{r}\sin^2 E\right), \frac{a\sin E}{r\cos\varphi}(\cos E - e), 0\right]^T \tag{4-78j}$$

$$\frac{\partial^2 x_1}{\partial a\,\partial i} = [\sin\Omega\sin\omega\sin i\,, \sin\Omega\cos\omega\sin i\,, \sin\Omega\cos i]\cdot$$
$$\cdot[\cos E - e\,, \cos\varphi\sin E\,, 0]^T \tag{4-78k}$$

$$\frac{\partial^2 x_1}{\partial \Omega^2} = [\sin\Omega \sin\omega \cos i - \cos\Omega \cos\omega, \cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i, -\sin\Omega \sin i] \cdot$$
$$\cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-78l}$$

$$\frac{\partial^2 x_1}{\partial \Omega \partial \omega} = [\sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i, \sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i, 0] \cdot$$
$$\cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-78m}$$

$$\frac{\partial^2 x_1}{\partial \Omega \partial e} = [-(\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i), \sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i, \cos\Omega \sin i] \cdot$$
$$\cdot \left[-a\left(1 + \frac{a}{r}\sin^2 E\right), \frac{a^2 \sin E}{r \cos\varphi}(\cos E - e), 0\right]^T \tag{4-78n}$$

$$\frac{\partial^2 x_1}{\partial \Omega \partial i} = [\cos\Omega \sin\omega \sin i, \cos\Omega \cos\omega \sin i, \cos\Omega \cos i] \cdot$$
$$\cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-78o}$$

$$\frac{\partial^2 x_1}{\partial \omega^2} = [\sin\Omega \sin\omega \cos i - \cos\Omega \cos\omega, \sin\Omega \cos\omega \cos i + \cos\Omega \sin\omega, 0] \cdot$$
$$\cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-78p}$$

$$\frac{\partial^2 x_1}{\partial \omega \partial e} = [-(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin\omega \cos i - \cos\Omega \cos\omega, 0] \cdot$$
$$\cdot \left[-a\left(1 + \frac{a}{r}\sin^2 E\right), \frac{a^2 \sin E}{r \cos\varphi}(\cos E - e), 0\right]^T \tag{4-78q}$$

$$\frac{\partial^2 x_1}{\partial \omega \partial i} = [\sin\Omega \cos\omega \sin i, -\sin\Omega \sin\omega \sin i, 0] \cdot$$
$$\cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-78r}$$

$$\frac{\partial^2 x_1}{\partial e^2} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot$$
$$\cdot \left[\frac{a^3}{r^2}\sin^2 E\left(\frac{a}{r} e \sin^2 E - 3\cos E\right), \frac{a^3 \sin E}{r^2 \cos\varphi}\left(\frac{r}{a} - 3\sin^2 E + \frac{a \tan\varphi}{r \cos\varphi} \cdot \right.\right.$$
$$\left.\left. \cdot (\cos E - e)^3\right), 0\right]^T \tag{4-78s}$$

$$\frac{\partial^2 x_1}{\partial e \partial i} = [\sin\Omega \sin\omega \sin i, \sin\Omega \cos\omega \sin i, \sin\Omega \cos i] \cdot$$
$$\cdot \left[-a\left(1 + \frac{a}{r}\sin^2 E\right), \frac{a^2 \sin E}{r \cos\varphi}(\cos E - e), 0\right]^T \tag{4-78t}$$

$$\frac{\partial^2 x_1}{\partial i^2} = [\sin\Omega \sin\omega \cos i, \sin\Omega \cos\omega \cos i, -\sin\Omega \sin i] \cdot$$
$$\cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-78u}$$

Matrix $\underline{X}_2$

Using (4-72a) and the vector $\underline{u}$ (4-69c) the matrix $\underline{X}_2$ is given by

$$\underline{X}_2 = \begin{bmatrix} \frac{\partial^2 x_2}{\partial M^2} & \frac{\partial^2 x_2}{\partial M\,\partial a} & \frac{\partial^2 x_2}{\partial M\,\partial \Omega} & \frac{\partial^2 x_2}{\partial M\,\partial \omega} & \frac{\partial^2 x_2}{\partial M\,\partial e} & \frac{\partial^2 x_2}{\partial M\,\partial i} \\ \frac{\partial^2 x_2}{\partial a\,\partial M} & \frac{\partial^2 x_2}{\partial a^2} & \frac{\partial^2 x_2}{\partial a\,\partial \Omega} & \frac{\partial^2 x_2}{\partial a\,\partial \omega} & \frac{\partial^2 x_2}{\partial a\,\partial e} & \frac{\partial^2 x_2}{\partial a\,\partial i} \\ \frac{\partial^2 x_2}{\partial \Omega\,\partial M} & \frac{\partial^2 x_2}{\partial \Omega\,\partial a} & \frac{\partial^2 x_2}{\partial^2 \Omega} & \frac{\partial^2 x_2}{\partial \Omega\,\partial \omega} & \frac{\partial^2 x_2}{\partial \Omega\,\partial e} & \frac{\partial^2 x_2}{\partial \Omega\,\partial i} \\ \frac{\partial^2 x_2}{\partial \omega\,\partial M} & \frac{\partial^2 x_2}{\partial \omega\,\partial a} & \frac{\partial^2 x_2}{\partial \omega\,\partial \Omega} & \frac{\partial^2 x_2}{\partial \omega^2} & \frac{\partial^2 x_2}{\partial \omega\,\partial e} & \frac{\partial^2 x_2}{\partial \omega\,\partial i} \\ \frac{\partial^2 x_2}{\partial e\,\partial M} & \frac{\partial^2 x_2}{\partial e\,\partial a} & \frac{\partial^2 x_2}{\partial e\,\partial \Omega} & \frac{\partial^2 x_2}{\partial e\,\partial \omega} & \frac{\partial^2 x_2}{\partial e^2} & \frac{\partial^2 x_2}{\partial e\,\partial i} \\ \frac{\partial^2 x_2}{\partial i\,\partial M} & \frac{\partial^2 x_2}{\partial i\,\partial a} & \frac{\partial^2 x_2}{\partial i\,\partial \Omega} & \frac{\partial^2 x_2}{\partial i\,\partial \omega} & \frac{\partial^2 x_2}{\partial i\,\partial e} & \frac{\partial^2 x_2}{\partial i^2} \end{bmatrix} \tag{4-79}$$

The elements of (4-79) are derived by partial differentiation of (2-77b), (2-78b), (2-79b), (2-80b), (2-81b) and (2-82b). $\underline{X}_2$ is uniquely determined by the elements of its upper triangle.

$$\frac{\partial^2 x_2}{\partial M^2} = [\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i\,,\ \cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega\,,\ -\cos\Omega\sin i]\cdot$$
$$\cdot\left[-\frac{a^4}{r^3}(\cos E - e), -\frac{a^4}{r^3}\cos\varphi\sin E\,,\ 0\right]^T \tag{4-80a}$$

$$\frac{\partial^2 x_2}{\partial M\,\partial a} = [\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i\,,\ \cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega\,,\ -\cos\Omega\sin i]\cdot$$
$$\cdot\left[-\frac{a}{r}\sin E\,,\ \frac{a}{r}\cos\varphi\cos E\,,\ 0\right]^T \tag{4-80b}$$

$$\frac{\partial^2 x_2}{\partial M\,\partial \Omega} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\,,\ -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\,,\ \sin\Omega\sin i]\cdot$$
$$\cdot\left[-\frac{a^2}{r}\sin E\,,\ \frac{a^2}{r}\cos\varphi\cos E\,,\ 0\right]^T \tag{4-80c}$$

$$\frac{\partial^2 x_2}{\partial M\,\partial \omega} = [\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i\,,\ -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\,,0]\cdot$$
$$\cdot\left[-\frac{a^2}{r}\sin E\,,\ \frac{a^2}{r}\cos\varphi\cos E\,,\ 0\right]^T \tag{4-80d}$$

$$\frac{\partial^2 x_2}{\partial M \partial e} = [\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i, \cos\Omega \cos\omega \cos i - \sin\Omega \sin\omega, -\cos\Omega \sin i] \cdot \cdot \left[-\frac{a^4}{r^3} \sin E \left(2\cos E - e\left(1+\cos^2 E\right)\right), \frac{a^4}{r^3 \cos\varphi} \left(e^2 + \cos E - e \cos E \left(1+\cos^2 E\right)\right), 0 \right]^T \tag{4-80e}$$

$$\frac{\partial^2 x_2}{\partial M \partial i} = [-\cos\Omega \sin\omega \sin i, -\cos\Omega \cos\omega \sin i, -\cos\Omega \cos i] \cdot \cdot \left[-\frac{a^2}{r} \sin E, \frac{a^2}{r} \cos\varphi \cos E, 0 \right]^T \tag{4-80f}$$

$$\frac{\partial^2 x_2}{\partial a^2} = 0 \tag{4-80g}$$

$$\frac{\partial^2 x_2}{\partial a \partial \Omega} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot \cdot [\cos E - e, \cos\varphi \sin E, 0]^T \tag{4-80h}$$

$$\frac{\partial^2 x_2}{\partial a \partial \omega} = [\cos\Omega \cos\omega \cos i - \sin\Omega \sin\omega, -(\cos\Omega \sin\omega \cos i + \sin\Omega \cos\omega), 0] \cdot \cdot [\cos E - e, \cos\varphi \sin E, 0]^T \tag{4-80i}$$

$$\frac{\partial^2 x_2}{\partial a \partial e} = [\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i, \cos\Omega \cos\omega \cos i - \sin\Omega \sin\omega, -\cos\Omega \sin i] \cdot \cdot \left[-\left(1 + \frac{a}{r} \sin^2 E\right), \frac{a \sin E}{r \cos\varphi} (\cos E - e), 0 \right]^T \tag{4-80j}$$

$$\frac{\partial^2 x_2}{\partial a \partial i} = [-\cos\Omega \sin\omega \sin i, -\cos\Omega \cos\omega \sin i, -\cos\Omega \cos i] \cdot \cdot [\cos E - e, \cos\varphi \sin E, 0]^T \tag{4-80k}$$

$$\frac{\partial^2 x_2}{\partial \Omega^2} = [-(\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i), \sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i, \cos\Omega \sin i] \cdot \cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-80l}$$

$$\frac{\partial^2 x_2}{\partial \Omega \partial \omega} = [-(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin\omega \cos i - \cos\Omega \cos\omega, 0] \cdot \cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-80m}$$

$$\frac{\partial^2 x_2}{\partial \Omega \partial e} = [\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i, -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i), \sin\Omega \sin i] \cdot \cdot \left[-a\left(1 + \frac{a}{r} \sin^2 E\right), \frac{a^2 \sin E}{r \cos\varphi} (\cos E - e), 0 \right]^T \tag{4-80n}$$

$$\frac{\partial^2 x_2}{\partial \Omega \partial i} = [\sin\Omega \sin\omega \sin i, \sin\Omega \cos\omega \sin i, \sin\Omega \cos i] \cdot \cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-80o}$$

$$\frac{\partial^2 x_2}{\partial \omega^2} = [-(\cos\Omega \sin\omega \cos i + \sin\Omega \cos\omega), \sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i, 0] \cdot \cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-80p}$$

$$\frac{\partial^2 x_2}{\partial\omega\,\partial e} = [\cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega\,,\ -(\cos\Omega\sin\omega\cos i + \sin\Omega\cos\omega)\,,0]\cdot$$
$$\cdot\left[-a\left(1+\frac{a}{r}\sin^2 E\right),\frac{a^2\sin E}{r\cos\varphi}(\cos E - e),0\right]^T \tag{4-80q}$$

$$\frac{\partial^2 x_2}{\partial\omega\,\partial i} = [-\cos\Omega\cos\omega\sin i\,,\ \cos\Omega\sin\omega\sin i\,,0]\cdot$$
$$\cdot[a(\cos E - e),\ a\cos\varphi\sin E\,,\ 0]^T \tag{4-80r}$$

$$\frac{\partial^2 x_2}{\partial e^2} = [\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i\,,\ \cos\Omega\cos\omega\cos i - \sin\Omega\sin\omega\,,\ -\cos\Omega\sin i]\cdot$$
$$\cdot\left[\frac{a^3}{r^2}\sin^2 E\left(\frac{a}{r}e\sin^2 E - 3\cos E\right),\frac{a^3\sin E}{r^2\cos\varphi}\left(\frac{r}{a} - 3\sin^2 E + \frac{a\tan\varphi}{r\cos\varphi}\cdot(\cos E - e)^3\right),0\right]^T \tag{4-80s}$$

$$\frac{\partial^2 x_2}{\partial e\,\partial i} = [-\cos\Omega\sin\omega\sin i\,,\ -\cos\Omega\cos\omega\sin i\,,-\cos\Omega\cos i]\cdot$$
$$\cdot\left[-a\left(1+\frac{a}{r}\sin^2 E\right),\frac{a^2\sin E}{r\cos\varphi}(\cos E - e),0\right]^T \tag{4-80t}$$

$$\frac{\partial^2 x_2}{\partial i^2} = [-\cos\Omega\sin\omega\cos i\,,\ -\cos\Omega\cos\omega\cos i\,,\cos\Omega\sin i]\cdot$$
$$\cdot[a(\cos E - e),\ a\cos\varphi\sin E\,,\ 0]^T \tag{4-80u}$$

Matrix $\underline{X}_3$

Using (4-72a) and the vector $\underline{u}$ (4-69c) the matrix $\underline{X}_3$ is given by

$$\underline{X}_3 = \begin{bmatrix}
\frac{\partial^2 x_3}{\partial M^2} & \frac{\partial^2 x_3}{\partial M\,\partial a} & \frac{\partial^2 x_3}{\partial M\,\partial\Omega} & \frac{\partial^2 x_3}{\partial M\,\partial\omega} & \frac{\partial^2 x_3}{\partial M\,\partial e} & \frac{\partial^2 x_3}{\partial M\,\partial i} \\
\frac{\partial^2 x_3}{\partial a\,\partial M} & \frac{\partial^2 x_3}{\partial a^2} & \frac{\partial^2 x_3}{\partial a\,\partial\Omega} & \frac{\partial^2 x_3}{\partial a\,\partial\omega} & \frac{\partial^2 x_3}{\partial a\,\partial e} & \frac{\partial^2 x_3}{\partial a\,\partial i} \\
\frac{\partial^2 x_3}{\partial\Omega\,\partial M} & \frac{\partial^2 x_3}{\partial\Omega\,\partial a} & \frac{\partial^2 x_3}{\partial^2\Omega} & \frac{\partial^2 x_3}{\partial\Omega\,\partial\omega} & \frac{\partial^2 x_3}{\partial\Omega\,\partial e} & \frac{\partial^2 x_3}{\partial\Omega\,\partial i} \\
\frac{\partial^2 x_3}{\partial\omega\,\partial M} & \frac{\partial^2 x_3}{\partial\omega\,\partial a} & \frac{\partial^2 x_3}{\partial\omega\,\partial\Omega} & \frac{\partial^2 x_3}{\partial\omega^2} & \frac{\partial^2 x_3}{\partial\omega\,\partial e} & \frac{\partial^2 x_3}{\partial\omega\,\partial i} \\
\frac{\partial^2 x_3}{\partial e\,\partial M} & \frac{\partial^2 x_3}{\partial e\,\partial a} & \frac{\partial^2 x_3}{\partial e\,\partial\Omega} & \frac{\partial^2 x_3}{\partial e\,\partial\omega} & \frac{\partial^2 x_3}{\partial e^2} & \frac{\partial^2 x_3}{\partial e\,\partial i} \\
\frac{\partial^2 x_3}{\partial i\,\partial M} & \frac{\partial^2 x_3}{\partial i\,\partial a} & \frac{\partial^2 x_3}{\partial i\,\partial\Omega} & \frac{\partial^2 x_3}{\partial i\,\partial\omega} & \frac{\partial^2 x_3}{\partial i\,\partial e} & \frac{\partial^2 x_3}{\partial i^2}
\end{bmatrix} \tag{4-81}$$

The elements of (4-81) are derived by partial differentiation of (2-77c), (2-78c), (2-79c), (2-80c), (2-81c) and (2-82c). Again, $\underline{X}_3$ is uniquely determined by the elements of its upper triangle.

$$\frac{\partial^2 x_3}{\partial M^2} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot$$
$$\cdot \left[-\frac{a^4}{r^3}(\cos E - e), -\frac{a^4}{r^3}\cos\varphi \sin E, 0 \right]^T \quad (4\text{-}82a)$$

$$\frac{\partial^2 x_3}{\partial M \partial a} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot$$
$$\cdot \left[-\frac{a}{r}\sin E, \frac{a}{r}\cos\varphi \cos E, 0 \right]^T \quad (4\text{-}82b)$$

$$\frac{\partial^2 x_3}{\partial M \partial \Omega} = 0 \quad (4\text{-}82c)$$

$$\frac{\partial^2 x_3}{\partial M \partial \omega} = [\cos\omega \sin i, -\sin\omega \sin i, 0] \cdot$$
$$\cdot \left[-\frac{a^2}{r}\sin E, \frac{a^2}{r}\cos\varphi \cos E, 0 \right]^T \quad (4\text{-}82d)$$

$$\frac{\partial^2 x_3}{\partial M \partial e} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot$$
$$\cdot \left[-\frac{a^4}{r^3}\sin E\left(2\cos E - e\left(1+\cos^2 E\right)\right), \frac{a^4}{r^3\cos\varphi}\left(e^2 + \cos E - e\cos E\left(1+\cos^2 E\right)\right), 0 \right]^T \quad (4\text{-}82e)$$

$$\frac{\partial^2 x_3}{\partial M \partial i} = [\sin\omega \cos i, \cos\omega \cos i, -\sin i] \cdot$$
$$\cdot \left[-\frac{a^2}{r}\sin E, \frac{a^2}{r}\cos\varphi \cos E, 0 \right]^T \quad (4\text{-}82f)$$

$$\frac{\partial^2 x_3}{\partial a^2} = 0 \quad (4\text{-}82g)$$

$$\frac{\partial^2 x_3}{\partial a \partial \Omega} = 0 \quad (4\text{-}82h)$$

$$\frac{\partial^2 x_3}{\partial a \partial \omega} = [\cos\omega \sin i, -\sin\omega \sin i, 0] \cdot$$
$$\cdot [\cos E - e, \cos\varphi \sin E, 0]^T \quad (4\text{-}82i)$$

$$\frac{\partial^2 x_3}{\partial a \partial e} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot$$
$$\cdot \left[-\left(1 + \frac{a}{r}\sin^2 E\right), \frac{a \sin E}{r\cos\varphi}(\cos E - e), 0 \right]^T \quad (4\text{-}82j)$$

$$\frac{\partial^2 x_3}{\partial a \partial i} = [\sin\omega \cos i, \cos\omega \cos i, -\sin i] \cdot$$
$$\cdot [\cos E - e, \cos\varphi \sin E, 0]^T \quad (4\text{-}82k)$$

$$\frac{\partial^2 x_3}{\partial \Omega^2} = 0 \tag{4-82l}$$

$$\frac{\partial^2 x_3}{\partial \Omega\, \partial \omega} = 0 \tag{4-82m}$$

$$\frac{\partial^2 x_3}{\partial \Omega\, \partial e} = 0 \tag{4-82n}$$

$$\frac{\partial^2 x_3}{\partial \Omega\, \partial i} = 0 \tag{4-82o}$$

$$\frac{\partial^2 x_3}{\partial \omega^2} = [-\sin\omega \sin i, -\cos\omega \sin i, 0] \cdot \\ \cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-82p}$$

$$\frac{\partial^2 x_3}{\partial \omega \partial e} = [\cos\omega \sin i, -\sin\omega \sin i, 0] \cdot \\ \cdot \left[-a\left(1 + \frac{a}{r}\sin^2 E\right), \frac{a^2 \sin E}{r\cos\varphi}(\cos E - e), 0\right]^T \tag{4-82q}$$

$$\frac{\partial^2 x_3}{\partial \omega \partial i} = [\cos\omega \cos i, -\sin\omega \cos i, 0] \cdot \\ \cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-82r}$$

$$\frac{\partial^2 x_3}{\partial e^2} = [\sin\omega \sin i, \cos\omega \sin i, \cos i] \cdot \\ \cdot \left[\frac{a^3}{r^2}\sin^2 E\left(\frac{a}{r} e\sin^2 E - 3\cos E\right), \frac{a^3 \sin E}{r^2 \cos\varphi}\left(\frac{r}{a} - 3\sin^2 E + \frac{a\tan\varphi}{r\cos\varphi} \cdot (\cos E - e)^3\right), 0\right]^T \tag{4-82s}$$

$$\frac{\partial^2 x_3}{\partial e \partial i} = [\sin\omega \cos i, \cos\omega \cos i, -\sin i] \cdot \\ \cdot \left[-a\left(1 + \frac{a}{r}\sin^2 E\right), \frac{a^2 \sin E}{r\cos\varphi}(\cos E - e), 0\right]^T \tag{4-82t}$$

$$\frac{\partial^2 x_3}{\partial i^2} = [-\sin\omega \sin i, -\cos\omega \sin i, -\cos i] \cdot \\ \cdot [a(\cos E - e), a\cos\varphi \sin E, 0]^T \tag{4-82u}$$

The matrices $\underline{U}_l$, $\underline{U}_g$, $\underline{U}_h$, $\underline{U}_L$, $\underline{U}_G$, $\underline{U}_H$

Only 21 of 36 column vectors the above mentioned matrices (4-74a) to (4-74f) consisting of, are independent. These vectors should be derived in the following.

Going for the second-order derivatives of the Kepler elements with respect to the Poincaré elements one has to consider that the first-order derivatives (2-69a,...,f), (2-70a,...,f), (2-71a,...,f), (2-72a,...,f), (2-73a, ...,f) and (2-74a,...,f) are given as function of the Kepler elements.

$$\frac{\partial u_i}{\partial p_j} = \frac{\partial u_i}{\partial p_j}(\underline{u}) \tag{4-83a}$$

$$\frac{\partial u_i}{\partial q_j} = \frac{\partial u_i}{\partial q_j}(\underline{u}) \tag{4-83b}$$

Thus, the second-order derivatives of (4-83a,b) can be computed using the rules

$$\frac{\partial^2 u_i}{\partial p_k\,\partial p_j} = \frac{\partial}{\partial u_1}\left(\frac{\partial u_i}{\partial p_j}\right)\frac{\partial u_1}{\partial p_k} + \frac{\partial}{\partial u_2}\left(\frac{\partial u_i}{\partial p_j}\right)\frac{\partial u_2}{\partial p_k} + \ldots\ldots + \frac{\partial}{\partial u_6}\left(\frac{\partial u_i}{\partial p_j}\right)\frac{\partial u_6}{\partial p_k} \tag{4-84a}$$

$$\frac{\partial^2 u_i}{\partial p_k\,\partial q_j} = \frac{\partial}{\partial u_1}\left(\frac{\partial u_i}{\partial q_j}\right)\frac{\partial u_1}{\partial p_k} + \frac{\partial}{\partial u_2}\left(\frac{\partial u_i}{\partial q_j}\right)\frac{\partial u_2}{\partial p_k} + \ldots\ldots + \frac{\partial}{\partial u_6}\left(\frac{\partial u_i}{\partial q_j}\right)\frac{\partial u_6}{\partial p_k} \tag{4-84b}$$

$$\frac{\partial^2 u_i}{\partial q_k\,\partial q_j} = \frac{\partial}{\partial u_1}\left(\frac{\partial u_i}{\partial q_j}\right)\frac{\partial u_1}{\partial q_k} + \frac{\partial}{\partial u_2}\left(\frac{\partial u_i}{\partial q_j}\right)\frac{\partial u_2}{\partial q_k} + \ldots\ldots + \frac{\partial}{\partial u_6}\left(\frac{\partial u_i}{\partial q_j}\right)\frac{\partial u_6}{\partial q_k} \tag{4-84c}$$

$$\frac{\partial^2 u_i}{\partial q_k\,\partial p_j} = \frac{\partial}{\partial u_1}\left(\frac{\partial u_i}{\partial p_j}\right)\frac{\partial u_1}{\partial q_k} + \frac{\partial}{\partial u_2}\left(\frac{\partial u_i}{\partial p_j}\right)\frac{\partial u_2}{\partial q_k} + \ldots\ldots + \frac{\partial}{\partial u_6}\left(\frac{\partial u_i}{\partial p_j}\right)\frac{\partial u_6}{\partial q_k} \tag{4-84d}$$

Using (4-84a,...,d) the 21 column vectors determining the matrices $\underline{U}_l$, $\underline{U}_g$, $\underline{U}_h$, $\underline{U}_L$, $\underline{U}_G$, $\underline{U}_H$ are given by (second-order derivatives which disappear are set already to zero)

$$\begin{bmatrix} \frac{\partial^2 M}{\partial l^2} \\ \frac{\partial^2 a}{\partial l^2} \\ \frac{\partial^2 \Omega}{\partial l^2} \\ \frac{\partial^2 \omega}{\partial l^2} \\ \frac{\partial^2 e}{\partial l^2} \\ \frac{\partial^2 i}{\partial l^2} \end{bmatrix} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial M}{\partial l} \\ \frac{\partial a}{\partial l} \\ \frac{\partial \Omega}{\partial l} \\ \frac{\partial \omega}{\partial l} \\ \frac{\partial e}{\partial l} \\ \frac{\partial i}{\partial l} \end{bmatrix} \tag{4-85a}$$

$$\begin{bmatrix} \frac{\partial^2 M}{\partial l\,\partial g} \\ \frac{\partial^2 a}{\partial l\,\partial g} \\ \frac{\partial^2 \Omega}{\partial l\,\partial g} \\ \frac{\partial^2 \omega}{\partial l\,\partial g} \\ \frac{\partial^2 e}{\partial l\,\partial g} \\ \frac{\partial^2 i}{\partial l\,\partial g} \end{bmatrix} = \begin{bmatrix} 0, & \frac{\partial}{\partial a}\left(\frac{\partial M}{\partial g}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial M}{\partial g}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial M}{\partial g}\right), & \frac{\partial}{\partial e}\left(\frac{\partial M}{\partial g}\right), & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial g}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial g}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial \omega}{\partial g}\right), & \frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial g}\right), & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial e}{\partial g}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial e}{\partial g}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial e}{\partial g}\right), & \frac{\partial}{\partial e}\left(\frac{\partial e}{\partial g}\right), & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial g}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial i}{\partial g}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial i}{\partial g}\right), & \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial g}\right), & \frac{\partial}{\partial i}\left(\frac{\partial i}{\partial g}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial M}{\partial l} \\ \frac{\partial a}{\partial l} \\ \frac{\partial \Omega}{\partial l} \\ \frac{\partial \omega}{\partial l} \\ \frac{\partial e}{\partial l} \\ \frac{\partial i}{\partial l} \end{bmatrix} \tag{4-85b}$$

$$\begin{bmatrix} \frac{\partial^2 M}{\partial l\,\partial h} \\ \frac{\partial^2 a}{\partial l\,\partial h} \\ \frac{\partial^2 \Omega}{\partial l\,\partial h} \\ \frac{\partial^2 \omega}{\partial l\,\partial h} \\ \frac{\partial^2 e}{\partial l\,\partial h} \\ \frac{\partial^2 i}{\partial l\,\partial h} \end{bmatrix} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial \Omega}{\partial h}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial \Omega}{\partial h}\right), & 0, & \frac{\partial}{\partial e}\left(\frac{\partial \Omega}{\partial h}\right), & \frac{\partial}{\partial i}\left(\frac{\partial \Omega}{\partial h}\right) \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial h}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial h}\right), & 0, & \frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial h}\right), & \frac{\partial}{\partial i}\left(\frac{\partial \omega}{\partial h}\right) \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial h}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial i}{\partial h}\right), & 0, & \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial h}\right), & \frac{\partial}{\partial i}\left(\frac{\partial i}{\partial h}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial M}{\partial l} \\ \frac{\partial a}{\partial l} \\ \frac{\partial \Omega}{\partial l} \\ \frac{\partial \omega}{\partial l} \\ \frac{\partial e}{\partial l} \\ \frac{\partial i}{\partial l} \end{bmatrix} \tag{4-85c}$$

$$\begin{bmatrix} \frac{\partial^2 M}{\partial l\,\partial L} \\ \frac{\partial^2 a}{\partial l\,\partial L} \\ \frac{\partial^2 \Omega}{\partial l\,\partial L} \\ \frac{\partial^2 \omega}{\partial l\,\partial L} \\ \frac{\partial^2 e}{\partial l\,\partial L} \\ \frac{\partial^2 i}{\partial l\,\partial L} \end{bmatrix} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial a}{\partial L}\right), & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial e}{\partial L}\right), & 0, & 0, & \frac{\partial}{\partial e}\left(\frac{\partial e}{\partial L}\right), & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial L}\right), & 0, & 0, & \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial L}\right), & \frac{\partial}{\partial i}\left(\frac{\partial i}{\partial L}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial M}{\partial l} \\ \frac{\partial a}{\partial l} \\ \frac{\partial \Omega}{\partial l} \\ \frac{\partial \omega}{\partial l} \\ \frac{\partial e}{\partial l} \\ \frac{\partial i}{\partial l} \end{bmatrix} \tag{4-85d}$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial l\,\partial G} \\[2ex] \dfrac{\partial^2 a}{\partial l\,\partial G} \\[2ex] \dfrac{\partial^2 \Omega}{\partial l\,\partial G} \\[2ex] \dfrac{\partial^2 \omega}{\partial l\,\partial G} \\[2ex] \dfrac{\partial^2 e}{\partial l\,\partial G} \\[2ex] \dfrac{\partial^2 i}{\partial l\,\partial G} \end{bmatrix}
=
\begin{bmatrix}
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial M}{\partial G}\right) , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial G}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial e}{\partial G}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial G}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial l} \\[2ex] \dfrac{\partial a}{\partial l} \\[2ex] \dfrac{\partial \Omega}{\partial l} \\[2ex] \dfrac{\partial \omega}{\partial l} \\[2ex] \dfrac{\partial e}{\partial l} \\[2ex] \dfrac{\partial i}{\partial l} \end{bmatrix}
\tag{4-85e}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial l\,\partial H} \\[2ex] \dfrac{\partial^2 a}{\partial l\,\partial H} \\[2ex] \dfrac{\partial^2 \Omega}{\partial l\,\partial H} \\[2ex] \dfrac{\partial^2 \omega}{\partial l\,\partial H} \\[2ex] \dfrac{\partial^2 e}{\partial l\,\partial H} \\[2ex] \dfrac{\partial^2 i}{\partial l\,\partial H} \end{bmatrix}
=
\begin{bmatrix}
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \Omega}{\partial H}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \Omega}{\partial H}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \Omega}{\partial H}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \Omega}{\partial H}\right) \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial H}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial H}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial H}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \omega}{\partial H}\right) \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial H}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial H}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial H}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial H}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial l} \\[2ex] \dfrac{\partial a}{\partial l} \\[2ex] \dfrac{\partial \Omega}{\partial l} \\[2ex] \dfrac{\partial \omega}{\partial l} \\[2ex] \dfrac{\partial e}{\partial l} \\[2ex] \dfrac{\partial i}{\partial l} \end{bmatrix}
\tag{4-85f}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial g^2} \\[2ex] \dfrac{\partial^2 a}{\partial g^2} \\[2ex] \dfrac{\partial^2 \Omega}{\partial g^2} \\[2ex] \dfrac{\partial^2 \omega}{\partial g^2} \\[2ex] \dfrac{\partial^2 e}{\partial g^2} \\[2ex] \dfrac{\partial^2 i}{\partial g^2} \end{bmatrix}
=
\begin{bmatrix}
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial M}{\partial g}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial M}{\partial g}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial M}{\partial g}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial M}{\partial g}\right) , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial g}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial g}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial \omega}{\partial g}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial g}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial e}{\partial g}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial e}{\partial g}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial e}{\partial g}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial e}{\partial g}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial g}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial g}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial i}{\partial g}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial g}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial g}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial g} \\[2ex] \dfrac{\partial a}{\partial g} \\[2ex] \dfrac{\partial \Omega}{\partial g} \\[2ex] \dfrac{\partial \omega}{\partial g} \\[2ex] \dfrac{\partial e}{\partial g} \\[2ex] \dfrac{\partial i}{\partial g} \end{bmatrix}
\tag{4-85g}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial g\,\partial h} \\[2ex] \dfrac{\partial^2 a}{\partial g\,\partial h} \\[2ex] \dfrac{\partial^2 \Omega}{\partial g\,\partial h} \\[2ex] \dfrac{\partial^2 \omega}{\partial g\,\partial h} \\[2ex] \dfrac{\partial^2 e}{\partial g\,\partial h} \\[2ex] \dfrac{\partial^2 i}{\partial g\,\partial h} \end{bmatrix}
=
\begin{bmatrix}
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \Omega}{\partial h}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \Omega}{\partial h}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \Omega}{\partial h}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \Omega}{\partial h}\right) \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial h}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial h}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial h}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \omega}{\partial h}\right) \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial h}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial h}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial h}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial h}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial g} \\[2ex] \dfrac{\partial a}{\partial g} \\[2ex] \dfrac{\partial \Omega}{\partial g} \\[2ex] \dfrac{\partial \omega}{\partial g} \\[2ex] \dfrac{\partial e}{\partial g} \\[2ex] \dfrac{\partial i}{\partial g} \end{bmatrix}
\tag{4-85h}
$$

$$\begin{bmatrix} \frac{\partial^2 M}{\partial g\,\partial L} \\ \frac{\partial^2 a}{\partial g\,\partial L} \\ \frac{\partial^2 \Omega}{\partial g\,\partial L} \\ \frac{\partial^2 \omega}{\partial g\,\partial L} \\ \frac{\partial^2 e}{\partial g\,\partial L} \\ \frac{\partial^2 i}{\partial g\,\partial L} \end{bmatrix} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial a}{\partial L}\right), & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial e}{\partial L}\right), & 0, & 0, & \frac{\partial}{\partial e}\left(\frac{\partial e}{\partial L}\right), & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial L}\right), & 0, & 0, & \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial L}\right), & \frac{\partial}{\partial i}\left(\frac{\partial i}{\partial L}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial M}{\partial g} \\ \frac{\partial a}{\partial g} \\ \frac{\partial \Omega}{\partial g} \\ \frac{\partial \omega}{\partial g} \\ \frac{\partial e}{\partial g} \\ \frac{\partial i}{\partial g} \end{bmatrix}$$

(4-85i)

$$\begin{bmatrix} \frac{\partial^2 M}{\partial g\,\partial G} \\ \frac{\partial^2 a}{\partial g\,\partial G} \\ \frac{\partial^2 \Omega}{\partial g\,\partial G} \\ \frac{\partial^2 \omega}{\partial g\,\partial G} \\ \frac{\partial^2 e}{\partial g\,\partial G} \\ \frac{\partial^2 i}{\partial g\,\partial G} \end{bmatrix} = \begin{bmatrix} 0, & \frac{\partial}{\partial a}\left(\frac{\partial M}{\partial G}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial M}{\partial G}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial M}{\partial G}\right), & \frac{\partial}{\partial e}\left(\frac{\partial M}{\partial G}\right), & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial G}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial G}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial \omega}{\partial G}\right), & \frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial G}\right), & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial e}{\partial G}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial e}{\partial G}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial e}{\partial G}\right), & \frac{\partial}{\partial e}\left(\frac{\partial e}{\partial G}\right), & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial G}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial i}{\partial G}\right), & \frac{\partial}{\partial \omega}\left(\frac{\partial i}{\partial G}\right), & \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial G}\right), & \frac{\partial}{\partial i}\left(\frac{\partial i}{\partial G}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial M}{\partial g} \\ \frac{\partial a}{\partial g} \\ \frac{\partial \Omega}{\partial g} \\ \frac{\partial \omega}{\partial g} \\ \frac{\partial e}{\partial g} \\ \frac{\partial i}{\partial g} \end{bmatrix}$$

(4-85j)

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial g\,\partial H} \\[2ex] \dfrac{\partial^2 a}{\partial g\,\partial H} \\[2ex] \dfrac{\partial^2 \Omega}{\partial g\,\partial H} \\[2ex] \dfrac{\partial^2 \omega}{\partial g\,\partial H} \\[2ex] \dfrac{\partial^2 e}{\partial g\,\partial H} \\[2ex] \dfrac{\partial^2 i}{\partial g\,\partial H} \end{bmatrix}
=
\begin{bmatrix}
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \Omega}{\partial H}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \Omega}{\partial H}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \Omega}{\partial H}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \Omega}{\partial H}\right) \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial H}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial H}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial H}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \omega}{\partial H}\right) \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial H}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial H}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial H}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial H}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial g} \\[2ex] \dfrac{\partial a}{\partial g} \\[2ex] \dfrac{\partial \Omega}{\partial g} \\[2ex] \dfrac{\partial \omega}{\partial g} \\[2ex] \dfrac{\partial e}{\partial g} \\[2ex] \dfrac{\partial i}{\partial g} \end{bmatrix}
\tag{4-85k}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial h^2} \\[2ex] \dfrac{\partial^2 a}{\partial h^2} \\[2ex] \dfrac{\partial^2 \Omega}{\partial h^2} \\[2ex] \dfrac{\partial^2 \omega}{\partial h^2} \\[2ex] \dfrac{\partial^2 e}{\partial h^2} \\[2ex] \dfrac{\partial^2 i}{\partial h^2} \end{bmatrix}
=
\begin{bmatrix}
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \Omega}{\partial h}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \Omega}{\partial h}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \Omega}{\partial h}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \Omega}{\partial h}\right) \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial h}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial h}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial h}\right), & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial h}\right) \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial h}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial h}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial h}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial h}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial h} \\[2ex] \dfrac{\partial a}{\partial h} \\[2ex] \dfrac{\partial \Omega}{\partial h} \\[2ex] \dfrac{\partial \omega}{\partial h} \\[2ex] \dfrac{\partial e}{\partial h} \\[2ex] \dfrac{\partial i}{\partial h} \end{bmatrix}
\tag{4-85l}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial h\,\partial L} \\[2ex] \dfrac{\partial^2 a}{\partial h\,\partial L} \\[2ex] \dfrac{\partial^2 \Omega}{\partial h\,\partial L} \\[2ex] \dfrac{\partial^2 \omega}{\partial h\,\partial L} \\[2ex] \dfrac{\partial^2 e}{\partial h\,\partial L} \\[2ex] \dfrac{\partial^2 i}{\partial h\,\partial L} \end{bmatrix}
=
\begin{bmatrix}
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial a}{\partial L}\right) , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial e}{\partial L}\right) , & 0 , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial e}{\partial L}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial L}\right) , & 0 , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial L}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial L}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial h} \\[2ex] \dfrac{\partial a}{\partial h} \\[2ex] \dfrac{\partial \Omega}{\partial h} \\[2ex] \dfrac{\partial \omega}{\partial h} \\[2ex] \dfrac{\partial e}{\partial h} \\[2ex] \dfrac{\partial i}{\partial h} \end{bmatrix}
\tag{4-85m}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial h\,\partial G} \\[2ex] \dfrac{\partial^2 a}{\partial h\,\partial G} \\[2ex] \dfrac{\partial^2 \Omega}{\partial h\,\partial G} \\[2ex] \dfrac{\partial^2 \omega}{\partial h\,\partial G} \\[2ex] \dfrac{\partial^2 e}{\partial h\,\partial G} \\[2ex] \dfrac{\partial^2 i}{\partial h\,\partial G} \end{bmatrix}
=
\begin{bmatrix}
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial M}{\partial G}\right) , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial G}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial e}{\partial G}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial G}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial h} \\[2ex] \dfrac{\partial a}{\partial h} \\[2ex] \dfrac{\partial \Omega}{\partial h} \\[2ex] \dfrac{\partial \omega}{\partial h} \\[2ex] \dfrac{\partial e}{\partial h} \\[2ex] \dfrac{\partial i}{\partial h} \end{bmatrix}
\tag{4-85n}
$$

$$\begin{bmatrix} \dfrac{\partial^2 M}{\partial h\,\partial H} \\ \dfrac{\partial^2 a}{\partial h\,\partial H} \\ \dfrac{\partial^2 \Omega}{\partial h\,\partial H} \\ \dfrac{\partial^2 \omega}{\partial h\,\partial H} \\ \dfrac{\partial^2 e}{\partial h\,\partial H} \\ \dfrac{\partial^2 i}{\partial h\,\partial H} \end{bmatrix} = \begin{bmatrix} 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \\ 0 & , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \Omega}{\partial H}\right) & , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \Omega}{\partial H}\right) & , & 0 & , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \Omega}{\partial H}\right) & , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \Omega}{\partial H}\right) \\ 0 & , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial H}\right) & , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial H}\right) & , & 0 & , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial H}\right) & , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \omega}{\partial H}\right) \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \\ 0 & , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial H}\right) & , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial H}\right) & , & 0 & , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial H}\right) & , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial H}\right) \end{bmatrix} \begin{bmatrix} \dfrac{\partial M}{\partial h} \\ \dfrac{\partial a}{\partial h} \\ \dfrac{\partial \Omega}{\partial h} \\ \dfrac{\partial \omega}{\partial h} \\ \dfrac{\partial e}{\partial h} \\ \dfrac{\partial i}{\partial h} \end{bmatrix} \tag{4-85o}$$

$$\begin{bmatrix} \dfrac{\partial^2 M}{\partial L^2} \\ \dfrac{\partial^2 a}{\partial L^2} \\ \dfrac{\partial^2 \Omega}{\partial L^2} \\ \dfrac{\partial^2 \omega}{\partial L^2} \\ \dfrac{\partial^2 e}{\partial L^2} \\ \dfrac{\partial^2 i}{\partial L^2} \end{bmatrix} = \begin{bmatrix} 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \\ 0 & , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial a}{\partial L}\right) & , & 0 & , & 0 & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \\ 0 & , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial e}{\partial L}\right) & , & 0 & , & 0 & , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial e}{\partial L}\right) & , & 0 \\ 0 & , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial L}\right) & , & 0 & , & 0 & , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial L}\right) & , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial L}\right) \end{bmatrix} \begin{bmatrix} \dfrac{\partial M}{\partial L} \\ \dfrac{\partial a}{\partial L} \\ \dfrac{\partial \Omega}{\partial L} \\ \dfrac{\partial \omega}{\partial L} \\ \dfrac{\partial e}{\partial L} \\ \dfrac{\partial i}{\partial L} \end{bmatrix} \tag{4-85p}$$

$$
\begin{bmatrix}
\dfrac{\partial^2 M}{\partial L\,\partial G} \\[2ex]
\dfrac{\partial^2 a}{\partial L\,\partial G} \\[2ex]
\dfrac{\partial^2 \Omega}{\partial L\,\partial G} \\[2ex]
\dfrac{\partial^2 \omega}{\partial L\,\partial G} \\[2ex]
\dfrac{\partial^2 e}{\partial L\,\partial G} \\[2ex]
\dfrac{\partial^2 i}{\partial L\,\partial G}
\end{bmatrix}
=
\begin{bmatrix}
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial M}{\partial G}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial M}{\partial G}\right), & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial M}{\partial G}\right), & \dfrac{\partial}{\partial e}\left(\dfrac{\partial M}{\partial G}\right), & 0 \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial G}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial G}\right), & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial \omega}{\partial G}\right), & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial G}\right), & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial e}{\partial G}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial e}{\partial G}\right), & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial e}{\partial G}\right), & \dfrac{\partial}{\partial e}\left(\dfrac{\partial e}{\partial G}\right), & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial G}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial G}\right), & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial i}{\partial G}\right), & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial G}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial G}\right)
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial M}{\partial L} \\[2ex]
\dfrac{\partial a}{\partial L} \\[2ex]
\dfrac{\partial \Omega}{\partial L} \\[2ex]
\dfrac{\partial \omega}{\partial L} \\[2ex]
\dfrac{\partial e}{\partial L} \\[2ex]
\dfrac{\partial i}{\partial L}
\end{bmatrix}
\tag{4-85q}
$$

$$
\begin{bmatrix}
\dfrac{\partial^2 M}{\partial L\,\partial H} \\[2ex]
\dfrac{\partial^2 a}{\partial L\,\partial H} \\[2ex]
\dfrac{\partial^2 \Omega}{\partial L\,\partial H} \\[2ex]
\dfrac{\partial^2 \omega}{\partial L\,\partial H} \\[2ex]
\dfrac{\partial^2 e}{\partial L\,\partial H} \\[2ex]
\dfrac{\partial^2 i}{\partial L\,\partial H}
\end{bmatrix}
=
\begin{bmatrix}
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \Omega}{\partial H}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \Omega}{\partial H}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \Omega}{\partial H}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \Omega}{\partial H}\right) \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial H}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial H}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial H}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \omega}{\partial H}\right) \\[2ex]
0, & 0, & 0, & 0, & 0, & 0 \\[2ex]
0, & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial H}\right), & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial H}\right), & 0, & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial H}\right), & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial H}\right)
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial M}{\partial L} \\[2ex]
\dfrac{\partial a}{\partial L} \\[2ex]
\dfrac{\partial \Omega}{\partial L} \\[2ex]
\dfrac{\partial \omega}{\partial L} \\[2ex]
\dfrac{\partial e}{\partial L} \\[2ex]
\dfrac{\partial i}{\partial L}
\end{bmatrix}
\tag{4-85r}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial G^2} \\[2ex] \dfrac{\partial^2 a}{\partial G^2} \\[2ex] \dfrac{\partial^2 \Omega}{\partial G^2} \\[2ex] \dfrac{\partial^2 \omega}{\partial G^2} \\[2ex] \dfrac{\partial^2 e}{\partial G^2} \\[2ex] \dfrac{\partial^2 i}{\partial G^2} \end{bmatrix}
=
\begin{bmatrix}
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial M}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial M}{\partial G}\right) , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial \omega}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial G}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial e}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial e}{\partial G}\right) , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial \omega}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial G}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial G}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial G} \\[2ex] \dfrac{\partial a}{\partial G} \\[2ex] \dfrac{\partial \Omega}{\partial G} \\[2ex] \dfrac{\partial \omega}{\partial G} \\[2ex] \dfrac{\partial e}{\partial G} \\[2ex] \dfrac{\partial i}{\partial G} \end{bmatrix}
\tag{4-85s}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 M}{\partial G\,\partial H} \\[2ex] \dfrac{\partial^2 a}{\partial G\,\partial H} \\[2ex] \dfrac{\partial^2 \Omega}{\partial G\,\partial H} \\[2ex] \dfrac{\partial^2 \omega}{\partial G\,\partial H} \\[2ex] \dfrac{\partial^2 e}{\partial G\,\partial H} \\[2ex] \dfrac{\partial^2 i}{\partial G\,\partial H} \end{bmatrix}
=
\begin{bmatrix}
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \Omega}{\partial H}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \Omega}{\partial H}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \Omega}{\partial H}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \Omega}{\partial H}\right) \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial \omega}{\partial H}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial \omega}{\partial H}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial \omega}{\partial H}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial \omega}{\partial H}\right) \\[2ex]
0 , & 0 , & 0 , & 0 , & 0 , & 0 \\[2ex]
0 , & \dfrac{\partial}{\partial a}\left(\dfrac{\partial i}{\partial H}\right) , & \dfrac{\partial}{\partial \Omega}\left(\dfrac{\partial i}{\partial H}\right) , & 0 , & \dfrac{\partial}{\partial e}\left(\dfrac{\partial i}{\partial H}\right) , & \dfrac{\partial}{\partial i}\left(\dfrac{\partial i}{\partial H}\right)
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial M}{\partial G} \\[2ex] \dfrac{\partial a}{\partial G} \\[2ex] \dfrac{\partial \Omega}{\partial G} \\[2ex] \dfrac{\partial \omega}{\partial G} \\[2ex] \dfrac{\partial e}{\partial G} \\[2ex] \dfrac{\partial i}{\partial G} \end{bmatrix}
\tag{4-85t}
$$

$$
\begin{bmatrix} \frac{\partial^2 M}{\partial H^2} \\ \frac{\partial^2 a}{\partial H^2} \\ \frac{\partial^2 \Omega}{\partial H^2} \\ \frac{\partial^2 \omega}{\partial H^2} \\ \frac{\partial^2 e}{\partial H^2} \\ \frac{\partial^2 i}{\partial H^2} \end{bmatrix} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial \Omega}{\partial H}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial \Omega}{\partial H}\right), & 0, & \frac{\partial}{\partial e}\left(\frac{\partial \Omega}{\partial H}\right), & \frac{\partial}{\partial i}\left(\frac{\partial \Omega}{\partial H}\right) \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial H}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial H}\right), & 0, & \frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial H}\right), & \frac{\partial}{\partial i}\left(\frac{\partial \omega}{\partial H}\right) \\ 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial H}\right), & \frac{\partial}{\partial \Omega}\left(\frac{\partial i}{\partial H}\right), & 0, & \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial H}\right), & \frac{\partial}{\partial i}\left(\frac{\partial i}{\partial H}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial M}{\partial H} \\ \frac{\partial a}{\partial H} \\ \frac{\partial \Omega}{\partial H} \\ \frac{\partial \omega}{\partial H} \\ \frac{\partial e}{\partial H} \\ \frac{\partial i}{\partial H} \end{bmatrix} \tag{4-85u}
$$

The vectors of first-order derivatives of Kepler elements with respect to the Poincaré elements on the right hand sides of (4-85a,...,u) are derived using the Jacobi matrix (2-68).

$$
\frac{\partial}{\partial a}\left(\frac{\partial M}{\partial g}\right) = -\frac{\kappa^2 \cos(\omega + \Omega)}{8\, L^{5/2} \sin\frac{\varphi}{2}} \qquad \kappa^2 = kM \tag{4-86a}
$$

$$
\frac{\partial}{\partial \Omega}\left(\frac{\partial M}{\partial g}\right) = -\frac{\sin(\omega + \Omega)}{2\, L^{1/2} \sin\frac{\varphi}{2}} \tag{4-86b}
$$

$$
\frac{\partial}{\partial \omega}\left(\frac{\partial M}{\partial g}\right) = -\frac{\sin(\omega + \Omega)}{2\, L^{1/2} \sin\frac{\varphi}{2}} \tag{4-86c}
$$

$$
\frac{\partial}{\partial e}\left(\frac{\partial M}{\partial g}\right) = -\frac{\cot\frac{\varphi}{2} \cos(\omega + \Omega)}{4\, L^{1/2} \sin\frac{\varphi}{2} \cos\varphi} \tag{4-86d}
$$

$$
\frac{\partial}{\partial a}\left(\frac{\partial M}{\partial G}\right) = -\frac{\kappa^2 \sin(\omega + \Omega)}{8\, L^{5/2} \sin\frac{\varphi}{2}} \qquad \kappa^2 = kM \tag{4-86e}
$$

$$
\frac{\partial}{\partial \Omega}\left(\frac{\partial M}{\partial G}\right) = -\frac{\cos(\omega + \Omega)}{2\, L^{1/2} \sin\frac{\varphi}{2}} \tag{4-86f}
$$

$$\frac{\partial}{\partial \omega}\left(\frac{\partial M}{\partial G}\right) = \frac{\cos(\omega+\Omega)}{2\,L^{1/2}\,\sin\frac{\varphi}{2}} \qquad (4\text{-}86g)$$

$$\frac{\partial}{\partial e}\left(\frac{\partial M}{\partial G}\right) = -\frac{\cot\frac{\varphi}{2}\,\sin(\omega+\Omega)}{4\,L^{1/2}\,\sin\frac{\varphi}{2}\,\cos\varphi} \qquad (4\text{-}86h)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial a}{\partial L}\right) = \frac{1}{L} \qquad (4\text{-}87a)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \Omega}{\partial h}\right) = \frac{\kappa^2\,\cos\Omega}{8\,L^{5/2}\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad \kappa^2 = kM \qquad (4\text{-}88a)$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial \Omega}{\partial h}\right) = \frac{\sin\Omega}{8\,L^{1/2}\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}88b)$$

$$\frac{\partial}{\partial e}\left(\frac{\partial \Omega}{\partial h}\right) = -\frac{\tan\varphi\,\cos\Omega}{4\,L^{1/2}\cos^{3/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}88c)$$

$$\frac{\partial}{\partial i}\left(\frac{\partial \Omega}{\partial h}\right) = \frac{\cos\Omega\,\cot\frac{i}{2}}{4\,L^{1/2}\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}88d)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \Omega}{\partial H}\right) = \frac{\kappa^2\,\sin\Omega}{8\,L^{5/2}\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad \kappa^2 = kM \qquad (4\text{-}88e)$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial \Omega}{\partial H}\right) = -\frac{\cos\Omega}{2\,L^{1/2}\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}88f)$$

$$\frac{\partial}{\partial e}\left(\frac{\partial \Omega}{\partial H}\right) = -\frac{\tan\varphi\,\sin\Omega}{4\,L^{1/2}\cos^{3/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}88g)$$

$$\frac{\partial}{\partial i}\left(\frac{\partial \Omega}{\partial H}\right) = \frac{\sin\Omega\,\cot\frac{i}{2}}{4\,L^{1/2}\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}88h)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial g}\right) = \frac{\kappa^2\,\cos(\omega+\Omega)}{8\,L^{5/2}\,\sin\frac{\varphi}{2}} \qquad \kappa^2 = kM \qquad (4\text{-}89a)$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial g}\right) = \frac{\sin(\omega+\Omega)}{2\,L^{1/2}\,\sin\frac{\varphi}{2}} \qquad (4\text{-}89b)$$

$$\frac{\partial}{\partial \omega}\left(\frac{\partial \omega}{\partial g}\right) = \frac{\sin(\omega+\Omega)}{2\,L^{1/2}\,\sin\frac{\varphi}{2}} \qquad (4\text{-}89c)$$

$$\frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial g}\right) = \frac{\cot\frac{\varphi}{2}\,\cos(\omega+\Omega)}{4\,L^{1/2}\,\sin\frac{\varphi}{2}\,\cos\varphi} \qquad (4\text{-}89d)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial h}\right) = -\frac{\kappa^2\,\cos\Omega}{8\,L^{5/2}\,\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad \kappa^2 = kM \qquad (4\text{-}89e)$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial h}\right) = -\frac{\sin\Omega}{8\,L^{1/2}\,\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}89f)$$

$$\frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial h}\right) = \frac{\tan\varphi\,\cos\Omega}{4\,L^{1/2}\,\cos^{3/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}89g)$$

$$\frac{\partial}{\partial i}\left(\frac{\partial \omega}{\partial h}\right) = -\frac{\cos\Omega\,\cot\frac{i}{2}}{4\,L^{1/2}\,\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}89h)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial G}\right) = \frac{\kappa^2\,\sin(\omega+\Omega)}{8\,L^{5/2}\,\sin\frac{\varphi}{2}} \qquad \kappa^2 = kM \qquad (4\text{-}89i)$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial G}\right) = -\frac{\cos(\omega+\Omega)}{2\,L^{1/2}\,\sin\frac{\varphi}{2}} \qquad (4\text{-}89j)$$

$$\frac{\partial}{\partial \omega}\left(\frac{\partial \omega}{\partial G}\right) = -\frac{\cos(\omega+\Omega)}{2\,L^{1/2}\,\sin\frac{\varphi}{2}} \qquad (4\text{-}89k)$$

$$\frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial G}\right) = \frac{\cot\frac{\varphi}{2}\,\sin(\omega+\Omega)}{4\,L^{1/2}\,\sin\frac{\varphi}{2}\,\cos\varphi} \qquad (4\text{-}89l)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \omega}{\partial h}\right) = -\frac{\kappa^2\,\cos\Omega}{8\,L^{5/2}\,\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad \kappa^2 = kM \qquad (4\text{-}89e)$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial \omega}{\partial H}\right) = \frac{\cos\Omega}{2\,L^{1/2}\,\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}89n)$$

$$\frac{\partial}{\partial e}\left(\frac{\partial \omega}{\partial H}\right) = \frac{\tan\varphi\,\sin\Omega}{4\,L^{1/2}\,\cos^{3/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}89o)$$

$$\frac{\partial}{\partial i}\left(\frac{\partial \omega}{\partial H}\right) = -\frac{\sin\Omega\,\cot\frac{i}{2}}{4\,L^{1/2}\,\cos^{1/2}\varphi\,\sin\frac{i}{2}} \qquad (4\text{-}89p)$$

$$\frac{\partial}{\partial a}\left(\frac{\partial e}{\partial h}\right) = \frac{\kappa^2 \cos\varphi \sin(\omega+\Omega)}{4 L^{5/2} \cos\frac{\varphi}{2}} \qquad \kappa^2 = kM \tag{4-90a}$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial e}{\partial g}\right) = -\frac{\cos\varphi \cos(\omega+\Omega)}{L^{1/2} \cos\frac{\varphi}{2}} \tag{4-90b}$$

$$\frac{\partial}{\partial \omega}\left(\frac{\partial e}{\partial g}\right) = -\frac{\cos\varphi \cos(\omega+\Omega)}{L^{1/2} \cos\frac{\varphi}{2}} \tag{4-90c}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial e}{\partial g}\right) = \frac{\tan\frac{\varphi}{2}\,(2+\cos\varphi) \sin(\omega+\Omega)}{2 L^{1/2} \cos\frac{\varphi}{2} \cos\varphi} \tag{4-90d}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial e}{\partial L}\right) = \frac{\kappa^2 \tan\frac{\varphi}{2} \cos\varphi}{2 L^3} \qquad \kappa^2 = kM \tag{4-90e}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial e}{\partial L}\right) = -\frac{1 - \sin\varphi \tan\varphi}{L\,(1+\cos\varphi)} \tag{4-90f}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial e}{\partial G}\right) = -\frac{\kappa^2 \cos\varphi \cos(\omega+\Omega)}{4 L^{5/2} \cos\frac{\varphi}{2}} \qquad \kappa^2 = kM \tag{4-90g}$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial e}{\partial G}\right) = -\frac{\cos\varphi \cos(\omega+\Omega)}{L^{1/2} \cos\frac{\varphi}{2}} \tag{4-90h}$$

$$\frac{\partial}{\partial \omega}\left(\frac{\partial e}{\partial G}\right) = -\frac{\cos\varphi \cos(\omega+\Omega)}{L^{1/2} \cos\frac{\varphi}{2}} \tag{4-90i}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial e}{\partial G}\right) = -\frac{\tan\frac{\varphi}{2}\,(2+\cos\varphi) \cos(\omega+\Omega)}{2 L^{1/2} \cos\frac{\varphi}{2} \cos\varphi} \tag{4-90j}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial i}{\partial g}\right) = \frac{\kappa^2 \sin\frac{\varphi}{2} \tan\frac{i}{2} \sin(\omega+\Omega)}{2 L^{5/2} \cos\varphi} \qquad \kappa^2 = kM \tag{4-91a}$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial i}{\partial g}\right) = -\frac{2 \sin\frac{\varphi}{2} \tan\frac{i}{2} \cos(\omega+\Omega)}{L^{1/2} \cos\varphi} \tag{4-91b}$$

$$\frac{\partial}{\partial \omega}\left(\frac{\partial i}{\partial g}\right) = -\frac{2 \sin\frac{\varphi}{2} \tan\frac{i}{2} \cos(\omega+\Omega)}{L^{1/2} \cos\varphi} \tag{4-91c}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial i}{\partial g}\right) = -\frac{\cos\frac{\varphi}{2}\,(2-\cos\varphi) \tan\frac{i}{2} \sin(\omega+\Omega)}{L^{1/2} \cos^3\varphi} \tag{4-91d}$$

$$\frac{\partial}{\partial i}\left(\frac{\partial i}{\partial g}\right) = -\frac{\sin\frac{\varphi}{2}\sin(\omega+\Omega)}{L^{1/2}\cos\varphi\cos^2\frac{i}{2}} \tag{4-91e}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial i}{\partial h}\right) = \frac{\kappa^2\sin\Omega}{4L^{5/2}\cos^{1/2}\varphi\cos\frac{i}{2}} \qquad \kappa^2 = kM \tag{4-91f}$$

$$\frac{\partial}{\partial\Omega}\left(\frac{\partial i}{\partial h}\right) = -\frac{\sin\Omega}{L^{1/2}\cos^{1/2}\varphi\cos\frac{i}{2}} \tag{4-91g}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial i}{\partial h}\right) = -\frac{\tan\varphi\sin\Omega}{2L^{1/2}\cos^{3/2}\varphi\cos\frac{i}{2}} \tag{4-91h}$$

$$\frac{\partial}{\partial i}\left(\frac{\partial i}{\partial h}\right) = -\frac{\tan\frac{i}{2}\sin\Omega}{2L^{1/2}\cos^{1/2}\varphi\cos\frac{i}{2}} \tag{4-91i}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial i}{\partial L}\right) = \frac{\kappa^2\tan\frac{i}{2}}{2L^3\cos\varphi} \qquad \kappa^2 = kM \tag{4-91j}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial i}{\partial L}\right) = -\frac{\tan\varphi\tan\frac{i}{2}}{L\cos^2\varphi} \tag{4-91k}$$

$$\frac{\partial}{\partial i}\left(\frac{\partial i}{\partial L}\right) = -\frac{1}{2L\cos\varphi\cos^2\frac{i}{2}} \tag{4-91l}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial i}{\partial G}\right) = -\frac{\kappa^2\sin\frac{\varphi}{2}\tan\frac{i}{2}\cos(\omega+\Omega)}{2L^{5/2}\cos\varphi} \qquad \kappa^2 = kM \tag{4-91m}$$

$$\frac{\partial}{\partial\Omega}\left(\frac{\partial i}{\partial G}\right) = -\frac{2\sin\frac{\varphi}{2}\tan\frac{i}{2}\sin(\omega+\Omega)}{L^{1/2}\cos\varphi} \tag{4-91n}$$

$$\frac{\partial}{\partial\omega}\left(\frac{\partial i}{\partial G}\right) = -\frac{2\sin\frac{\varphi}{2}\tan\frac{i}{2}\sin(\omega+\Omega)}{L^{1/2}\cos\varphi} \tag{4-91o}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial i}{\partial G}\right) = \frac{\cos\frac{\varphi}{2}(2-\cos\varphi)\tan\frac{i}{2}\cos(\omega+\Omega)}{L^{1/2}\cos^3\varphi} \tag{4-91p}$$

$$\frac{\partial}{\partial i}\left(\frac{\partial i}{\partial G}\right) = \frac{\sin\frac{\varphi}{2}\cos(\omega+\Omega)}{L^{1/2}\cos\varphi\cos^2\frac{i}{2}} \tag{4-91q}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial i}{\partial H}\right) = -\frac{\kappa^2\cos\Omega}{4L^{5/2}\cos^{1/2}\varphi\cos\frac{i}{2}} \qquad \kappa^2 = kM \tag{4-91r}$$

$$\frac{\partial}{\partial \Omega}\left(\frac{\partial i}{\partial H}\right) = -\frac{\sin\Omega}{L^{1/2}\cos^{1/2}\varphi\cos\frac{i}{2}} \tag{4-91s}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial i}{\partial H}\right) = \frac{\tan\varphi\cos\Omega}{2L^{1/2}\cos^{3/2}\varphi\cos\frac{i}{2}} \tag{4-91t}$$

$$\frac{\partial}{\partial i}\left(\frac{\partial i}{\partial H}\right) = \frac{\tan\frac{i}{2}\cos\Omega}{2L^{1/2}\cos^{1/2}\varphi\cos\frac{i}{2}} \tag{4-91u}$$

Matrix $\partial \underline{Y}/\partial l$

$\partial \underline{Y}/\partial l$ is derived by partial differentiations of $\underline{Y}$ (2-99b).

$$\frac{\partial \underline{Y}}{\partial l} = \begin{bmatrix} -\frac{\partial^2 x_1}{\partial l\,\partial L} + 3\tau\frac{\partial^2 x_1}{\partial l^2} & , & -\frac{\partial^2 x_2}{\partial l\,\partial L} + 3\tau\frac{\partial^2 x_2}{\partial l^2} & , & -\frac{\partial^2 x_3}{\partial l\,\partial L} + 3\tau\frac{\partial^2 x_3}{\partial l^2} \\ -\frac{\partial^2 x_1}{\partial l\,\partial G} & , & -\frac{\partial^2 x_2}{\partial l\,\partial G} & , & -\frac{\partial^2 x_3}{\partial l\,\partial G} \\ -\frac{\partial^2 x_1}{\partial l\,\partial H} & , & -\frac{\partial^2 x_2}{\partial l\,\partial H} & , & -\frac{\partial^2 x_3}{\partial l\,\partial H} \\ \frac{\partial^2 x_1}{\partial l^2} & , & \frac{\partial^2 x_2}{\partial l^2} & , & \frac{\partial^2 x_3}{\partial l^2} \\ \frac{\partial^2 x_1}{\partial l\,\partial g} & , & \frac{\partial^2 x_2}{\partial l\,\partial g} & , & \frac{\partial^2 x_3}{\partial l\,\partial g} \\ \frac{\partial^2 x_1}{\partial l\,\partial h} & , & \frac{\partial^2 x_2}{\partial l\,\partial h} & , & \frac{\partial^2 x_3}{\partial l\,\partial h} \end{bmatrix} \tag{4-92}$$

where the 18 elements are defined by (4-71a,...,d).

$$\frac{\partial^2 x_1}{\partial l\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial L} \tag{4-93a}$$

$$\frac{\partial^2 x_2}{\partial l\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial L} \tag{4-93b}$$

$$\frac{\partial^2 x_3}{\partial l\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial L} \tag{4-93c}$$

$$\frac{\partial^2 x_1}{\partial l\,\partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial G} \tag{4-93d}$$

$$\frac{\partial^2 x_2}{\partial l\,\partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial G} \tag{4-93e}$$

$$\frac{\partial^2 x_3}{\partial l\,\partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial G} \tag{4-93f}$$

$$\frac{\partial^2 x_1}{\partial l\,\partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial H} \tag{4-93g}$$

$$\frac{\partial^2 x_2}{\partial l\,\partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial H} \tag{4-93h}$$

$$\frac{\partial^2 x_3}{\partial l\,\partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial H} \tag{4-93i}$$

$$\frac{\partial^2 x_1}{\partial l^2} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial l^2} \tag{4-93j}$$

$$\frac{\partial^2 x_2}{\partial l^2} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial l^2} \tag{4-93k}$$

$$\frac{\partial^2 x_3}{\partial l^2} = \frac{\partial \underline{\underline{u}}^T}{\partial l}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial l^2} \tag{4-93l}$$

$$\frac{\partial^2 x_1}{\partial l\,\partial g} = \frac{\partial \underline{u}^T}{\partial l}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial g} \tag{4-93m}$$

$$\frac{\partial^2 x_2}{\partial l\,\partial g} = \frac{\partial \underline{u}^T}{\partial l}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial g} \tag{4-93n}$$

$$\frac{\partial^2 x_3}{\partial l\,\partial g} = \frac{\partial \underline{u}^T}{\partial l}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial g} \tag{4-93o}$$

$$\frac{\partial^2 x_1}{\partial l\,\partial h} = \frac{\partial \underline{u}^T}{\partial l}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial h} \tag{4-93p}$$

$$\frac{\partial^2 x_2}{\partial l\,\partial h} = \frac{\partial \underline{u}^T}{\partial l}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial h} \tag{4-93q}$$

$$\frac{\partial^2 x_3}{\partial l\,\partial h} = \frac{\partial \underline{u}^T}{\partial l}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial l\,\partial h} \tag{4-93r}$$

Matrix $\partial \underline{Y}/\partial g$

$\partial \underline{Y}/\partial g$ is derived by partial differentiations of $\underline{Y}$ (2-97b).

$$\frac{\partial \underline{Y}}{\partial g} = \begin{bmatrix} -\frac{\partial^2 x_1}{\partial g\,\partial L} + 3\tau\frac{\partial^2 x_1}{\partial g\,\partial l} , & -\frac{\partial^2 x_2}{\partial g\,\partial L} + 3\tau\frac{\partial^2 x_2}{\partial g\,\partial l} , & -\frac{\partial^2 x_3}{\partial g\,\partial L} + 3\tau\frac{\partial^2 x_3}{\partial g\,\partial l} \\ -\frac{\partial^2 x_1}{\partial g\,\partial G} , & -\frac{\partial^2 x_2}{\partial g\,\partial G} , & -\frac{\partial^2 x_3}{\partial g\,\partial G} \\ -\frac{\partial^2 x_1}{\partial g\,\partial H} , & -\frac{\partial^2 x_2}{\partial g\,\partial H} , & -\frac{\partial^2 x_3}{\partial g\,\partial H} \\ \frac{\partial^2 x_1}{\partial g\,\partial l} , & \frac{\partial^2 x_2}{\partial g\,\partial l} , & \frac{\partial^2 x_3}{\partial g\,\partial l} \\ \frac{\partial^2 x_1}{\partial g^2} , & \frac{\partial^2 x_2}{\partial g^2} , & \frac{\partial^2 x_3}{\partial g^2} \\ \frac{\partial^2 x_1}{\partial g\,\partial h} , & \frac{\partial^2 x_2}{\partial g\,\partial h} , & \frac{\partial^2 x_3}{\partial g\,\partial h} \end{bmatrix} \tag{4-94}$$

where the 18 elements are defined by

$$\frac{\partial^2 x_1}{\partial g\, \partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial g\, \partial L} \qquad (4\text{-}95a)$$

$$\frac{\partial^2 x_2}{\partial g\, \partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial g\, \partial L} \qquad (4\text{-}95b)$$

$$\frac{\partial^2 x_3}{\partial g\, \partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial g\, \partial L} \qquad (4\text{-}95c)$$

$$\frac{\partial^2 x_1}{\partial g\, \partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial g\, \partial G} \qquad (4\text{-}95d)$$

$$\frac{\partial^2 x_2}{\partial g\, \partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial g\, \partial G} \qquad (4\text{-}95e)$$

$$\frac{\partial^2 x_3}{\partial g\, \partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial g\, \partial G} \qquad (4\text{-}95f)$$

$$\frac{\partial^2 x_1}{\partial g\, \partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial g\, \partial H} \qquad (4\text{-}95g)$$

$$\frac{\partial^2 x_2}{\partial g\, \partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial g\, \partial H} \qquad (4\text{-}95h)$$

$$\frac{\partial^2 x_3}{\partial g\, \partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial g\, \partial H} \qquad (4\text{-}95i)$$

$$\frac{\partial^2 x_1}{\partial g\, \partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial g\, \partial l} \qquad (4\text{-}95j)$$

$$\frac{\partial^2 x_2}{\partial g\, \partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial g\, \partial l} \qquad (4\text{-}95k)$$

$$\frac{\partial^2 x_3}{\partial g\, \partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial g} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial g\, \partial l} \qquad (4\text{-}95l)$$

$$\frac{\partial^2 x_1}{\partial g^2} = \frac{\partial \underline{u}^T}{\partial g} \underline{X}_1 \frac{\partial \underline{u}}{\partial g} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial g^2} \qquad (4\text{-}95m)$$

$$\frac{\partial^2 x_1}{\partial g^2} = \frac{\partial \underline{u}^T}{\partial g} \underline{X}_2 \frac{\partial \underline{u}}{\partial g} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial g^2} \qquad (4\text{-}95n)$$

$$\frac{\partial^2 x_1}{\partial g^2} = \frac{\partial \underline{u}^T}{\partial g} \underline{X}_3 \frac{\partial \underline{u}}{\partial g} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial g^2} \qquad (4\text{-}95o)$$

$$\frac{\partial^2 x_1}{\partial g\,\partial h} = \frac{\partial \underline{u}^T}{\partial g} \underline{X}_1 \frac{\partial \underline{u}}{\partial h} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial g\,\partial h} \qquad (4\text{-}95p)$$

$$\frac{\partial^2 x_2}{\partial g\,\partial h} = \frac{\partial \underline{u}^T}{\partial g} \underline{X}_2 \frac{\partial \underline{u}}{\partial h} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial g\,\partial h} \qquad (4\text{-}95q)$$

$$\frac{\partial^2 x_3}{\partial g\,\partial h} = \frac{\partial \underline{u}^T}{\partial g} \underline{X}_3 \frac{\partial \underline{u}}{\partial h} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial g\,\partial h} \qquad (4\text{-}95r)$$

Matrix $\partial \underline{Y}/\partial h$

$\partial \underline{Y}/\partial h$ is derived by partial differentiations of $\underline{Y}$ (2-97b).

$$\frac{\partial \underline{Y}}{\partial h} = \begin{bmatrix} -\frac{\partial^2 x_1}{\partial h\,\partial L} + 3\tau \frac{\partial^2 x_1}{\partial h\,\partial l} , & -\frac{\partial^2 x_2}{\partial h\,\partial L} + 3\tau \frac{\partial^2 x_2}{\partial h\,\partial l} , & -\frac{\partial^2 x_3}{\partial h\,\partial L} + 3\tau \frac{\partial^2 x_3}{\partial h\,\partial l} \\ -\frac{\partial^2 x_1}{\partial h\,\partial G} , & -\frac{\partial^2 x_2}{\partial h\,\partial G} , & -\frac{\partial^2 x_3}{\partial h\,\partial G} \\ -\frac{\partial^2 x_1}{\partial h\,\partial H} , & -\frac{\partial^2 x_2}{\partial h\,\partial H} , & -\frac{\partial^2 x_3}{\partial h\,\partial H} \\ \frac{\partial^2 x_1}{\partial h\,\partial l} , & \frac{\partial^2 x_2}{\partial h\,\partial l} , & \frac{\partial^2 x_3}{\partial h\,\partial l} \\ \frac{\partial^2 x_1}{\partial h\,\partial g} , & \frac{\partial^2 x_2}{\partial h\,\partial g} , & \frac{\partial^2 x_3}{\partial h\,\partial g} \\ \frac{\partial^2 x_1}{\partial h^2} , & \frac{\partial^2 x_2}{\partial h^2} , & \frac{\partial^2 x_3}{\partial h^2} \end{bmatrix} \qquad (4\text{-}96)$$

where the 18 elements are defined by (4-71a,...,d)

$$\frac{\partial^2 x_1}{\partial h\,\partial L} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial L} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial L} \qquad (4\text{-}97a)$$

$$\frac{\partial^2 x_2}{\partial h\,\partial L} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial L} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial L} \qquad (4\text{-}97b)$$

$$\frac{\partial^2 x_3}{\partial h\,\partial L} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial L} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial L} \qquad (4\text{-}97c)$$

$$\frac{\partial^2 x_1}{\partial h\,\partial G} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial G} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial G} \qquad (4\text{-}97d)$$

$$\frac{\partial^2 x_2}{\partial h\,\partial G} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial G} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial G} \qquad (4\text{-}97e)$$

$$\frac{\partial^2 x_3}{\partial h\,\partial G} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial G} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial G} \qquad (4\text{-}97f)$$

$$\frac{\partial^2 x_1}{\partial h\,\partial H} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial H} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial H} \qquad (4\text{-}97g)$$

$$\frac{\partial^2 x_2}{\partial h\,\partial H} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial H} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial H} \qquad (4\text{-}97h)$$

$$\frac{\partial^2 x_3}{\partial h\,\partial H} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial H} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial H} \qquad (4\text{-}97i)$$

$$\frac{\partial^2 x_1}{\partial h\,\partial l} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial l} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial l} \qquad (4\text{-}97j)$$

$$\frac{\partial^2 x_2}{\partial h\,\partial l} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial l} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial l} \qquad (4\text{-}97k)$$

$$\frac{\partial^2 x_3}{\partial h\,\partial l} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial l} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial l} \qquad (4\text{-}97l)$$

$$\frac{\partial^2 x_1}{\partial h\,\partial g} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial g} \tag{4-97m}$$

$$\frac{\partial^2 x_2}{\partial h\,\partial g} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial g} \tag{4-97n}$$

$$\frac{\partial^2 x_3}{\partial h\,\partial g} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial h\,\partial g} \tag{4-97o}$$

$$\frac{\partial^2 x_1}{\partial h^2} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial h^2} \tag{4-97p}$$

$$\frac{\partial^2 x_1}{\partial h^2} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial h^2} \tag{4-97q}$$

$$\frac{\partial^2 x_1}{\partial h^2} = \frac{\partial \underline{u}^T}{\partial h}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial h^2} \tag{4-97r}$$

Matrix $\partial \underline{Y}/\partial L$

Again, $\partial \underline{Y}/\partial L$ is derived by partial differentiations of $\underline{Y}$ (2-97b). Thereby one has to consider that $\tau = \kappa^2 t/L^4$ with $\kappa^2 = kM$ is also a function

$$\frac{\partial \underline{Y}}{\partial L} = \begin{bmatrix} -\frac{\partial^2 x_1}{\partial L^2} + 3\left(\frac{\partial \tau}{\partial L}\,\frac{\partial x_1}{\partial l} + \tau\frac{\partial^2 x_1}{\partial L\,\partial l}\right), & -\frac{\partial^2 x_2}{\partial L^2} + 3\left(\frac{\partial \tau}{\partial L}\,\frac{\partial x_2}{\partial l} + \tau\frac{\partial^2 x_2}{\partial L\,\partial l}\right), & -\frac{\partial^2 x_2}{\partial L^2} + 3\left(\frac{\partial \tau}{\partial L}\,\frac{\partial x_2}{\partial l} + \tau\frac{\partial^2 x_2}{\partial L\,\partial l}\right) \\ -\frac{\partial^2 x_1}{\partial L\,\partial G} & -\frac{\partial^2 x_2}{\partial L\,\partial G} & -\frac{\partial^2 x_3}{\partial L\,\partial G} \\ -\frac{\partial^2 x_1}{\partial L\,\partial H} & -\frac{\partial^2 x_2}{\partial L\,\partial H} & -\frac{\partial^2 x_3}{\partial L\,\partial H} \\ \frac{\partial^2 x_1}{\partial L\,\partial l} & \frac{\partial^2 x_2}{\partial L\,\partial l} & \frac{\partial^2 x_3}{\partial L\,\partial l} \\ \frac{\partial^2 x_1}{\partial L\,\partial g} & \frac{\partial^2 x_2}{\partial L\,\partial g} & \frac{\partial^2 x_3}{\partial L\,\partial g} \\ \frac{\partial^2 x_1}{\partial L\,\partial h} & \frac{\partial^2 x_2}{\partial L\,\partial h} & \frac{\partial^2 x_3}{\partial L\,\partial h} \end{bmatrix} \tag{4-98}$$

The second-order derivatives in (4-98) are defined by (4-71a,...,d). For the first-order derivatieves of the position vector with respect to l see (2-76) and (-68).

$$\frac{\partial \tau}{\partial L} = -4\frac{\kappa^4}{L^5}\, t \tag{4-99}$$

$$\frac{\partial^2 x_1}{\partial L^2} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_1\, \frac{\partial \underline{u}}{\partial L} + \underline{x}_1^T\, \frac{\partial^2 \underline{u}}{\partial L^2} \tag{4-100a}$$

$$\frac{\partial^2 x_2}{\partial L^2} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_2\, \frac{\partial \underline{u}}{\partial L} + \underline{x}_2^T\, \frac{\partial^2 \underline{u}}{\partial L^2} \tag{4-100b}$$

$$\frac{\partial^2 x_3}{\partial L^2} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_3\, \frac{\partial \underline{u}}{\partial L} + \underline{x}_3^T\, \frac{\partial^2 \underline{u}}{\partial L^2} \tag{4-100c}$$

$$\frac{\partial^2 x_1}{\partial L\, \partial G} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_1\, \frac{\partial \underline{u}}{\partial G} + \underline{x}_1^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial G} \tag{4-100d}$$

$$\frac{\partial^2 x_2}{\partial L\, \partial G} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_2\, \frac{\partial \underline{u}}{\partial G} + \underline{x}_2^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial G} \tag{4-100e}$$

$$\frac{\partial^2 x_3}{\partial L\, \partial G} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_3\, \frac{\partial \underline{u}}{\partial G} + \underline{x}_3^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial G} \tag{4-100f}$$

$$\frac{\partial^2 x_1}{\partial L\, \partial H} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_1\, \frac{\partial \underline{u}}{\partial H} + \underline{x}_1^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial H} \tag{4-100g}$$

$$\frac{\partial^2 x_2}{\partial L\, \partial H} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_2\, \frac{\partial \underline{u}}{\partial H} + \underline{x}_2^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial H} \tag{4-100h}$$

$$\frac{\partial^2 x_3}{\partial L\, \partial H} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_3\, \frac{\partial \underline{u}}{\partial H} + \underline{x}_3^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial H} \tag{4-100i}$$

$$\frac{\partial^2 x_1}{\partial L\, \partial l} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_1\, \frac{\partial \underline{u}}{\partial l} + \underline{x}_1^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial l} \tag{4-100j}$$

$$\frac{\partial^2 x_2}{\partial L\, \partial l} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_2\, \frac{\partial \underline{u}}{\partial l} + \underline{x}_2^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial l} \tag{4-100k}$$

$$\frac{\partial^2 x_3}{\partial L\, \partial l} = \frac{\partial \underline{u}^T}{\partial L}\, \underline{X}_3\, \frac{\partial \underline{u}}{\partial l} + \underline{x}_3^T\, \frac{\partial^2 \underline{u}}{\partial L\, \partial l} \tag{4-100l}$$

$$\frac{\partial^2 x_1}{\partial L\,\partial g} = \frac{\partial \underline{u}^T}{\partial L} \underline{X}_1 \frac{\partial \underline{u}}{\partial g} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial L\,\partial g} \qquad (4\text{-}100m)$$

$$\frac{\partial^2 x_2}{\partial L\,\partial g} = \frac{\partial \underline{u}^T}{\partial L} \underline{X}_2 \frac{\partial \underline{u}}{\partial g} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial L\,\partial g} \qquad (4\text{-}100n)$$

$$\frac{\partial^2 x_3}{\partial L\,\partial g} = \frac{\partial \underline{u}^T}{\partial L} \underline{X}_3 \frac{\partial \underline{u}}{\partial g} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial L\,\partial g} \qquad (4\text{-}100o)$$

$$\frac{\partial^2 x_1}{\partial L\,\partial h} = \frac{\partial \underline{u}^T}{\partial L} \underline{X}_1 \frac{\partial \underline{u}}{\partial h} + \underline{x}_1^T \frac{\partial^2 \underline{u}}{\partial L\,\partial h} \qquad (4\text{-}100p)$$

$$\frac{\partial^2 x_2}{\partial L\,\partial h} = \frac{\partial \underline{u}^T}{\partial L} \underline{X}_2 \frac{\partial \underline{u}}{\partial h} + \underline{x}_2^T \frac{\partial^2 \underline{u}}{\partial L\,\partial h} \qquad (4\text{-}100q)$$

$$\frac{\partial^2 x_3}{\partial L\,\partial h} = \frac{\partial \underline{u}^T}{\partial L} \underline{X}_3 \frac{\partial \underline{u}}{\partial h} + \underline{x}_3^T \frac{\partial^2 \underline{u}}{\partial L\,\partial h} \qquad (4\text{-}100r)$$

Matrix $\partial \underline{Y}/\partial G$

Again, $\partial \underline{Y}/\partial G$ is derived by partial differentiation of $\underline{Y}$ (2-97b).

$$\frac{\partial \underline{Y}}{\partial G} = \begin{bmatrix} -\frac{\partial^2 x_1}{\partial G\,\partial L} + 3\tau\frac{\partial^2 x_1}{\partial G\,\partial l} & , & -\frac{\partial^2 x_2}{\partial G\,\partial L} + 3\tau\frac{\partial^2 x_2}{\partial G\,\partial l} & , & -\frac{\partial^2 x_3}{\partial G\,\partial L} + 3\tau\frac{\partial^2 x_3}{\partial G\,\partial l} \\ -\frac{\partial^2 x_1}{\partial G^2} & , & -\frac{\partial^2 x_2}{\partial G^2} & , & -\frac{\partial^2 x_3}{\partial G^2} \\ -\frac{\partial^2 x_1}{\partial G\,\partial H} & , & -\frac{\partial^2 x_2}{\partial G\,\partial H} & , & -\frac{\partial^2 x_3}{\partial G\,\partial H} \\ \frac{\partial^2 x_1}{\partial G\,\partial l} & , & \frac{\partial^2 x_2}{\partial G\,\partial l} & , & \frac{\partial^2 x_3}{\partial G\,\partial l} \\ \frac{\partial^2 x_1}{\partial G\,\partial g} & , & \frac{\partial^2 x_2}{\partial G\,\partial g} & , & \frac{\partial^2 x_3}{\partial G\,\partial g} \\ \frac{\partial^2 x_1}{\partial G\,\partial h} & , & \frac{\partial^2 x_2}{\partial G\,\partial h} & , & \frac{\partial^2 x_3}{\partial G\,\partial h} \end{bmatrix} \qquad (4\text{-}101)$$

where the 18 elements are defined by (4-71a,...,d)

$$\frac{\partial^2 x_1}{\partial G\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial L} \qquad (4\text{-}102a)$$

$$\frac{\partial^2 x_2}{\partial G\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial L} \qquad (4\text{-}102b)$$

$$\frac{\partial^2 x_3}{\partial G\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial L} \qquad (4\text{-}102c)$$

$$\frac{\partial^2 x_1}{\partial G^2} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial G^2} \qquad (4\text{-}102d)$$

$$\frac{\partial^2 x_2}{\partial G^2} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial G^2} \qquad (4\text{-}102e)$$

$$\frac{\partial^2 x_3}{\partial G^2} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial G^2} \qquad (4\text{-}102f)$$

$$\frac{\partial^2 x_1}{\partial G\,\partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial H} \qquad (4\text{-}102g)$$

$$\frac{\partial^2 x_2}{\partial G\,\partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial H} \qquad (4\text{-}102h)$$

$$\frac{\partial^2 x_3}{\partial G\,\partial H} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial H} \qquad (4\text{-}102i)$$

$$\frac{\partial^2 x_1}{\partial G\,\partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_1\,\frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial l} \qquad (4\text{-}102j)$$

$$\frac{\partial^2 x_2}{\partial G\,\partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_2\,\frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial l} \qquad (4\text{-}102k)$$

$$\frac{\partial^2 x_3}{\partial G\,\partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial G}\,\underline{X}_3\,\frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial l} \qquad (4\text{-}102l)$$

$$\frac{\partial^2 x_1}{\partial G\,\partial g} = \frac{\partial \underline{u}^T}{\partial G}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial g} \tag{4-102m}$$

$$\frac{\partial^2 x_2}{\partial G\,\partial g} = \frac{\partial \underline{u}^T}{\partial G}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial g} \tag{4-102n}$$

$$\frac{\partial^2 x_3}{\partial G\,\partial g} = \frac{\partial \underline{u}^T}{\partial G}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial g} \tag{4-102o}$$

$$\frac{\partial^2 x_1}{\partial G\,\partial h} = \frac{\partial \underline{u}^T}{\partial G}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial h} \tag{4-102p}$$

$$\frac{\partial^2 x_2}{\partial G\,\partial h} = \frac{\partial \underline{u}^T}{\partial G}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial h} \tag{4-102q}$$

$$\frac{\partial^2 x_3}{\partial G\,\partial h} = \frac{\partial \underline{u}^T}{\partial G}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial G\,\partial h} \tag{4-102r}$$

Matrix $\partial \underline{Y}/\partial H$

$\partial \underline{Y}/\partial H$ is derived by partial differentiation of $\underline{Y}$ (2-97b).

$$\frac{\partial \underline{Y}}{\partial H} = \begin{bmatrix} -\frac{\partial^2 x_1}{\partial H\,\partial L} + 3\tau\frac{\partial^2 x_1}{\partial H\,\partial l} & , & -\frac{\partial^2 x_2}{\partial H\,\partial L} + 3\tau\frac{\partial^2 x_2}{\partial H\,\partial l} & , & -\frac{\partial^2 x_3}{\partial H\,\partial L} + 3\tau\frac{\partial^2 x_3}{\partial H\,\partial l} \\ -\frac{\partial^2 x_1}{\partial H\,\partial G} & , & -\frac{\partial^2 x_2}{\partial H\,\partial G} & , & -\frac{\partial^2 x_3}{\partial H\,\partial G} \\ -\frac{\partial^2 x_1}{\partial H^2} & , & --\frac{\partial^2 x_2}{\partial H^2} & , & -\frac{\partial^2 x_3}{\partial H^2} \\ \frac{\partial^2 x_1}{\partial H\,\partial l} & , & \frac{\partial^2 x_2}{\partial H\,\partial l} & , & \frac{\partial^2 x_3}{\partial H\,\partial l} \\ \frac{\partial^2 x_1}{\partial H\,\partial g} & , & \frac{\partial^2 x_2}{\partial H\,\partial g} & , & \frac{\partial^2 x_3}{\partial H\,\partial g} \\ \frac{\partial^2 x_1}{\partial H\,\partial h} & , & \frac{\partial^2 x_2}{\partial H\,\partial h} & , & \frac{\partial^2 x_3}{\partial H\,\partial h} \end{bmatrix} \tag{4-103}$$

where the 18 elements are defined by (4-71a,...,d)

$$\frac{\partial^2 x_1}{\partial H\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_1^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial L} \qquad (4\text{-}104a)$$

$$\frac{\partial^2 x_2}{\partial H\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_2^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial L} \qquad (4\text{-}104b)$$

$$\frac{\partial^2 x_3}{\partial H\,\partial L} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial L} + \underline{x}_3^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial L} \qquad (4\text{-}104c)$$

$$\frac{\partial^2 x_1}{\partial H\,\partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial G} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_1^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial G} \qquad (4\text{-}104d)$$

$$\frac{\partial^2 x_2}{\partial H\,\partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial G} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_2^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial G} \qquad (4\text{-}104e)$$

$$\frac{\partial^2 x_3}{\partial H\,\partial G} = \frac{\partial \underline{\underline{u}}^T}{\partial G} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial G} + \underline{x}_3^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial G} \qquad (4\text{-}104f)$$

$$\frac{\partial^2 x_1}{\partial H^2} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_1^T \frac{\partial^2 \underline{\underline{u}}}{\partial H^2} \qquad (4\text{-}104g)$$

$$\frac{\partial^2 x_1}{\partial H^2} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_2^T \frac{\partial^2 \underline{\underline{u}}}{\partial H^2} \qquad (4\text{-}104h)$$

$$\frac{\partial^2 x_1}{\partial H^2} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial H} + \underline{x}_3^T \frac{\partial^2 \underline{\underline{u}}}{\partial H^2} \qquad (4\text{-}104i)$$

$$\frac{\partial^2 x_1}{\partial H\,\partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_1 \frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_1^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial l} \qquad (4\text{-}104j)$$

$$\frac{\partial^2 x_2}{\partial H\,\partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_2 \frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_2^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial l} \qquad (4\text{-}104k)$$

$$\frac{\partial^2 x_3}{\partial H\,\partial l} = \frac{\partial \underline{\underline{u}}^T}{\partial H} \underline{X}_3 \frac{\partial \underline{\underline{u}}}{\partial l} + \underline{x}_3^T \frac{\partial^2 \underline{\underline{u}}}{\partial H\,\partial l} \qquad (4\text{-}104l)$$

$$\frac{\partial^2 x_1}{\partial H\,\partial g} = \frac{\partial \underline{u}^T}{\partial H}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial H\,\partial g} \tag{4-104m}$$

$$\frac{\partial^2 x_2}{\partial H\,\partial g} = \frac{\partial \underline{u}^T}{\partial H}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial H\,\partial g} \tag{4-104n}$$

$$\frac{\partial^2 x_3}{\partial H\,\partial g} = \frac{\partial \underline{u}^T}{\partial H}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial g} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial H\,\partial g} \tag{4-104o}$$

$$\frac{\partial^2 x_1}{\partial H\,\partial h} = \frac{\partial \underline{u}^T}{\partial H}\,\underline{X}_1\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_1^T\,\frac{\partial^2 \underline{u}}{\partial H\,\partial h} \tag{4-104p}$$

$$\frac{\partial^2 x_2}{\partial H\,\partial h} = \frac{\partial \underline{u}^T}{\partial H}\,\underline{X}_2\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_2^T\,\frac{\partial^2 \underline{u}}{\partial H\,\partial h} \tag{4-104q}$$

$$\frac{\partial^2 x_3}{\partial H\,\partial h} = \frac{\partial \underline{u}^T}{\partial H}\,\underline{X}_3\,\frac{\partial \underline{u}}{\partial h} + \underline{x}_3^T\,\frac{\partial^2 \underline{u}}{\partial H\,\partial h} \tag{4-104r}$$

The Jacobi matrix $\partial \underline{j}_1/\partial \underline{p}_2$

The model gravity vector $\underline{j}_1$ (3-14) is not only implicitly dependent via the spherical coordinates (r, φ, λ) and the coordinates $\underline{v}$ from the model parameters $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$, it is also an explicit function of $\underline{p}_2$.

Using for $\underline{p}_2$ the expression (4-11b) the vector $\partial \underline{j}_1/\partial \underline{p}_2$ has the following form

$$\frac{\partial \underline{j}_1}{\partial \underline{p}_2} = \left[\frac{\partial \underline{j}_1}{\partial c_{20}}, \frac{\partial \underline{j}_1}{\partial c_{21}}, \ldots, \frac{\partial \underline{j}_1}{\partial c_{nn}}, \frac{\partial \underline{j}_1}{\partial s_{20}}, \frac{\partial \underline{j}_1}{\partial s_{21}}, \ldots, \frac{\partial \underline{j}_1}{\partial s_{nn}} \right] \tag{4-105}$$

where the partial derivatives of $\underline{j}_1$ with respect to the spherical harmonics coefficients $c_{\mu\nu}$ and $s_{\mu\nu}$ - see (3-16) - of the model gravity potential U_1 are given using (3-14) by

$$\frac{\partial \underline{j}_1}{\partial c_{\mu\nu}} = \underline{R}\,\frac{\partial \underline{r}^T}{\partial y}\,\frac{\partial}{\partial c_{\mu\nu}} \left[\frac{\partial U_1}{\partial r} \quad \frac{\partial U_1}{\partial \varphi} \quad \frac{\partial U_1}{\partial \lambda} \right]^T \tag{4-106a}$$

$$\frac{\partial \underline{j}_1}{\partial s_{\mu\nu}} = \underline{R}\,\frac{\partial \underline{r}^T}{\partial y}\,\frac{\partial}{\partial s_{\mu\nu}} \left[\frac{\partial U_1}{\partial r} \quad \frac{\partial U_1}{\partial \varphi} \quad \frac{\partial U_1}{\partial \lambda} \right]^T \tag{4-106b}$$

$$\frac{\partial}{\partial c_{\mu\nu}}\left(\frac{\partial U_1}{\partial r}\right) = -kM \frac{R^{\mu}}{r^{\mu+2}} (\mu+1)\, p_{\mu\nu}(\sin\varphi) \cos\nu\lambda \tag{4-107a}$$

$$\frac{\partial}{\partial c_{\mu\nu}}\left(\frac{\partial U_1}{\partial \varphi}\right) = kM \frac{R^{\mu}}{r^{\mu+1}} \frac{\partial p_{\mu\nu}(\sin\varphi)}{\partial\varphi} \cos\nu\lambda \tag{4-107b}$$

$$\frac{\partial}{\partial c_{\mu\nu}}\left(\frac{\partial U_1}{\partial \lambda}\right) = -kM \frac{R^{\mu}}{r^{\mu+1}}\, \nu\, p_{\mu\nu}(\sin\varphi) \sin\nu\lambda \tag{4-107c}$$

$$\frac{\partial}{\partial s_{\mu\nu}}\left(\frac{\partial U_1}{\partial r}\right) = -kM \frac{R^{\mu}}{r^{\mu+2}} (\mu+1)\, p_{\mu\nu}(\sin\varphi) \sin\nu\lambda \tag{4-108a}$$

$$\frac{\partial}{\partial s_{\mu\nu}}\left(\frac{\partial U_1}{\partial \varphi}\right) = kM \frac{R^{\mu}}{r^{\mu+1}} \frac{\partial p_{\mu\nu}(\sin\varphi)}{\partial\varphi} \sin\nu\lambda \tag{4-108b}$$

$$\frac{\partial}{\partial s_{\mu\nu}}\left(\frac{\partial U_1}{\partial \lambda}\right) = kM \frac{R^{\mu}}{r^{\mu+1}}\, \nu\, p_{\mu\nu}(\sin\varphi) \cos\nu\lambda \tag{4-108c}$$

The vectors $\partial \underline{j}_1/\partial\xi$, $\partial \underline{j}_1/\partial\eta$, $\partial \underline{j}_1/\partial\Theta$

The above mentioned partial derivatives of $\underline{j}_1$ with respect to the pole coordinates ξ, η, and the sideral time Θ ,

$$\frac{\partial \underline{j}}{\partial p_4} = \left[\frac{\partial \underline{j}_1}{\partial\xi} , \frac{\partial \underline{j}_1}{\partial\eta} , \frac{\partial \underline{j}_1}{\partial\Theta} \right] \tag{4-109a}$$

and derived from the rotation matrix $\underline{R}(t)$ in (3-14).

$$\frac{\partial \underline{j}_1}{\partial\xi} = \frac{\partial \underline{R}}{\partial\xi} \frac{\partial \underline{r}^T}{\partial \underline{y}} \left[\frac{\partial U_1}{\partial r} , \frac{\partial U_1}{\partial\varphi} , \frac{\partial U_1}{\partial\lambda} \right]^T \tag{4-109b}$$

$$\frac{\partial \underline{j}_1}{\partial\eta} = \frac{\partial \underline{R}}{\partial\eta} \frac{\partial \underline{r}^T}{\partial \underline{y}} \left[\frac{\partial U_1}{\partial r} , \frac{\partial U_1}{\partial\varphi} , \frac{\partial U_1}{\partial\lambda} \right]^T \tag{4-109c}$$

$$\frac{\partial \underline{j}_1}{\partial\Theta} = \frac{\partial \underline{R}}{\partial\Theta} \frac{\partial \underline{r}^T}{\partial \underline{y}} \left[\frac{\partial U_1}{\partial r} , \frac{\partial U_1}{\partial\varphi} , \frac{\partial U_1}{\partial\lambda} \right]^T \tag{4-109d}$$

where (see also (3-5c), appendix A, (A-3), appendix D, (D-4))

$$\frac{\partial \underline{R}(t)}{\partial \xi} = \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \xi} \qquad (4\text{-}100a)$$

$$\frac{\partial \underline{R}(t)}{\partial \eta} = \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \eta} \qquad (4\text{-}100b)$$

$$\frac{\partial \underline{R}(t)}{\partial \Theta} = \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \Theta} \qquad (4\text{-}100c)$$

$$\frac{\partial \underline{S}}{\partial \xi} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (4\text{-}111a)$$

$$\frac{\partial \underline{S}}{\partial \eta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad (4\text{-}111b)$$

$$\frac{\partial \underline{R}_3(-\Theta)}{\partial \Theta} = \begin{bmatrix} -\sin\Theta & -\cos\Theta & 0 \\ \cos\Theta & -\sin\Theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}111c)$$

The Jacobi matrix $\partial \underline{f}/\partial \underline{p}_3$

Also the resultant acceleration vector $\underline{f}$ depends not opnly implicitly via the transformation $\underline{x} = \underline{x}\Big(\underline{v}\big(\underline{p}_1, \underline{p}_2, \underline{p}_3, \underline{p}_4\big)\Big)$ and $\dot{\underline{x}} = \dot{\underline{x}}\Big(\underline{v}\big(\underline{p}_1, \underline{p}_2, \underline{p}_3, \underline{p}_4\big)\Big)$ on the model parameters $\underline{p}_1, \underline{p}_2, \underline{p}_3, \underline{p}_4$, it is also an explicit function of $\underline{p}_3$. Using (4-11c) we get

$$\underline{f} = \underline{f}_1 + \underline{f}_2 + \underline{f}_3 + \underline{f}_4 \qquad (4\text{-}112a)$$

$$\underline{f}_1 = \underline{f}_1\,(\dot{\Theta}_A, C_D, \rho_1, \bar{\mu}, \bar{\lambda}) \qquad (4\text{-}112b)$$

$$\underline{f}_2 = \underline{f}_2\,(k) \qquad (4\text{-}112c)$$

$$\underline{f}_4 = \underline{f}_4\,(k_2, \Delta t) \qquad (4\text{-}112d)$$

The acceleration $\underline{f}_3$ is not introduced as parameter in this context. $\underline{f}_3$ is a function of the masses of sun and moon and their orbits, as well as of the gravitational constant. It is assumed that $\underline{f}_3$ can be sufficiently accurate determined by other methods of celestial mechanics.

The matrix $\partial \underline{f} / \partial \underline{p}_3$ is then determined by

$$\frac{\partial \underline{f}}{\partial \underline{p}_3} = \left[\frac{\partial \underline{f}_1}{\partial \dot{\Theta}_A} \quad \frac{\partial \underline{f}_1}{\partial C_D} \quad \frac{\partial \underline{f}_1}{\partial \bar{\rho}_1} \quad \frac{\partial \underline{f}_1}{\partial \bar{\mu}} \quad \frac{\partial \underline{f}_1}{\partial \bar{\lambda}} \quad \frac{\partial \underline{f}_2}{\partial k} \quad \frac{\partial \underline{f}_2}{\partial k_2} \quad \frac{\partial \underline{f}_4}{\partial \Delta t} \right] \tag{4-113}$$

where the partial derivatives of $\underline{f}_1$ are given using (3-21), (3-31a,b), (3-33) as well as (3-34a,...,i).

$$\frac{\partial \underline{f}_1}{\partial \dot{\Theta}_A} = -\frac{1}{2} C_D \frac{A}{m} \rho \left(\frac{\partial \dot{\underline{x}}_{rel}}{\partial \dot{\Theta}_A} \left| \dot{\underline{x}}_{rel} \right| + \dot{\underline{x}}_{rel} \frac{\partial \left| \dot{\underline{x}}_{rel} \right|}{\partial \dot{\Theta}_A} \right) \tag{4-114}$$

$$\frac{\partial \dot{\underline{x}}_{rel}}{\partial \dot{\Theta}_A} = [x_2, -x_1, 0]^T \tag{4-114a}$$

$$\frac{\partial \left| \dot{\underline{x}}_{rel} \right|}{\partial \dot{\Theta}_A} = \frac{x_2 \dot{x}_1 - x_1 \dot{x}_2 + 2\dot{\Theta}_A \left(\underline{x}^T \underline{x} - x_3^2 \right)}{2 \left| \dot{\underline{x}}_{rel} \right|} \tag{4-114b}$$

$$\frac{\partial \underline{f}_1}{\partial C_D} = -\frac{1}{2} \frac{A}{m} \rho \, \dot{\underline{x}}_{rel} \left| \dot{\underline{x}}_{rel} \right| \tag{4-115}$$

$$\frac{\partial \underline{f}_1}{\partial \bar{\rho}_1} = -\frac{1}{2} C_D \frac{A}{m} \frac{\partial \rho}{\partial \bar{\rho}_1} \dot{\underline{x}}_{rel} \left| \dot{\underline{x}}_{rel} \right| \tag{4-116}$$

$$\frac{\partial \rho}{\partial \bar{\rho}_1} = \cos^{\bar{\mu}} \frac{\psi}{2} \left\{ \rho_m(h) + [\rho_M(h) - \rho_m(h)] \cos^{\bar{\mu}} \frac{\psi}{2} \right\} \tag{4-116a}$$

$$\frac{\partial \underline{f}_1}{\partial \bar{\mu}} = -\frac{1}{2} C_D \frac{A}{m} \frac{\partial \rho}{\partial \bar{\mu}} \dot{\underline{x}}_{rel} \left| \dot{\underline{x}}_{rel} \right| \tag{4-117}$$

$$\frac{\partial \rho}{\partial \bar{\mu}} = \ln \cos \frac{\psi}{2} \cos^{\bar{\mu}} \frac{\psi}{2} \cdot$$

$$\cdot \left\{ \bar{\rho}_1 \rho_m(h) + [\rho_M(h) - \rho_m(h)] + 2 \cdot \bar{\rho}_1 [\rho_M(h) - \rho_m(h)] \cos^{\bar{\mu}} \frac{\psi}{2} \right\} \tag{4-117a}$$

ψ is the angle between the position vector of the satellite and the point of maximum density expansion of the earth's atmosphere. For $\psi < \pi$, $\cos(\psi/2) > \sigma$ the ln - function in (4-117a) exists. For the case $\psi = \pi$, it holds after (3-33) $\rho = \rho_m(k)$ which means that the density at the orbit of the satellite is a minimum. Thus, it holds $\partial\rho/\partial\bar{\mu} = \sigma$.

$$\frac{\partial \underline{f}_1}{\partial \bar{\lambda}} = -\frac{1}{2} C_D \frac{A}{m} \frac{\partial \rho}{\partial \bar{\lambda}} \dot{\underline{x}}_{rel} \left| \dot{\underline{x}}_{rel} \right| \tag{4-118}$$

$$\frac{\partial \rho}{\partial \bar{\lambda}} = \frac{\partial \rho}{\partial \psi} \frac{\partial \psi}{\partial \bar{\lambda}} \tag{4-118a}$$

$$\frac{\partial \rho}{\partial \psi} = -\frac{\bar{\mu}}{2} \sin\frac{\psi}{2} \cos^{\bar{\mu}-1}\frac{\psi}{2} \left[\rho_M(h) - \rho_m(h) + \bar{\rho}_1 \rho_m + 2\bar{\rho}_1 \left(\rho_M(h) - \rho_m(h) \right) \cos^{\mu}\frac{\psi}{2} \right] \tag{4-118b}$$

$$\frac{\partial \psi}{\partial \bar{\lambda}} = -\frac{1}{\sin\psi} \frac{\frac{\partial \underline{u}^T}{\partial \bar{\lambda}} \underline{x}}{|\underline{x}|} \tag{4-118c}$$

$$\frac{\partial \underline{u}^T}{\partial \bar{\lambda}} = [-\cos\delta_s \sin(\alpha_s + \bar{\lambda}) , \quad \cos\delta_s \cos(\alpha_s + \bar{\lambda}) , \quad 0] \tag{4-118d}$$

The partial derivative of $\underline{f}_2$ with respect to k is given by using (3-35)

$$\frac{\partial \underline{f}_2}{\partial k} = \nu\, a_s^2 \frac{\underline{x} - \underline{x}_s}{\left| \underline{x} - \underline{x}_s \right|^3} \tag{4-119}$$

whereas the partial derivative of $\underline{f}_4$ with respect to the Love number k_2 can be found using (3-45b)

$$\frac{\partial \underline{f}_4}{\partial k_2} = \frac{R^5}{|\underline{x}|^4} \left\{ \frac{km_S}{\left| \hat{\underline{x}}_S \right|^3} \left[p_2' (\cos\hat{\delta}_S) \frac{\hat{\underline{x}}_S}{\left| \hat{\underline{x}}_S \right|} - p_3' (\cos\hat{\delta}_S) \frac{\underline{x}}{|\underline{x}|} \right] + \right.$$

$$\left. + \frac{km_M}{\left| \hat{\underline{x}}_M \right|^3} \left[p_2' (\cos\hat{\delta}_M) \frac{\hat{\underline{x}}_M}{\left| \hat{\underline{x}}_M \right|} - p_3' (\cos\hat{\delta}_M) \frac{\underline{x}}{|\underline{x}|} \right] \right\} \tag{4-120}$$

The contributions of sun (S) and moon (M) are represented separately in the derivative $\partial \underline{f}_4 / \partial \Delta t$ (in contrast to (3-45b)),

$$\frac{\partial \underline{f}_4}{\partial \Delta t} = \frac{\partial \underline{f}_{4S}}{\partial \Delta t} + \frac{\partial \underline{f}_{4M}}{\partial \Delta t} \tag{4-121}$$

where the part coming from the sun is given by

$$\frac{\partial \underline{f}_{4S}}{\partial \Delta t} = 3\, k m_S\, k_2 \frac{R^5}{|\underline{x}|^4 \, |\hat{\underline{x}}_S|^4} \left[\frac{\underline{x}_S \cdot \underline{x}^T}{|\hat{\underline{x}}_S|\,|\underline{x}|} + \cos \hat{\delta}_S \left(\underline{I} - 5 \frac{\hat{\underline{x}}_S \cdot \hat{\underline{x}}_S^T}{|\hat{\underline{x}}_S|^2} - 5 \frac{\underline{x} \cdot \underline{x}^T}{|\underline{x}|^2} \right) + \right.$$

$$\left. + \left(25 \cos^2 \hat{\delta}_S - 3\right) \frac{\underline{x} \cdot \hat{\underline{x}}_S^T}{2\,|\underline{x}|\,|\hat{\underline{x}}_S|} \right] \frac{\partial \hat{\underline{x}}_S}{\partial \Delta t} \tag{4-122a}$$

$$\frac{\partial \hat{\underline{x}}_S}{\partial \Delta t} = \frac{\partial \hat{\underline{x}}_S}{\partial \hat{\Omega}_S} \frac{\partial \hat{\Omega}_S}{\partial \Delta t} + \frac{\partial \hat{\underline{x}}_S}{\partial \hat{\omega}_S} \frac{\partial \hat{\omega}_S}{\partial \Delta t} + \frac{\partial \hat{\underline{x}}_S}{\partial \hat{M}_S} \frac{\partial \hat{M}_S}{\partial \Delta t} \tag{4-122b}$$

$$\frac{\partial \hat{\underline{x}}_S}{\partial \hat{\Omega}_S} = a_S \frac{\partial \underline{R}_3(-\hat{\Omega}_S)}{\partial \hat{\Omega}_S} \underline{R}_1(-i_S)\, \underline{R}_3(-\hat{\omega}_S) \left[\cos \hat{E}_S - e_S,\ \left(1-e_S^2\right)^{1/2} \sin \hat{E}_S,\ 0 \right]^T \tag{4-122c}$$

$$\frac{\partial \hat{\underline{x}}_S}{\partial \hat{\omega}_S} = a_S\, \underline{R}_3(-\hat{\Omega}_S)\, \underline{R}_1(-i_S) \frac{\partial \underline{R}_3(-\hat{\omega}_S)}{\partial \hat{\omega}_S} \left[\cos \hat{E}_S - e_S,\ \left(1-e_S^2\right)^{1/2} \sin \hat{E}_S,\ 0 \right]^T \tag{4-122d}$$

$$\frac{\partial \hat{\underline{x}}_S}{\partial \hat{M}_S} = a_S\, \underline{R}_3(-\hat{\Omega}_S)\, \underline{R}_1(-i_S)\, \underline{R}_3(-\hat{\omega}_S) \frac{\partial \hat{E}_S}{\partial \hat{M}_S} \left[-\sin \hat{E}_S,\ \left(1-e_S^2\right)^{1/2} \cos \hat{E}_S,\ 0 \right]^T \tag{4-122e}$$

$$\frac{\partial \underline{R}_3(-\hat{\Omega}_S)}{\partial \hat{\Omega}_S} = \begin{bmatrix} -\sin \hat{\Omega}_S & -\cos \hat{\Omega}_S & 0 \\ \cos \hat{\Omega}_S & -\sin \hat{\Omega}_S & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4-122f}$$

$$\frac{\partial \underline{R}_3(-\hat{\omega}_S)}{\partial \hat{\omega}_S} = \begin{bmatrix} -\sin\hat{\omega}_S & -\cos\hat{\omega}_S & 0 \\ \cos\hat{\omega}_S & -\sin\hat{\omega}_S & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4-122g}$$

$$\frac{\partial \hat{E}_S}{\partial \hat{M}_S} = \frac{1}{1 - e_S \cos\hat{E}_S} \tag{4-122h}$$

$$\frac{\partial \hat{\Omega}_S}{\partial \Delta t} = \dot{\Theta} - \dot{\Omega}_S \tag{4-122i}$$

$$\frac{\partial \hat{\omega}_S}{\partial \Delta t} = -\dot{\omega}_S \tag{4-122j}$$

$$\frac{\partial \hat{M}_S}{\partial \Delta t} = -\dot{M}_S \tag{4-122k}$$

and the part coming from the moon is given analogously

$$\frac{\partial \underline{f}_{4M}}{\partial \Delta t} = 3\, k m_M\, k_2 \frac{R^5}{|\underline{x}|^4 |\hat{\underline{x}}_M|^4} \left[\frac{\underline{x}_M \cdot \underline{x}^T}{|\hat{\underline{x}}_M| |\underline{x}|} + \cos\hat{\delta}_M \left(\underline{I} - 5 \frac{\hat{\underline{x}}_M \cdot \hat{\underline{x}}_M^T}{|\hat{\underline{x}}_M|^2} - 5 \frac{\underline{x} \cdot \underline{x}^T}{|\underline{x}|^2} \right) + \right.$$

$$\left. + \left(25 \cos^2\hat{\delta}_M - 3\right) \frac{\underline{x} \cdot \hat{\underline{x}}_M^T}{2 |\underline{x}| |\hat{\underline{x}}_M|} \right] \frac{\partial \hat{\underline{x}}_M}{\partial \Delta t} \tag{4-123a}$$

$$\frac{\partial \hat{\underline{x}}_M}{\partial \Delta t} = \frac{\partial \hat{\underline{x}}_M}{\partial \hat{\Omega}_M} \frac{\partial \hat{\Omega}_M}{\partial \Delta t} + \frac{\partial \hat{\underline{x}}_M}{\partial \hat{\omega}_M} \frac{\partial \hat{\omega}_M}{\partial \Delta t} + \frac{\partial \hat{\underline{x}}_M}{\partial \hat{M}_M} \frac{\partial \hat{M}_M}{\partial \Delta t} \tag{4-123b}$$

The partial derivatives in (4-123b) can be found using (3-42a,...,c) and (3-41a,...,c).

$$\frac{\partial \hat{\underline{x}}_M}{\partial \hat{\Omega}_M} = a_M \frac{\partial \underline{R}_3(-\hat{\Omega}_M)}{\partial \hat{\Omega}_M} \underline{R}_1(-i_M) \underline{R}_3(-\hat{\omega}_M) \left[\cos\hat{E}_M - e_M, \left(1 - e_M^2\right)^{1/2} \sin\hat{E}_M, 0 \right]^T \tag{4-123c}$$

$$\frac{\partial \hat{\underline{x}}_M}{\partial \hat{\omega}_M} = a_M \underline{R}_3(-\hat{\Omega}_M)\, \underline{R}_1(-i_M)\, \frac{\partial \underline{R}_3(-\hat{\omega}_M)}{\partial \hat{\omega}_M} \left[\cos\hat{E}_M - e_M,\, (1-e_M^2)^{1/2}\sin\hat{E}_M,\, 0\right]^T \tag{4-123d}$$

$$\frac{\partial \hat{\underline{x}}_M}{\partial \hat{M}_M} = a_M \underline{R}_3(-\hat{\Omega}_M)\, \underline{R}_1(-i_M)\, \underline{R}_3(-\hat{\omega}_M)\, \frac{\partial \hat{E}_M}{\partial \hat{M}_M} \left[-\sin\hat{E}_M,\, (1-e_M^2)^{1/2}\cos\hat{E}_M,\, 0\right]^T \tag{4-123e}$$

$$\frac{\partial \underline{R}_3(-\hat{\Omega}_M)}{\partial \hat{\Omega}_M} = \begin{bmatrix} -\sin\hat{\Omega}_M & -\cos\hat{\Omega}_M & 0 \\ \cos\hat{\Omega}_M & -\sin\hat{\Omega}_M & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4-123f}$$

$$\frac{\partial \underline{R}_3(-\hat{\omega}_M)}{\partial \hat{\omega}_M} = \begin{bmatrix} -\sin\hat{\omega}_M & -\cos\hat{\omega}_M & 0 \\ \cos\hat{\omega}_M & -\sin\hat{\omega}_M & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4-123g}$$

$$\frac{\partial \hat{E}_M}{\partial \hat{M}_M} = \frac{1}{1 - e_M \cos\hat{E}_M} \tag{4-123h}$$

$$\frac{\partial \hat{\Omega}_M}{\partial \Delta t} = \dot{\Theta} - \dot{\Omega}_M \tag{4-123i}$$

$$\frac{\partial \hat{\omega}_M}{\partial \Delta t} = -\dot{\omega}_M \tag{4-123j}$$

$$\frac{\partial \hat{M}_M}{\partial \Delta t} = -\dot{M}_M \tag{4-123k}$$

4.5 Numerical determination of the derivatives of the coordinate vector $\underline{v}$ with respect to the parameters $\underline{p}_1,\dots,\underline{p}_4$

Analogously to the computation of $\underline{v}$ the above-mentioned Jacobi matrices of $\underline{v}$ have to be derived by the method of successive approximations (Picard's method). Thereby the integrals in (4-31), (4-32) and (4-33) have to be computed numerically. Assuming that in iteration step $k+1=m$ the necessary accuracy is derived, we get

$$\frac{\partial \underline{v}}{\partial \underline{p}_1} \doteq \frac{\partial \underline{v}_m}{\partial \underline{p}_1} \tag{4-124a}$$

$$\frac{\partial \underline{v}}{\partial \underline{p}_2} \doteq \frac{\partial \underline{v}_m}{\partial \underline{p}_2} \tag{4-124b}$$

$$\frac{\partial \underline{v}}{\partial \underline{p}_3} \doteq \frac{\partial \underline{v}_m}{\partial \underline{p}_3} \tag{4-124c}$$

$$\frac{\partial \underline{v}}{\partial \underline{p}_4} \doteq \frac{\partial \underline{v}_m}{\partial \underline{p}_4} \tag{4-124d}$$

$$\frac{\partial \underline{v}}{\partial \underline{\dot{p}}_4} \doteq \frac{\partial \underline{v}_m}{\partial \underline{\dot{p}}_4} \tag{4-124e}$$

Using, for example, the trapezoid rule for the numerical integration of the integrals in (4-31), (4-32), (4-33), we have $(j = \{1,...,4\})$

$$\frac{\partial \underline{v}}{\partial \underline{p}_j} k+1 (t_n) =$$

$$= \frac{\partial \underline{v}_0}{\partial \underline{p}_j} +$$

$$+ \Delta t \sum_{i=0}^{n} \boldsymbol{\alpha}_i \, \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{j}_1}{\partial \underline{v}} \left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{v}_k}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \Delta t \sum_{i=0}^{n} \boldsymbol{\alpha}_i \, \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{f}}{\partial \underline{v}} \left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{v}_k}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \Delta t \sum_{i=0}^{n} \boldsymbol{\alpha}_i \, \frac{\partial \underline{Y}}{\partial v_1} \left(\underline{v}_k(t_i), t_i\right) \left\{ \underline{j}_1 \left(\underline{v}_k(t_i), t_i\right) + \underline{f}\left(\underline{v}_k(t_i), t_i\right) \right\} \frac{\partial \underline{v}_k}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad +$$

$$+ \Delta t \sum_{i=0}^{n} \boldsymbol{\alpha}_i \frac{\partial \underline{Y}}{\partial v_6} \left(\underline{v}_k(t_i), t_i\right) \left\{\underline{j}_1\left(\underline{v}_k(t_i), t_i\right) + \underline{f}\left(\underline{v}_k(t_i), t_i\right)\right\} \frac{\partial \underline{v}_k}{\partial \underline{p}_j}\left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \Delta t \sum_{i=0}^{n} \boldsymbol{\alpha}_i \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{j}_1}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \Delta t \sum_{i=0}^{n} \boldsymbol{\alpha}_i \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{f}}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) \quad (4\text{-}125a)$$

where

$$\Delta t = \left(t - t_0\right)/n \quad (4\text{-}125b)$$

and $\boldsymbol{\alpha}_i$ is defined by (only considering the trapezoid rule)

$$\boldsymbol{\alpha}_i = \begin{cases} 0.5 & \text{for} \quad i = 0 \quad \text{and} \quad i = n \\ 1 & \text{for} \quad i \neq 0 \quad \text{and} \quad i \neq n \end{cases} \quad (4\text{-}125c)$$

Analogously to (2-104a) a simple rule can be found for the determination of $\partial \underline{v}_{k+1} / \partial \underline{p}_j$, $j = \{1, ..., 4\}$

$$\frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_j}(t_{i+1}) =$$

$$= \frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_j}(t_i) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{j}_1}{\partial \underline{v}} \left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{v}_k}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \frac{\partial \underline{j}_1}{\partial \underline{v}} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \frac{\partial \underline{v}_k}{\partial \underline{p}_j} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{f}}{\partial \underline{v}} \left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{v}_k}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \frac{\partial \underline{f}}{\partial \underline{v}} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \frac{\partial \underline{v}_k}{\partial \underline{p}_j} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) +$$

$$+ \frac{\Delta t}{2} \frac{\partial \underline{Y}}{\partial v_1} \left(\underline{v}_k(t_i), t_i\right) \left\{ \underline{j}_1 \left(\underline{v}_k(t_i), t_i\right) + \underline{f}\left(\underline{v}_k(t_i), t_i\right)\right\} \frac{\partial \underline{v}_k}{\partial \underline{p}_j}\left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \frac{\Delta t}{2} \frac{\partial \underline{Y}}{\partial v_1} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \left\{ \underline{j}_1 \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) + \underline{f}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right)\right\} \frac{\partial \underline{v}_k}{\partial \underline{p}_j}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right) +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \frac{\Delta t}{2} \frac{\partial \underline{Y}}{\partial v_6} \left(\underline{v}_k(t_i), t_i\right) \left\{ \underline{j}_1 \left(\underline{v}_k(t_i), t_i\right) + \underline{f}\left(\underline{v}_k(t_i), t_i\right)\right\} \frac{\partial \underline{v}_k}{\partial \underline{p}_j}\left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \frac{\Delta t}{2} \frac{\partial \underline{Y}}{\partial v_6} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \left\{ \underline{j}_1 \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) + \underline{f}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right)\right\} \frac{\partial \underline{v}_k}{\partial \underline{p}_j}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{j}_1}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \frac{\partial \underline{j}_1}{\partial \underline{p}_j} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_i), t_i\right) \frac{\partial \underline{f}}{\partial \underline{p}_j} \left(\underline{v}_k(t_i), t_i\right) +$$

$$+ \frac{\Delta t}{2} \underline{Y}\left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \frac{\partial \underline{f}}{\partial \underline{p}_j} \left(\underline{v}_k(t_{i+1}), t_{i+1}\right) \qquad (4\text{-}126)$$

(4-126) is only valid when using the trapezoid rule for the integration. For higher-order quadrature formulas similar expressions can be found.

Generally, the algorithm (4-126) can be used in the following way:

(i) For all matrices and vectors th initial states ($k = i = 0$) have to be put in.

(ii) Computation of $\partial \underline{v}_1/\partial p_j\,(t_n)$ using (4-126) in the integration interval $[t_n, t_0]$

(iii) Intermediate storage of matrices $\partial \underline{v}_1/\partial p_j\,(t_i)$ for all knots t_i

(iv) New iteration with initial matrices $\partial \underline{v}_1/\partial \underline{p}_j\,(t_i)$, resulting in $\partial \underline{v}_2/\partial \underline{p}_j\,(t_n)$, etc.

The intermediate storage of $\partial \underline{v}_1/\partial \underline{p}_j$ can be avoided when - similar to the computation of $\underline{v}_k$ - first the $k+1$ iterations at t_i are carried out, and then by using this sequence of matrices the $k+1$ iterations at t_{i+1} are performed. Of course, in that way the $k+1$ matrices at t_i have to be stored. However, they can be scratched after having done the computations at t_{i+1} . In addition, $k+1$ will be much smaller as n . From experiences with analytical orbit integration one can expect that with $k+1 = 2$ or 3 the necessary accuracy is obtained.

For $t = t_0 \rightarrow \Delta t = 0$ according to (4-125b). This means, that all initial values $\partial \underline{v}_1/\partial p_j\,(k+1)$ at t_0 are the same for al steps of iteration.

$$\frac{\partial \underline{v}_0}{\partial \underline{p}_j}(t_0) = \frac{\partial \underline{v}_1}{\partial \underline{p}_j}(t_0) = \frac{\partial \underline{v}_2}{\partial \underline{p}_j}(t_0) = \ldots\ldots = \frac{\partial \underline{v}_{k+1}}{\partial \underline{p}_j} \qquad (4\text{-}127)$$

For $j = \{1,...,4\}$ we have according to (4-31), (4-32), (4-33)

$$\frac{\partial \underline{v}_0}{\partial \underline{p}_1} = \underline{I} + \underline{Q}_3(t_0) \qquad \text{see (4-37a)}$$

$$\frac{\partial \underline{v}_0}{\partial \underline{p}_2} = \frac{\partial \underline{v}_0}{\partial \underline{p}_3} = \frac{\partial \underline{v}_0}{\partial \underline{p}_4} = 0$$

4.6 Numerical integration of the vector $\delta \underline{g}$

The vector $\delta \underline{v}(t)$ needed for the set-up of the linearized observation equations, is besides $\delta \underline{p}$ dependent also on the vector $\delta \underline{g}$ (4-17c) defined as the gradient of the disturbing potential T .

The vector $\delta \underline{g}$ is needed after multiplication with matrix $\underline{Y}$ (2-97a,b) and subsequent integration over the interval $[t_0, t]$ for defining $\delta \underline{v}(t)$. Expressing the part in (4-24b) generated through $\delta \underline{g}$ by $\underline{\sigma}(t)$,

$$\underline{\sigma}(t) = \int_{t_0}^{n} \underline{Y}\left(\underline{v}_k(t),t\right) \delta\underline{g}\left(\underline{v}_k(t),t\right) dt \tag{4-128}$$

we get using the trapezoid rule,

$$\underline{\sigma}(t) = \Delta t \sum_{i=0}^{n} \alpha_i \underline{Y}\left(\underline{v}_k(t_i),t_i\right) \delta\underline{g}\left(\underline{v}_k(t_i),t_i\right) \tag{4-129a}$$

where

$$\alpha_i = \begin{Bmatrix} 0.5 & \text{for} & i = 0 & \text{and} & i = n \\ 1 & \text{for} & i \neq 0 & \text{and} & i \neq n \end{Bmatrix} \tag{4-129b}$$

and

$$\Delta t = \left(t - t_0\right)/n \tag{4-129c}$$

For practical reasons (use of covariance functions) it is convenient to transform the vector $\delta\underline{g} = \delta\underline{g}(\underline{x},t)$ in the earth-fixed frame with spherical coordinates $\underline{r} = [r,\varphi,\lambda]^T$ (see also chapter 3.2). With $\delta\underline{g} = \text{grad}\,T$ we get according to (3-14)

$$\delta\underline{g} = \underline{R}(t) \frac{\partial\underline{\tilde{r}}^T}{\partial\underline{v}} \left[\frac{\partial T}{\partial r} , \frac{1}{r}\frac{\partial T}{\partial\varphi} , \frac{1}{r\cos\varphi}\frac{\partial T}{\partial\lambda} \right]^T \tag{4-130}$$

The rotation matrix $\underline{R}(t)$ is given by (3-5c). $\frac{\partial\underline{\tilde{r}}^T}{\partial\underline{v}}$ is found to be (see 3-15)

$$\frac{\partial\underline{\tilde{r}}^T}{\partial\underline{v}} = \begin{bmatrix} \cos\varphi\cos\lambda & , & -\sin\varphi\cos\lambda & , & -\sin\lambda \\ \cos\varphi\sin\lambda & , & -\sin\varphi\sin\lambda & , & \cos\lambda \\ \sin\varphi & , & \cos\varphi & , & 0 \end{bmatrix} \tag{4-131}$$

The transformation of the components of the position vector $\underline{x}$ into spherical coordinates is already given by (3-10a,...,d).

4.7 The position vector of the ground station, its linearized form and corresponding time-derivative

Most of the observations in satellite geodesy are between a ground station $\underline{x}_G$ and a satellite. Thus, the corresponding observation equations are non-linear functions of the position and velocity vectors of both.

The vectors $\underline{x}_G$, $\underline{\dot{x}}_G$ and their linearized forms $\delta\underline{x}_G$, $\delta\underline{\dot{x}}_G$ are referring to the inertial system when introduced into the observation equations. Since those vectors then due to earth's rotation are functions of time, it is reasonable to express it in the earth-fixed reference frame. Via the necessary rotation matrix $\underline{R}(t)$ in the transformation between earth-fixed and inertial coordinate system the parameters Θ, ξ, η of earth's rotation can be introduced in the observation equations as unknowns.

For the position vector $\underline{x}_G$ and the velocity vector $\underline{\dot{x}}_G$ of the ground station we have according to (3-5a)

$$\underline{x}_G = \underline{R}(t)\,\underline{y}_G \tag{4-132a}$$

$$\underline{\dot{x}}_G = \underline{\dot{R}}(t)\,\underline{y}_G + \underline{R}(t)\,\underline{\dot{y}}_G \tag{4-132b}$$

We will assume in this context that the position $\underline{y}_G$ of the ground station in the earth-fixed reference system is time-independent, e.g., the earth is assumed to be rigid.

$$\underline{\dot{y}}_G = 0 \tag{4-132c}$$

From (4-132b) follows with (4-132c) that

$$\underline{\dot{x}}_G = \underline{\dot{R}}(t)\,\underline{y}_G \tag{4-132d}$$

Using

$$\underline{p}_4 = [\xi, \eta, \Theta]^T \tag{4-133a}$$

$$\underline{\dot{p}}_4 = [\dot{\xi}, \dot{\eta}, \dot{\Theta}]^T \tag{4-133b}$$

and

$$\underline{y}_G = [y_{1G}, y_{2G}, y_{3G}]^T \tag{4-133c}$$

we can express the vectors $\delta\underline{x}_G$ and $\delta\underline{\dot{x}}_G$ by

$$\delta\underline{x}_G = \frac{\partial \underline{x}_G}{\partial \underline{p}}\,\delta\underline{p} + \frac{\partial \underline{x}_G}{\partial \underline{y}_G}\,\delta\underline{y}_G \tag{4-134a}$$

$$\delta\underline{\dot{x}}_G = \frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}}\,\delta\underline{p} + \frac{\partial \underline{\dot{x}}_G}{\partial \underline{y}_G}\,\delta\underline{y}_G \tag{4-134b}$$

where $\delta \underline{p}$ is defined by (4-10) and the necessary Jacobi matrices are given by

$$\frac{\partial \underline{x}_G}{\partial \underline{y}_G} = \underline{R}(t) \tag{4-135a}$$

$$\frac{\partial \underline{\dot{x}}_G}{\partial \underline{y}_G} = \underline{\dot{R}}(t) \tag{4-135b}$$

$$\frac{\partial \underline{x}_G}{\partial \underline{p}} = \left[\frac{\partial \underline{x}_G}{\partial \underline{p}_1} , \frac{\partial \underline{x}_G}{\partial \underline{p}_2} , \frac{\partial \underline{x}_G}{\partial \underline{p}_3} , \frac{\partial \underline{x}_G}{\partial \underline{p}_4} , \frac{\partial \underline{x}_G}{\partial \underline{\dot{p}}_4} \right] \tag{4-135c}$$

$$\frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}} = \left[\frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_1} , \frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_2} , \frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_3} , \frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_4} , \frac{\partial \underline{\dot{x}}_G}{\partial \underline{\dot{p}}_4} \right] \tag{4-135d}$$

$$\frac{\partial \underline{x}_G}{\partial \underline{p}_1} = \underline{0} \tag{4-135e}$$

$$\frac{\partial \underline{x}_G}{\partial \underline{p}_2} = \underline{0} \tag{4-135f}$$

$$\frac{\partial \underline{x}_G}{\partial \underline{p}_3} = \underline{0} \tag{4-135g}$$

$$\frac{\partial \underline{x}_G}{\partial \underline{p}_4} = \left[\frac{\partial \underline{R}}{\partial \xi} \underline{y}_G , \frac{\partial \underline{R}}{\partial \eta} \underline{y}_G , \frac{\partial \underline{R}}{\partial \Theta} \underline{y}_G \right] \tag{4-135h}$$

$$\frac{\partial \underline{x}_G}{\partial \underline{\dot{p}}_4} = \underline{0} \tag{4-135i}$$

$$\frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_1} = \underline{0} \tag{4-135j}$$

$$\frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_2} = \underline{0} \qquad (4\text{-}135k)$$

$$\frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_3} = \underline{0} \qquad (4\text{-}135l)$$

$$\frac{\partial \underline{\dot{x}}_G}{\partial \underline{p}_4} = \left[\frac{\partial \underline{\dot{R}}}{\partial \xi} \underline{y}_G \,,\, \frac{\partial \underline{\dot{R}}}{\partial \eta} \underline{y}_G \,,\, \frac{\partial \underline{\dot{R}}}{\partial \Theta} \underline{y}_G \right] \qquad (4\text{-}135m)$$

$$\frac{\partial \underline{\dot{x}}_G}{\partial \underline{\dot{p}}_4} = \left[\frac{\partial \underline{\dot{R}}}{\partial \dot{\xi}} \underline{y}_G \,,\, \frac{\partial \underline{\dot{R}}}{\partial \dot{\eta}} \underline{y}_G \,,\, \frac{\partial \underline{\dot{R}}}{\partial \dot{\Theta}} \underline{y}_G \right] \qquad (4\text{-}135n)$$

The partial derivatives of the rotation matrices $\underline{R}(t)$ and $\underline{\dot{R}}(t)$ in (4-1a,...,d) can be found using (3-5c) and (3-28).

$$\frac{\partial \underline{R}(t)}{\partial \xi} = \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{P}^T(t)\, \underline{N}^T(t)\, \underline{R}_3\big(-\Theta(t)\big) \frac{\partial \underline{S}(t)}{\partial \xi} \qquad (4\text{-}136a)$$

$$\frac{\partial \underline{R}(t)}{\partial \eta} = \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{P}^T(t)\, \underline{N}^T(t)\, \underline{R}_3\big(-\Theta(t)\big) \frac{\partial \underline{S}(t)}{\partial \eta} \qquad (4\text{-}136b)$$

$$\frac{\partial \underline{R}(t)}{\partial \Theta} = \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{P}^T(t)\, \underline{N}^T(t)\, \frac{\partial \underline{R}_3\big(-\Theta(t)\big)}{\partial \Theta}\, \underline{S}(t) \qquad (4\text{-}136c)$$

$$\begin{aligned}
\frac{\partial \underline{\dot{R}}}{\partial \xi} &= \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{\dot{P}}^T(t)\, \underline{N}^T(t)\, \underline{R}_3\big(-\Theta(t)\big) \frac{\partial \underline{S}(t)}{\partial \xi} + \\
&+ \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{P}^T(t)\, \underline{\dot{N}}^T(t)\, \underline{R}_3\big(-\Theta(t)\big) \frac{\partial \underline{S}(t)}{\partial \xi} + \\
&+ \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{P}^T(t)\, \underline{N}^T(t)\, \underline{\dot{R}}_3\big(-\Theta(t)\big) \frac{\partial \underline{S}(t)}{\partial \xi} + \\
&+ \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{P}^T(t)\, \underline{N}^T(t)\, \underline{R}_3\big(-\Theta(t)\big) \frac{\partial \underline{\dot{S}}(t)}{\partial \xi} \qquad (4\text{-}136d)
\end{aligned}$$

$$\frac{\partial \underline{\dot{R}}}{\partial \eta} = \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{\dot{P}}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \eta} +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{\dot{N}}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \eta} +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{\dot{R}}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \eta} +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{\dot{S}}(t)}{\partial \eta} \qquad (4\text{-}136e)$$

$$\frac{\partial \underline{\dot{R}}}{\partial \Theta} = \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{\dot{P}}^T(t)\,\underline{N}^T(t)\,\frac{\partial \underline{R}_3\big(-\Theta(t)\big)}{\partial \Theta}\,\underline{S}(t) +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{\dot{N}}^T(t)\,\frac{\partial \underline{R}_3\big(-\Theta(t)\big)}{\partial \Theta}\,\underline{S}(t) +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\frac{\partial \underline{\dot{R}}_3\big(-\Theta(t)\big)}{\partial \Theta}\,\underline{S}(t) +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\frac{\partial \underline{R}_3\big(-\Theta(t)\big)}{\partial \Theta}\,\underline{\dot{S}}(t) \qquad (4\text{-}136f)$$

$$\frac{\partial \underline{\dot{R}}}{\partial \dot{\xi}} = \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{\dot{P}}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \dot{\xi}} +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{\dot{N}}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \dot{\xi}} +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{\dot{R}}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{S}(t)}{\partial \dot{\xi}} +$$

$$+ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{R}_3\big(-\Theta(t)\big)\,\frac{\partial \underline{\dot{S}}(t)}{\partial \dot{\xi}} \qquad (4\text{-}136g)$$

$$\frac{\partial \dot{\underline{R}}}{\partial \dot{\eta}} = \underline{N}(t_0)\,\underline{P}(t_0)\,\dot{\underline{P}}^T(t)\,\underline{N}^T(t)\,\underline{R}_3(-\Theta(t))\,\frac{\partial \underline{S}(t)}{\partial \dot{\eta}} +$$

$$+\ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\dot{\underline{N}}^T(t)\,\underline{R}_3(-\Theta(t))\,\frac{\partial \underline{S}(t)}{\partial \dot{\eta}} +$$

$$+\ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\dot{\underline{R}}_3(-\Theta(t))\,\frac{\partial \underline{S}(t)}{\partial \dot{\eta}} +$$

$$+\ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\underline{R}_3(-\Theta(t))\,\frac{\partial \dot{\underline{S}}(t)}{\partial \dot{\eta}} \qquad (4\text{-}136h)$$

$$\frac{\partial \dot{\underline{R}}}{\partial \dot{\Theta}} = \underline{N}(t_0)\,\underline{P}(t_0)\,\dot{\underline{P}}^T(t)\,\underline{N}^T(t)\,\frac{\partial \underline{R}_3(-\Theta(t))}{\partial \dot{\Theta}}\,\underline{S}(t) +$$

$$+\ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\dot{\underline{N}}^T(t)\,\frac{\partial \underline{R}_3(-\Theta(t))}{\partial \dot{\Theta}}\,\underline{S}(t) +$$

$$+\ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\frac{\partial \dot{\underline{R}}_3(-\Theta(t))}{\partial \dot{\Theta}}\,\underline{S}(t) +$$

$$+\ \underline{N}(t_0)\,\underline{P}(t_0)\,\underline{P}^T(t)\,\underline{N}^T(t)\,\frac{\partial \underline{R}_3(-\Theta(t))}{\partial \dot{\Theta}}\,\dot{\underline{S}}(t) \qquad (4\text{-}136i)$$

We have further according to appendix D, eq. (D-4), the matrix $\underline{S}(t)$ defined to be

$$\underline{S}(t) = \begin{bmatrix} 1 & 0 & -\xi \\ 0 & 1 & +\eta \\ \xi & -\eta & 1 \end{bmatrix} \qquad (4\text{-}137a)$$

$$\dot{\underline{S}}(t) = \begin{bmatrix} 0 & 0 & -\dot{\xi} \\ 0 & 0 & +\dot{\eta} \\ \dot{\xi} & -\dot{\eta} & 1 \end{bmatrix} \qquad (4\text{-}137b)$$

$$\frac{\partial \underline{S}}{\partial \xi} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (4\text{-}137c)$$

$$\frac{\partial \underline{S}}{\partial \eta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{4-137d}$$

$$\frac{\partial \underline{S}}{\partial \dot{\eta}} = \underline{0} \tag{4-137e}$$

$$\frac{\partial \underline{S}}{\partial \dot{\xi}} = \underline{0} \tag{4-137f}$$

$$\frac{\partial \underline{\dot{S}}}{\partial \xi} = \underline{0} \tag{4-137g}$$

$$\frac{\partial \underline{\dot{S}}}{\partial \eta} = \underline{0} \tag{4-137h}$$

$$\frac{\partial \underline{\dot{S}}}{\partial \dot{\xi}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{4-137i}$$

$$\frac{\partial \underline{\dot{S}}}{\partial \dot{\eta}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{4-137j}$$

According to appendox A, eq. (A-3) we have for $\underline{R}_3(-\Theta)$

$$\underline{R}_3(-\Theta) = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4-138a}$$

The time derivative $\underline{\dot{R}}_3(-\Theta)$ is then

$$\underline{\dot{R}}_3(-\Theta) = \dot{\Theta} \begin{bmatrix} -\sin\Theta & -\cos\Theta & 0 \\ \cos\Theta & -\sin\Theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4-138b}$$

and all other necessary partial derivatives of $\underline{R}_3(-\Theta)$ are

$$\frac{\partial \underline{R}_3}{\partial \Theta} = \begin{bmatrix} -\sin\Theta & -\cos\Theta & 0 \\ \cos\Theta & -\sin\Theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}138c)$$

$$\frac{\partial \underline{R}_3}{\partial \dot{\Theta}} = \underline{0} \qquad (4\text{-}138d)$$

$$\frac{\partial \dot{\underline{R}}_3}{\partial \Theta} = \dot{\Theta} \begin{bmatrix} -\cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & -\cos\Theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}138e)$$

$$\frac{\partial \dot{\underline{R}}_3}{\partial \dot{\Theta}} = \begin{bmatrix} -\sin\Theta & -\cos\Theta & 0 \\ \cos\Theta & -\sin\Theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}138f)$$

4.8 Further parametrization of the vector of earth rotation parameters and of its time-derivative

The linearized form of the earth rotation parameter vector $\underline{p}_4$ and its time derivative $\dot{\underline{p}}_4$ is given by (4-133a,b)

$$\delta\underline{p}_4 = [\delta\xi , \delta\eta , \delta\Theta]^T \qquad (4\text{-}139a)$$

$$\delta\dot{\underline{p}}_4 = [\delta\dot{\xi} , \delta\dot{\eta} , \delta\dot{\Theta}]^T \qquad (4\text{-}139b)$$

where

$$\underline{p}_4 = \underline{p}_4(t) \qquad (4\text{-}140a)$$

$$\dot{\underline{p}}_4 = \dot{\underline{p}}_4(t) \qquad (4\text{-}140b)$$

For the determination of $\underline{p}_4$, $\dot{\underline{p}}_4$ one can assume that approximate values $\underline{p}_4^o(t)$, $\dot{\underline{p}}_4^o(t)$ for the pole coordinates and for the sideral time as well as for their time-derivatives are available. By introducing (4-139a), (4-139b) as unknowns, it is possible to estimate improved earth-rotation parameters by satellite geodesy.

In general, two possibilities can be mentioned:

(i) Determination of $\delta \underline{p}_4$, $\delta \dot{\underline{p}}_4$ as discrete functions of time, which means, that for every discrete time t_k the pair of vectors $\delta \underline{p}_{4k} = [\delta\xi_k, \delta\eta_k, \delta\Theta_k]^T$ and $\delta \dot{\underline{p}}_{4k} = [\delta\dot{\xi}_k, \delta\dot{\eta}_k, \delta\dot{\Theta}_k]^T$ has to be introduced as unknowns.

(ii) Determination of $\delta \underline{p}_4(t)$ and $\delta \dot{\underline{p}}_4(t)$ by a regression series.

For example, the following expansion of ξ, η, Θ and $\dot{\xi}, \dot{\eta}, \dot{\Theta}$ might be reasonable.

$$\delta\xi = \delta\xi_0 + (t-t_0)\,\delta\xi_1 + (t-t_0)^2\,\delta\xi_2 + \\ + \sum_{k=1}^{n} [\sin k\,\nu_1(t-t_0)\cdot\delta\xi_{1k} + \cos k\,\nu_1(t-t_0)\cdot\delta\xi_{2k}] \tag{4-141a}$$

$$\delta\eta = \delta\eta_0 + (t-t_0)\,\delta\eta_1 + (t-t_0)^2\,\delta\eta_2 + \\ + \sum_{k=1}^{n} [\sin k\,\nu_2(t-t_0)\cdot\delta\eta_{1k} + \cos k\,\nu_2(t-t_0)\cdot\delta\eta_{2k}] \tag{4-141b}$$

$$\delta\Theta = \delta\Theta_0 + (t-t_0)\,\delta\Theta_1 + (t-t_0)^2\,\delta\Theta_2 + \\ + \sum_{k=1}^{n} [\sin k\,\nu_3(t-t_0)\cdot\delta\Theta_{1k} + \cos k\,\nu_3(t-t_0)\cdot\delta\Theta_{2k}] \tag{4-141c}$$

$$\delta\dot{\xi} = \delta\xi_1 + 2(t-t_0)\,\delta\xi_2 + \\ + \sum_{k=1}^{n} k\,\nu_1 [\cos k\,\nu_1(t-t_0)\cdot\delta\xi_{1k} - \sin k\,\nu_1(t-t_0)\cdot\delta\xi_{2k}] \tag{4-141d}$$

$$\delta\dot{\eta} = \delta\eta_1 + 2(t-t_0)\,\delta\eta_2 + \\ + \sum_{k=1}^{n} k\,\nu_2 [\cos k\,\nu_2(t-t_0)\cdot\delta\eta_{1k} - \sin k\,\nu_2(t-t_0)\cdot\delta\eta_{2k}] \tag{4-141e}$$

$$\delta\dot{\Theta} = \delta\Theta_1 + 2(t-t_0)\,\delta\Theta_2 + \\ + \sum_{k=1}^{n} k\,\nu_3 [\cos k\,\nu_3(t-t_0)\cdot\delta\Theta_{1k} - \sin k\,\nu_3(t-t_0)\cdot\delta\Theta_{2k}] \tag{4-141f}$$

4.9 Final expressions for the linear observation equations of satellite measurements

By using the derivations in the paragraphs before it is now possible to write down explicitly the linear observation equations for the satellite techniques presently available. We are using index Q for quantities (especially the position and velocity vectors) referring to the satellite, and index G for those ones referring to the ground station.

Since we are able now to express all satellite observations by the vectors $\underline{x}_Q$, $\dot{\underline{x}}_Q$, $\underline{x}_G$, $\dot{\underline{x}}_G$ derived before, we repeat them again for clearness. Although the integrals in (4-142a,b) have to be computed numerically - integrals will be sums - we still have the integrals for the reason of shortness.

Thus, we have by (4-9c,d) using $\delta\underline{v}$ (4-24b) the following expression for the position and velocity vectors:

$$\delta\underline{x}_Q(t) = \frac{\partial\underline{x}_Q(t)}{\partial\underline{v}(t)} \frac{\partial\underline{v}(t)}{\partial\underline{p}} \delta\underline{p} + \frac{\partial\underline{x}_Q(t)}{\partial\underline{v}(t)} \int_{t_0}^{t} \underline{Y}(t)\,\delta\underline{g}(t)\,dt \tag{4-142a}$$

$$\delta\dot{\underline{x}}_Q(t) = \frac{\partial\dot{\underline{x}}_Q(t)}{\partial\underline{v}(t)} \frac{\partial\underline{v}(t)}{\partial\underline{p}} \delta\underline{p} + \frac{\partial\dot{\underline{x}}_Q(t)}{\partial\underline{v}(t)} \int_{t_0}^{t} \underline{Y}(t)\,\delta\underline{g}(t)\,dt \tag{4-142b}$$

$$\delta\underline{x}_G(t) = \frac{\partial\underline{x}_G(t)}{\partial\underline{p}} \delta\underline{p} + \underline{R}(t)\,\delta\underline{y}_G \tag{4-142c}$$

$$\delta\dot{\underline{x}}_G(t) = \frac{\partial\dot{\underline{x}}_G(t)}{\partial\underline{p}} \delta\underline{p} + \dot{\underline{R}}(t)\,\delta\underline{y}_G \tag{4-142d}$$

$$\delta\underline{g}(t) = \underline{R}(t) \frac{\partial\tilde{\underline{r}}^T(t)}{\partial\underline{y}(t)} \left[\frac{\partial T}{\partial r}, \frac{\partial T}{r\,\partial\varphi}, \frac{1}{r\cos\varphi}\frac{\partial T}{\partial\lambda}\right]^T_{(t)} = \underline{R}(t) \frac{\partial\tilde{\underline{r}}^T(t)}{\partial\underline{y}(t)} \underline{t}(t) \tag{4-142e}$$

Type S1: $\underline{u}(t)$ $(\alpha(t), \delta(t))$***...direction measurements*** [1)]

The topocentric unit vector $\underline{u}$ from a ground station G to the satellite Q is given by

$$\underline{u}(t) = \frac{\underline{x}_Q(t) - \underline{x}_G(t)}{\left|\underline{x}_Q(t) - \underline{x}_G(t)\right|} \; ; \qquad \left|\underline{u}(t)\right| = 1 \tag{4-143a}$$

---

1) Note that by $\underline{u}$ we do not denote here the Kepler elements as used in the first chapters.

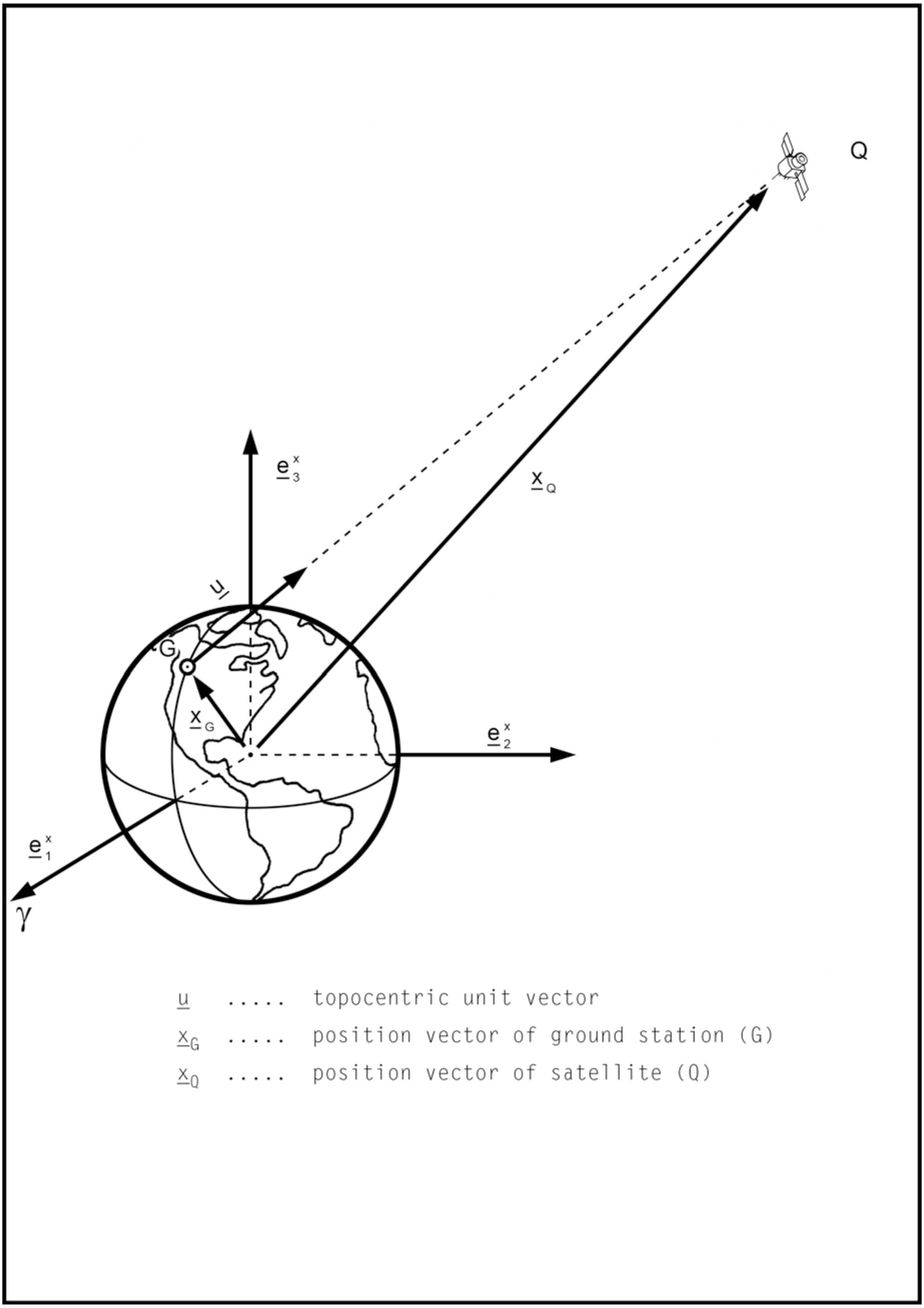


Fig. 2: Direction measurements (Type S1)

In general, $\underline{u}(t)$ is determined photographically with respect to the surrounding stars. In other words, rectascension $\alpha(t)$ and declination $\delta(t)$ with respect to the inertial system defined earlier are computed from the photographic image. Thus we get $\underline{u}(t)$ from

$$\underline{u}(t) = \begin{vmatrix} \cos\delta(t)\ \cos\alpha(t) \\ \cos\delta(t)\ \sin\alpha(t) \\ \delta\ \sin\delta(t)\ \alpha \end{vmatrix} \tag{4-143b}$$

and the inverse relations are found from the three components of $\underline{u}(t)$,

$$\underline{u}(t) = [u_1(t), u_2(t), u_3(t)]^T \tag{4-143c}$$

$$\alpha(t) = \arctan \frac{u_2(t)}{u_1(t)} \tag{4-144a}$$

$$\delta(t) = \arccos \left[u_1^2(t) + u_2^2(t)\right]^{0.5} \tag{4-144b}$$

$$\delta(t) = \arcsin u_3(t) \tag{4-144c}$$

where ($\underline{e}_1, \underline{e}_2, \underline{e}_3$ are the unit vectors forming an orthonormal basis of the inertial system)

$$u_1(t) = \underline{e}_1^T\, \underline{u}(t) \tag{4-145a}$$

$$u_2(t) = \underline{e}_2^T\, \underline{u}(t) \tag{4-145b}$$

$$u_3(t) = \underline{e}_3^T\, \underline{u}(t) \tag{4-145c}$$

$$\delta u_1(t) = \underline{e}_1^T\, \delta\underline{u}(t) \tag{4-146a}$$

$$\delta u_2(t) = \underline{e}_2^T\, \delta\underline{u}(t) \tag{4-146b}$$

$$\delta u_3(t) = \underline{e}_3^T\, \delta\underline{u}(t) \tag{4-146c}$$

For $\delta\underline{u}(t)$ we get by differentiation of (4-143a)

$$\delta\underline{u}(t) = \frac{\partial\underline{u}(t)}{\partial\underline{x}_Q}\, \delta\underline{x}_Q(t) + \frac{\partial\underline{u}(t)}{\partial\underline{x}_G}\, \delta\underline{x}_G(t) \tag{4-147}$$

where the Jacobi matrices are explicitly defined by

$$\frac{\partial \underline{u}}{\partial \underline{x}_Q} = \frac{1}{\left| \underline{x}_Q - \underline{x}_G \right|} \left[\underline{I} - \frac{\left(\underline{x}_Q - \underline{x}_G \right) \left(\underline{x}_Q - \underline{x}_G \right)^T}{\left| \underline{x}_Q - \underline{x}_G \right|^2} \right] \tag{4-148a}$$

$$\frac{\partial \underline{u}}{\partial \underline{x}_G} = - \frac{1}{\left| \underline{x}_Q - \underline{x}_G \right|} \left[\underline{I} - \frac{\left(\underline{x}_Q - \underline{x}_G \right) \left(\underline{x}_Q - \underline{x}_G \right)^T}{\left| \underline{x}_Q - \underline{x}_G \right|^2} \right] \tag{4-148a}$$

Using (4-142a,c) we get for (4-147)

$$\frac{\partial \underline{u}}{\partial \underline{x}_G} = - \frac{\partial \underline{u}}{\partial \underline{x}_Q} \tag{4-148c}$$

and

$$\delta \underline{u}(t) = \frac{\partial \underline{u}(t)}{\partial \underline{x}_Q(t)} \left\{ \left[\frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \frac{\partial \underline{v}(t)}{\partial \underline{p}} - \frac{\partial \underline{x}_G(t)}{\partial \underline{p}} \right] \delta \underline{p} - \right.$$

$$- \underline{R}(t) \, \delta \underline{y}_G +$$

$$\left. + \frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \int_{t_0}^{t} \underline{Y}(t) \, \delta \underline{g}(t) \, dt \right\} \tag{4-149}$$

The observation equation for $\alpha(t)$ and $\delta(t)$ are derived by decomposing the corresponding quantities into approximate values $\alpha^o(t)$, $\delta^o(t)$, and linear variations $\delta\alpha(t)$ and $\delta\delta(t)$

$$\alpha(t) = \alpha^o(t) + \delta\alpha(t) \tag{4-150a}$$

$$\delta(t) = \delta^o(t) + \delta\delta(t) \tag{4-150b}$$

where the approximate values are given by

$$\alpha^o(t) = \arctan \frac{u_2^o(t)}{u_1^o(t)} \tag{4-151a}$$

$$\delta^o(t) = \arcsin u_3^o(t) \tag{4-151b}$$

and $\delta\alpha(t)$, $\delta\delta(t)$ by

$$\delta\alpha(t) = \frac{1}{\cos\delta(t)} \left[\cos\alpha(t)\, \underline{e}_2^T - \sin\alpha(t)\, \underline{e}_1^T\right] \delta\underline{u}(t) \tag{4-152a}$$

$$\delta\delta(t) = -\frac{1}{\sin\delta(t)} \left[\cos\alpha(t)\, \underline{e}_1^T + \sin\alpha(t)\, \underline{e}_2^T\right] \delta\underline{u}(t) \tag{4-152b}$$

Inserting (4-149) into (4-152a,b) yields the final form of the observation equations

$$\begin{aligned}\delta\alpha(t) = \; & \frac{\cos\alpha(t)\,\underline{e}_2^T - \sin\alpha(t)\,\underline{e}_1^T}{\left|\underline{x}_Q - \underline{x}_G\right| \cos\delta(t)} \frac{\partial\underline{u}(t)}{\partial\underline{x}_Q} \cdot \\ & \cdot \left\{\left[\frac{\partial\underline{x}_Q(t)}{\partial\underline{v}} \frac{\partial\underline{v}(t)}{\partial\underline{p}} - \frac{\partial\underline{x}_G(t)}{\partial\underline{p}}\right] \delta\underline{p} - \right. \\ & - \underline{R}(t)\, \delta\underline{y}_G + \\ & \left. + \frac{\partial\underline{x}_Q(t)}{\partial\underline{v}} \int_{t_0}^{t} \underline{Y}(t)\, \delta\underline{g}(t)\, dt \right\}\end{aligned} \tag{4-153a}$$

$$\begin{aligned}\delta\delta(t) = - & \frac{\cos\alpha(t)\,\underline{e}_1^T - \sin\alpha(t)\,\underline{e}_2^T}{\left|\underline{x}_Q - \underline{x}_G\right| \sin\delta(t)} \frac{\partial\underline{u}(t)}{\partial\underline{x}_Q} \cdot \\ & \cdot \left\{\left[\frac{\partial\underline{x}_Q(t)}{\partial\underline{v}} \frac{\partial\underline{v}(t)}{\partial\underline{p}} - \frac{\partial\underline{x}_G(t)}{\partial\underline{p}}\right] \delta\underline{p} - \right. \\ & - \underline{R}(t)\, \delta\underline{y}_G + \\ & \left. + \frac{\partial\underline{x}_Q(t)}{\partial\underline{v}} \int_{t_0}^{t} \underline{Y}(t)\, \delta\underline{g}(t)\, dt \right\}\end{aligned} \tag{4-153b}$$

Type S2: S ... *distance measurements from a terrestrial ground station to a satellite*

The distance $S(t)$, usually observed by laser, is given by the norm of the difference of the corresponding position vectors $\underline{x}_G(t)$, $\underline{x}_Q(t)$.

$$S(t) = \left| \underline{x}_Q(t) - \underline{x}_G(t) \right| \tag{4-154}$$

Decomposing $S(t)$ by

$$S(t) = S^o(t) + \delta S(t) \tag{4-155}$$

where the approximate value $S^o(t)$ is given by

$$S^o(t) = \left| \underline{x}_Q^o(t) - \underline{x}_G^o(t) \right| \tag{4-156}$$

results in the linear part $\delta S(t)$

$$\delta S(t) = \frac{\partial S(t)}{\partial \underline{x}_Q} \delta \underline{x}_Q(t) + \frac{\partial S(t)}{\partial \underline{x}_G} \delta \underline{x}_G(t) \tag{4-157}$$

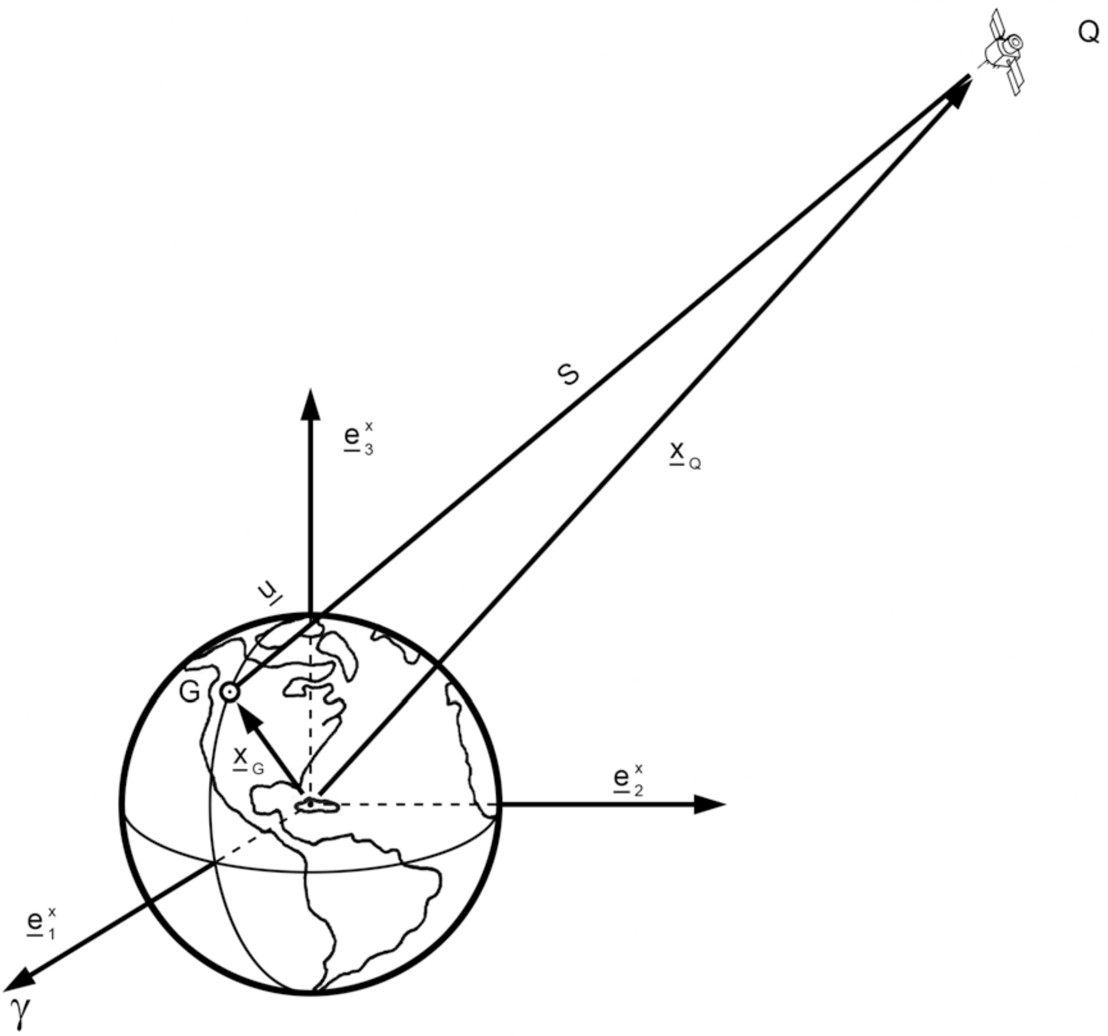
where the gradients explicitly defined by

$$\frac{\partial S}{\partial \underline{x}_Q} = \frac{\left(\underline{x}_Q - \underline{x}_G \right)^T}{\left| \underline{x}_Q - \underline{x}_G \right|} = \frac{\left(\underline{x}_Q - \underline{x}_G \right)^T}{S} \tag{4-158a}$$

$$\frac{\partial S}{\partial \underline{x}_G} = \frac{\left(\underline{x}_Q - \underline{x}_G \right)^T}{\left| \underline{x}_Q - \underline{x}_G \right|} = - \frac{\left(\underline{x}_Q - \underline{x}_G \right)^T}{S} \tag{4-158b}$$

The final form of the linear observation equations is derived by inserting again (4.142a,c) into (4-157).

$$\frac{\partial S}{\partial \underline{x}_Q} = - \frac{\partial S}{\partial \underline{x}_G} \tag{4-158c}$$



S ..... distance from terrestrial ground station to a satellite

$\underline{x}_G$ ..... position vector of ground station (G)

$\underline{x}_Q$ ..... position vector of satellite (Q)

Fig. 3: Distance measurements (Type S2)

$$\delta S(t) = \frac{\partial S(t)}{\partial \underline{x}_Q(t)} \left\{ \left[\frac{\partial \underline{x}_Q(t)}{\partial \underline{y}(t)} \frac{\partial \underline{y}(t)}{\partial \underline{p}} - \frac{\partial x_G(t)}{\partial \underline{p}} \right] \delta \underline{p} - \right.$$

$$- \underline{R}(t)\, \delta \underline{y}_G +$$

$$\left. + \frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \int_{t_0}^{t} \underline{Y}(t)\, \delta \underline{g}(t)\, dt \right\} \qquad (4\text{-}159)$$

Type S3.1: $\Delta f(t)$ ***... Doppler observation of a satellite at a ground station: Doppler frequency shift*** Δf ***(range-rate*** $\dot{S}$ ***)***

With respect to the Doppler frequency shift between a received satellite signal and a reference signal in the receiver of the ground station two observables can be introduced.

In the first the instantaneous frequency shift $\Delta f(t)$ is used which is porportional to the time-derivative $\dot{S}$ of the distance S between ground station and satellite. Neglecting atmospherical time delays of the satellite signal we have the basic relation

$$\Delta f(t) = -\frac{f}{c}\, \dot{S}(t) \qquad (4\text{-}160)$$

where f is the transmitted frequency of the satellite signal and c is the velocity of light.

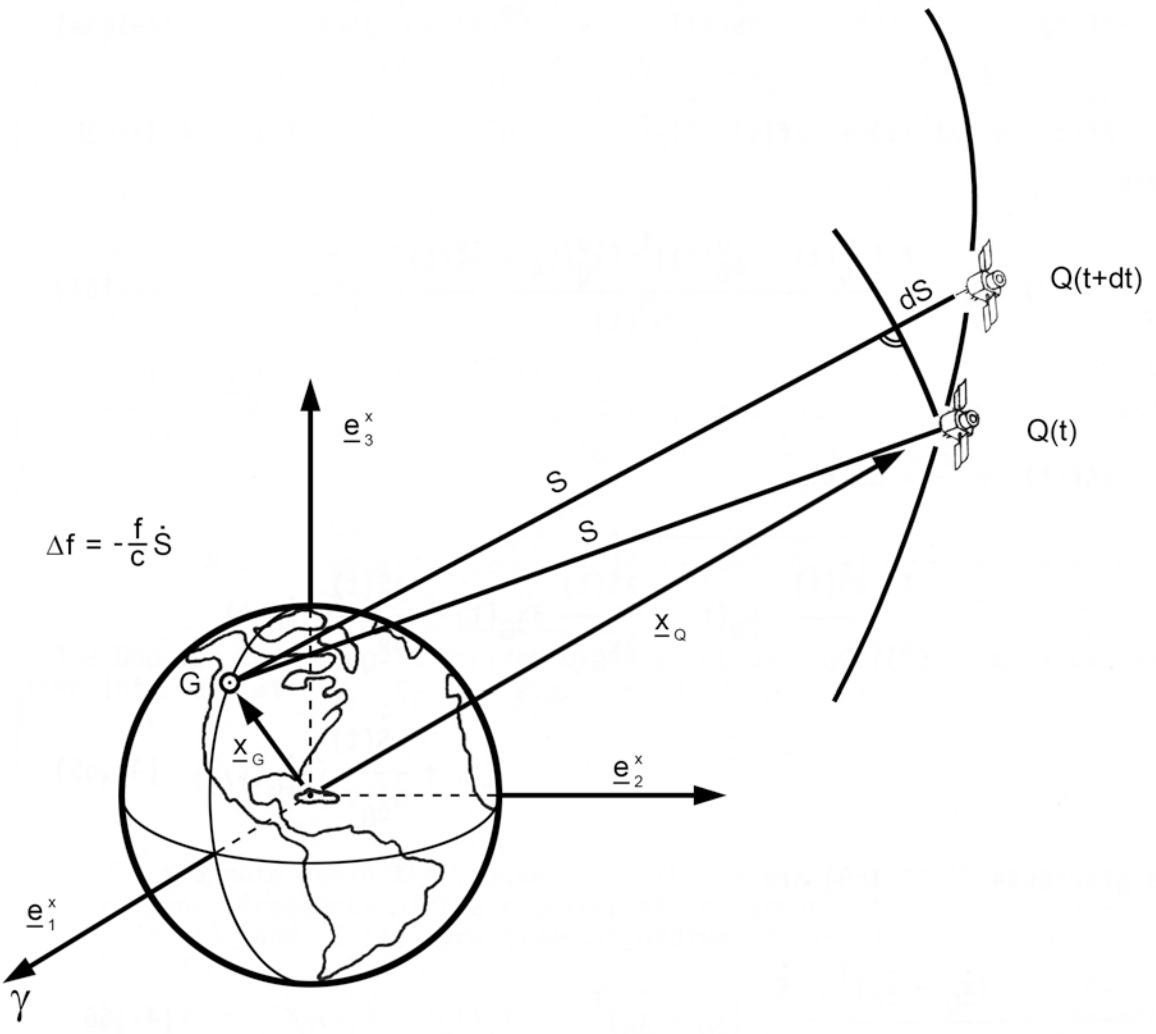
In order to get the linear observation equation of (4-160) we have to determine the time-derivative $\dot{S}$ first. For the distance S we have according to (4-154)

$$S(t) = \left| \underline{x}_Q(t) - \underline{x}_G(t) \right| = \left\{ \left[\underline{x}_Q^T(t) - \underline{x}_G^T(t) \right] \left[\underline{x}_Q(t) - \underline{x}_G(t) \right] \right\}^{0.5} \qquad (4\text{-}161a)$$

$$S(t) = \left[\underline{x}_Q^T(t)\, \underline{x}_Q(t) - 2\, \underline{x}_Q^T(t)\, \underline{x}_G(t) + \underline{x}_G^T(t)\, \underline{x}_G(t) \right]^{0.5} \qquad (4\text{-}161b)$$

and for $\dot{S}(t)$

$$\frac{dS}{dt} = \dot{S}(t) = \frac{\left[\underline{x}_Q(t) - \underline{x}_G(t) \right]^T \left[\dot{\underline{x}}_Q(t) - \dot{\underline{x}}_G(t) \right]}{S(t)} \qquad (4\text{-}162)$$



Δf ..... doppler frequency shift

f ..... frequency

c ..... velocity of light

$\dot{S}$ ..... range-rate

$\underline{x}_G$ ..... position vector of ground station (G)

$\underline{x}_Q$ ..... position vector of satellite (Q)

Fig. 4: Doppler frequency shift (Type S3.1)

Decomposing $\dot{S}(t)$ again in an approximate part $\dot{S}(t)$ and a residual $\delta\dot{S}(t)$, and inserting it into (4-160) we get

$$\Delta f(t) = -\frac{f}{c}\left[\dot{S}^o(t)+\delta\dot{S}(t)\right] = -\frac{f}{c}\dot{S}^o(t) - \frac{f}{c}\delta\dot{S}(t) \qquad (4\text{-}163a)$$

or

$$\Delta f(t) = \Delta f^o(t)+\delta\Delta f(t) \qquad (4\text{-}163b)$$

where

$$\Delta f^o(t) = -\frac{f}{c}\frac{\left[\underline{x}_Q^o(t)-\underline{x}_G^o(t)\right]^T\left[\dot{\underline{x}}_Q^o(t)-\dot{\underline{x}}_G^o(t)\right]}{S^o(t)} \qquad (4\text{-}164)$$

and

$$\delta\Delta f(t) = -\frac{f}{c}\delta\dot{S}(t)$$

$$= -\frac{f}{c}\left[\frac{\partial\dot{S}(t)}{\partial\underline{x}_Q}\delta\underline{x}_Q(t)+\frac{\partial\dot{S}(t)}{\partial\underline{x}_G}\delta\underline{x}_G(t)+\frac{\partial\dot{S}(t)}{\partial\dot{\underline{x}}_Q}\delta\dot{\underline{x}}_Q(t)+\right.$$

$$\left.+\frac{\partial\dot{S}(t)}{\partial\dot{\underline{x}}_G}\delta\dot{\underline{x}}_G(t)\right] \qquad (4\text{-}165)$$

The gradients in (4-165) are

$$\frac{\partial\dot{S}}{\partial\underline{x}_Q} = \frac{\left(\dot{\underline{x}}_Q-\dot{\underline{x}}_G\right)^T}{S} - \frac{\dot{S}}{S^2}\left(\underline{x}_Q-\underline{x}_G\right)^T \qquad (4\text{-}166a)$$

$$\frac{\partial\dot{S}}{\partial\underline{x}_G} = -\frac{\left(\dot{\underline{x}}_Q-\dot{\underline{x}}_G\right)^T}{S} + \frac{\dot{S}}{S^2}\left(\underline{x}_Q-\underline{x}_G\right)^T = -\frac{\partial\dot{S}}{\partial\underline{x}_Q} \qquad (4\text{-}166a)$$

$$\frac{\partial\dot{S}}{\partial\dot{\underline{x}}_Q} = \frac{\left(\underline{x}_Q-\underline{x}_G\right)^T}{S} \qquad (4\text{-}166c)$$

$$\frac{\partial\dot{S}}{\partial\dot{\underline{x}}_G} = -\frac{\left(\underline{x}_Q-\underline{x}_G\right)^T}{S} = -\frac{\partial\dot{S}}{\partial\underline{x}_Q} \qquad (4\text{-}166d)$$

Inserting of (4-166a,...,d) and (4-142a,...,d) into (4-165) results in the final observation equation for $\Delta f(t)$

$$\delta\Delta f(t) =$$

$$= -\frac{f}{c}\left\{\frac{\partial\dot{S}(t)}{\partial\underline{x}_Q(t)}\left[\frac{\partial\underline{x}_Q(t)}{\partial\underline{v}(t)}\frac{\partial\underline{v}(t)}{\partial\underline{p}} - \frac{\partial\underline{x}_G(t)}{\partial\underline{p}}\right] + \frac{\partial\dot{S}(t)}{\partial\underline{x}_Q(t)}\left[\frac{\partial\dot{\underline{x}}_Q(t)}{\partial\underline{v}(t)}\frac{\partial\underline{v}(t)}{\partial\underline{p}} - \frac{\partial\dot{\underline{x}}_G(t)}{\partial\underline{p}}\right]\right\}\delta\underline{p} +$$

$$+\frac{f}{c}\left[\frac{\partial\dot{S}(t)}{\partial\underline{x}_Q(t)}\underline{R}(t) + \frac{\partial\dot{S}(t)}{\partial\dot{\underline{x}}_Q(t)}\dot{\underline{R}}(t)\right]\delta\underline{y}_G -$$

$$-\frac{f}{c}\left[\frac{\partial\dot{S}(t)}{\partial\underline{x}_Q(t)}\frac{\partial\underline{x}_Q(t)}{\partial\underline{v}(t)} + \frac{\partial\dot{S}(t)}{\partial\dot{\underline{x}}_Q(t)}\frac{\partial\dot{\underline{x}}_Q(t)}{\partial\underline{v}(t)}\right]\int_{t_0}^{t}\underline{Y}(t)\,\delta\underline{g}(t)\,dt \qquad (4\text{-}167)$$

Type S3.2: B ... ***Doppler observation of a satellite at a ground station: Doppler count*** B

The Doppler count B is derived by integration of $\Delta f(t)$ over a certain time interval $\Delta t = t_2 - t_1$ ***(see e.g. Arnold, 1970, p. 75)***

$$B = (f_0 - f)(t_2 - t_1) + \frac{f}{c}(S_2 - S_1) \qquad (4\text{-}168)$$

By f we denote again the frequency of the satellite signal and by f_0 the reference frequency of the receiver at the gropund station. The distances $S_1(t)$ and $S_2(t)$ are given according to (4-154) by

$$S_1(t_1) = \left|\underline{x}_Q(t_1) - \underline{x}_G(t_1)\right| \qquad (4\text{-}169a)$$

$$S_2(t_2) = \left|\underline{x}_Q(t_2) - \underline{x}_G(t_2)\right| \qquad (4\text{-}169b)$$

Decomposing (4-168) into

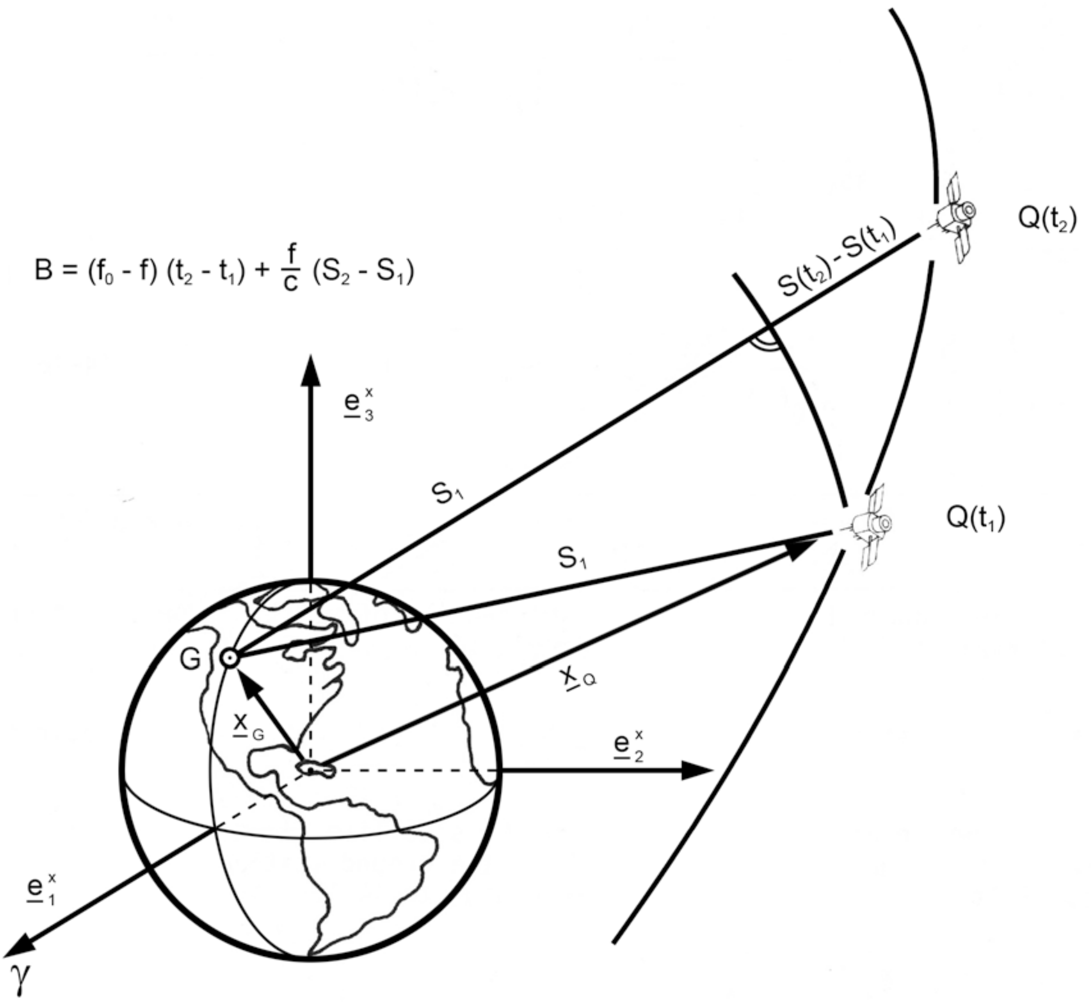
$$B = B^o + \delta B \qquad (4\text{-}170)$$

where the approximate value B^o of the Doppler count is given by

$$B^o = (f_0 - f)(t_2 - t_1) + \frac{f}{c}\left(S_2^o - S_1^0\right) \qquad (4\text{-}171a)$$

$$S_1^0 = \left|\underline{x}_Q^o(t_1) - \underline{x}_G^o(t_1)\right| \qquad (4\text{-}171b)$$

$$S_2^0 = \left|\underline{x}_Q^o(t_2) - \underline{x}_G^o(t_2)\right| \qquad (4\text{-}171c)$$



B	.....	doppler count
f	.....	signal frequency
f_0	.....	receiver reference frequency
t_1, t_2	.....	time epochs
c	.....	velocity of light
$\underline{x}_G$	.....	position vector of ground station (G)
$\underline{x}_Q$	.....	position vector of satellite (Q)

Fig. 5: Doppler count (Type S3.2)

results in the final observation equation for the residual δB when considering (4-159) and neglecting atmospherical parameters.

$$\delta B = \frac{f}{c}(\delta S_2 - \delta S_1) \tag{4-172}$$

$$\begin{aligned} \delta B = \frac{f}{c} \Bigg\{ & \frac{\partial S(t_2)}{\partial \underline{x}_Q(t_2)} \left[\frac{\partial \underline{x}_Q(t_2)}{\partial \underline{v}(t_2)} \frac{\partial \underline{v}(t_2)}{\partial \underline{p}} - \frac{\partial \underline{x}_G(t_2)}{\partial \underline{p}} \right] - \\ & - \frac{\partial S(t_1)}{\partial \underline{x}_Q(t_1)} \left[\frac{\partial \underline{x}_Q(t_1)}{\partial \underline{v}(t_1)} \frac{\partial \underline{v}(t_1)}{\partial \underline{p}} - \frac{\partial \underline{x}_G(t_1)}{\partial \underline{p}} \right] \Bigg\} \delta \underline{p} + \\ & + \frac{f}{c} \left[\frac{\partial S(t_1)}{\partial \underline{x}_Q(t_1)} \underline{R}(t_1) - \frac{\partial S(t_2)}{\partial \underline{x}_Q(t_2)} \underline{R}(t_2) \right] \delta \underline{y}_G + \\ & + \frac{f}{c} \Bigg[\frac{\partial S(t_2)}{\partial \underline{x}_Q(t_2)} \frac{\partial \underline{x}_Q(t_2)}{\partial \underline{v}(t_2)} \int_{t_0}^{t_2} \underline{Y}(t)\, \delta \underline{g}(t)\, dt - \\ & - \frac{\partial S(t_1)}{\partial \underline{x}_Q(t_1)} \frac{\partial \underline{x}_Q(t_1)}{\partial \underline{v}(t_1)} \int_{t_0}^{t_1} \underline{Y}(t)\, \delta \underline{g}(t)\, dt \Bigg] \end{aligned} \tag{4-173}$$

Type S4.1: s ... ***Satellite-to-satellite tracking (SST):*** ***Intersatellite laser distances*** s

Doppler observations as well as laser distances cannot only be obtained between a ground station and a satellite, but also between two satellites if appropriate equipment is on board of the satellites. In principle we may have two different observation configurations: high-low and low-low mode depending on the attitude of the two satellites. For satellites in high altitudes the acceleration model, in particular, the acceleration due to the earth's gravity field, can be simplified. Thus, the vector $\delta \underline{p}$ may consist then only of a few coefficients of a spherical harmonics expansion of the earth's gravity potential.

The abobe-mentioned satellite configuration are not distinguished in the formulas given below. In all cases the parameter vector $\delta \underline{p}$ is introduced.

Let $\underline{x}_{Q_1}$ be the position vector of satellite Q_1 and $\underline{x}_{Q_2}$ the corresponding one of satellite Q_2. Then, the distance s between Q_1 and Q_2 is given by (see also (4-154))

$$s(t) = \left| \underline{x}_{Q_2}(t) - \underline{x}_{Q_1}(t) \right| \; . \qquad (4\text{-}174)$$

Linearizing of (4-174) yields

$$s(t) = s^o(t) + \delta s(t) \qquad (4\text{-}175a)$$

where the approximate value s^o of the distance $\overline{Q_1 Q_2}$ is determined by the approximate positions $\underline{x}_{Q_1}$, $\underline{x}_{Q_2}$ of the satellites Q_1 , Q_2 ,

$$s^o(t) = \left| \underline{x}^o_{Q_2}(t) - \underline{x}^o_{Q_1}(t) \right| \; . \qquad (4\text{-}175b)$$

This results in the linear part $\delta s(t)$

$$\delta s(t) = \frac{\partial s(t)}{\partial \underline{x}_{Q_2}} \, \delta \underline{x}_{Q_2}(t) + \frac{\partial s(t)}{\partial \underline{x}_{Q_1}} \, \delta \underline{x}_{Q_1}(t) \qquad (4\text{-}175c)$$

The gradients in (4-175c) are determined analogously to (4-158a,b)

$$\frac{\partial s}{\partial \underline{x}_{Q_2}} = \frac{\left(\underline{x}_{Q_2} - \underline{x}_{Q_1} \right)^T}{S} \qquad (4\text{-}176a)$$

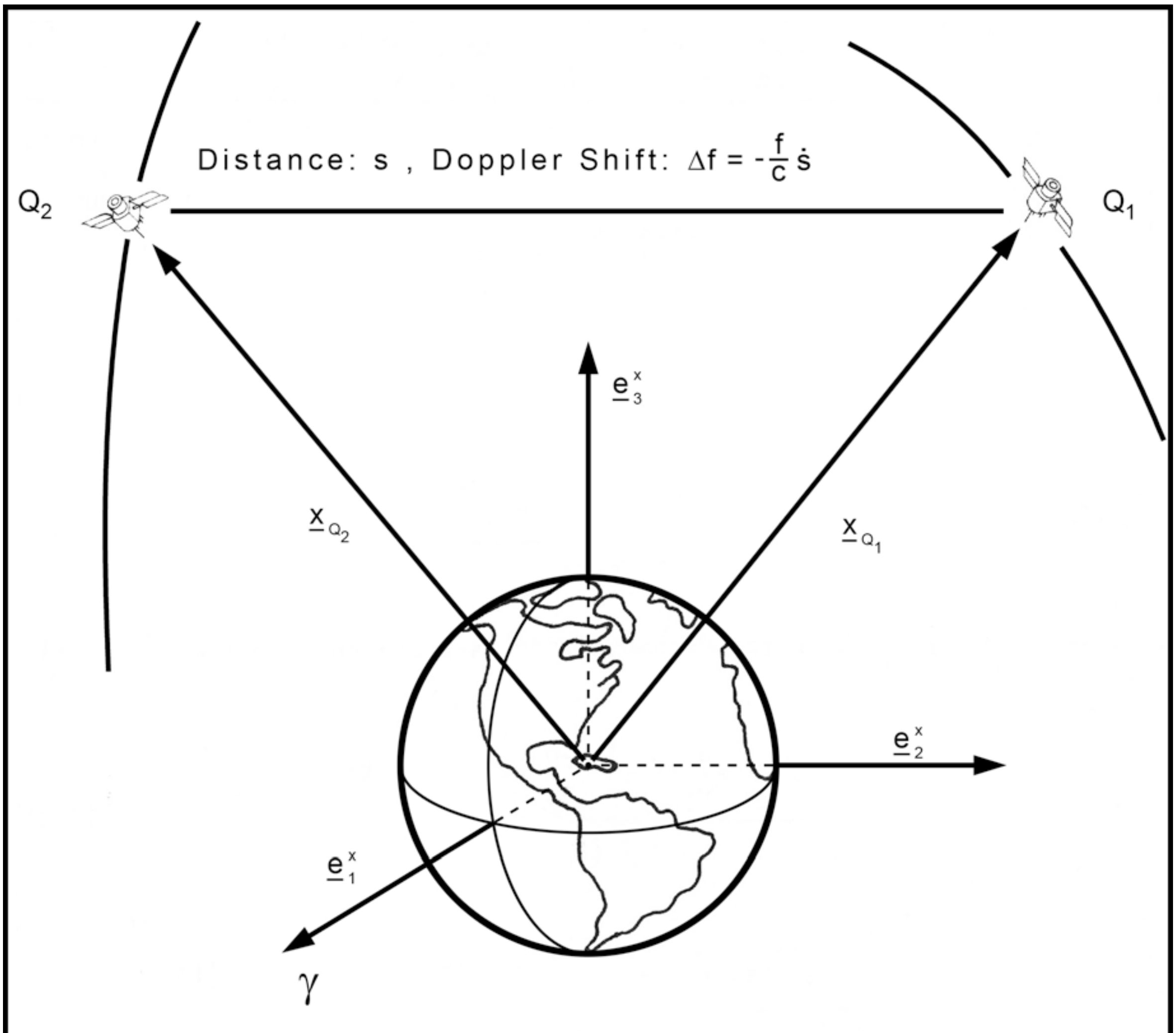
$$\frac{\partial s}{\partial \underline{x}_{Q_1}} = - \frac{\left(\underline{x}_{Q_2} - \underline{x}_{Q_1} \right)^T}{S} \qquad (4\text{-}176b)$$

Thus, the final form of the observation equation for $\delta s(t)$ is found by inserting (4-176a,b) and (4-142a) into (4-175c).

$$\delta s(t) = \frac{\partial s(t)}{\partial \underline{x}_{Q_2}(t)} \left[\frac{\partial \underline{x}_{Q_2}(t)}{\partial \underline{v}_{Q_2}(t)} \frac{\partial \underline{v}_{Q_2}(t)}{\partial \underline{p}} - \frac{\partial \underline{x}_{Q_1}(t)}{\partial \underline{v}_{Q_1}(t)} \frac{\partial \underline{v}_{Q_1}(t)}{\partial \underline{p}} \right] \delta \underline{p} \; +$$

$$+ \frac{\partial s(t)}{\partial \underline{x}_{Q_2}(t)} \left[\frac{\partial \underline{x}_{Q_2}(t)}{\partial \underline{v}_{Q_2}(t)} \int_{t_0}^{t} \underline{Y}_{Q_2}(t) \, \delta \underline{g}_{Q_2}(t) \, dt \; - \right.$$

$$\left. - \frac{\partial \underline{x}_{Q_1}(t)}{\partial \underline{v}_{Q_1}(t)} \int_{t_0}^{t} \underline{Y}_{Q_1}(t) \, \delta \underline{g}_{Q_1}(t) \, dt \right] \qquad (4\text{-}177)$$



- s ..... intersatellite laser distance
- $\dot{s}$ ..... range rate
- Δf ..... doppler shift
- f ..... time epochs
- c ..... velocity of light
- $\underline{x}_{Q_1}$ ..... position vector of satellite (Q_1)
- $\underline{x}_{Q_2}$ ..... position vector of satellite (Q_2)

Fig. 6: Satellite-to-satellite tracking (Type S4.1, S4.2)

Type S4.2: $\Delta f(t)$ ... ***Satellite-to-satellite tracking (SST): Doppler type (range-rate*** $\dot{s}$ ***)***

In order to get the observation equation for the Doppler frequency shift $\Delta f(t)$ observed between two satellites we first derive the time derivative $\dot{s}$ of the distance s between the satellites Q_1 and Q_2 . Analogously to (4-162) we get for the distance s

$$s(t) = \left| \underline{x}_{Q_2}(t) - \underline{x}_{Q_1}(t) \right| \tag{4-178}$$

and for its time-derivative

$$\dot{s}(t) = \frac{ds(t)}{dt} = \frac{\left[\underline{x}_{Q_2}(t) - \underline{x}_{Q_1}(t)\right]^T \left[\underline{\dot{x}}_{Q_2}(t) - \underline{\dot{x}}_{Q_1}(t)\right]}{s(t)} \ . \tag{4-179}$$

Further, in analogy to the decomposition (4-163b) and the formulas (4-165) we can write

$$\Delta f^o(t) = -\frac{f}{c} \frac{\left[\underline{x}_{Q_2}(t) - \underline{x}_{Q_1}(t)\right]^T \left[\underline{\dot{x}}_{Q_2}(t) - \underline{\dot{x}}_{Q_1}(t)\right]}{s^o(t)} \tag{4-180a}$$

$$\delta\Delta f(t) = -\frac{f}{c} \left[\frac{\partial \dot{s}(t)}{\partial \underline{x}_{Q_2}} \delta\underline{x}_{Q_2}(t) + \frac{\partial \dot{s}(t)}{\partial \underline{x}_{Q_1}} \delta\underline{x}_{Q_1}(t) + \frac{\partial \dot{s}(t)}{\partial \underline{x}_{Q_2}} \delta\underline{x}_{Q_2}(t) + \right.$$
$$\left. + \frac{\partial \dot{s}(t)}{\partial \underline{x}_{Q_1}} \delta\underline{x}_{Q_1}(t) \right] \tag{4-180b}$$

where the gradients in (4-180b) are

$$\frac{\partial \dot{s}}{\partial \underline{x}_{Q_2}} = -\frac{\left(\underline{\dot{x}}_{Q_2} - \underline{\dot{x}}_{Q_1}\right)^T}{s_0} - \frac{\dot{s}}{s^2} \left(\underline{x}_{Q_2} - \underline{x}_{Q_1}\right)^T \tag{4-181a}$$

$$\frac{\partial \dot{s}}{\partial \underline{x}_{Q_1}} = -\frac{\left(\underline{\dot{x}}_{Q_2} - \underline{\dot{x}}_{Q_1}\right)^T}{s_0} + \frac{\dot{s}}{s^2} \left(\underline{x}_{Q_2} - \underline{x}_{Q_1}\right)^T = -\frac{\partial \dot{s}}{\partial \underline{x}_{Q_2}} \tag{4-181b}$$

$$\frac{\partial \dot{s}}{\partial \underline{\dot{x}}_{Q_2}} = \frac{\left(\underline{x}_{Q_2} - \underline{x}_{Q_1}\right)^T}{s} \qquad (4\text{-}181c)$$

$$\frac{\partial \dot{s}}{\partial \underline{\dot{x}}_{Q_1}} = -\frac{\left(\underline{x}_{Q_2} - \underline{x}_{Q_1}\right)^T}{s} = -\frac{\partial \dot{s}}{\partial \underline{\dot{x}}_{Q_1}} \qquad (4\text{-}181d)$$

The final observation equation can be derived by inserting of (4-181a,...., d) and (4-142a,b) into (4-180b).

$$\delta\Delta f(t) =$$

$$= -\frac{f}{c}\left\{\frac{\partial \dot{s}(t)}{\partial \underline{x}_{Q_2}(t)}\left[\frac{\partial \underline{x}_{Q_2}(t)}{\partial \underline{v}_{Q_2}(t)}\,\frac{\partial \underline{v}_{Q_2}(t)}{\partial \underline{p}} - \frac{\partial \underline{x}_{Q_1}(t)}{\partial \underline{v}_{Q_1}(t)}\,\frac{\partial \underline{v}_{Q_1}(t)}{\partial \underline{p}}\right] + \right.$$

$$\left. + \frac{\partial \dot{s}(t)}{\partial \underline{\dot{x}}_{Q_2}(t)}\left[\frac{\partial \underline{\dot{x}}_{Q_2}(t)}{\partial \underline{v}_{Q_2}(t)}\,\frac{\partial \underline{v}_{Q_2}(t)}{\partial \underline{p}} - \frac{\partial \underline{\dot{x}}_{Q_1}(t)}{\partial \underline{v}_{Q_1}(t)}\,\frac{\partial \underline{v}_{Q_1}(t)}{\partial \underline{p}}\right]\right\}\delta\underline{p} -$$

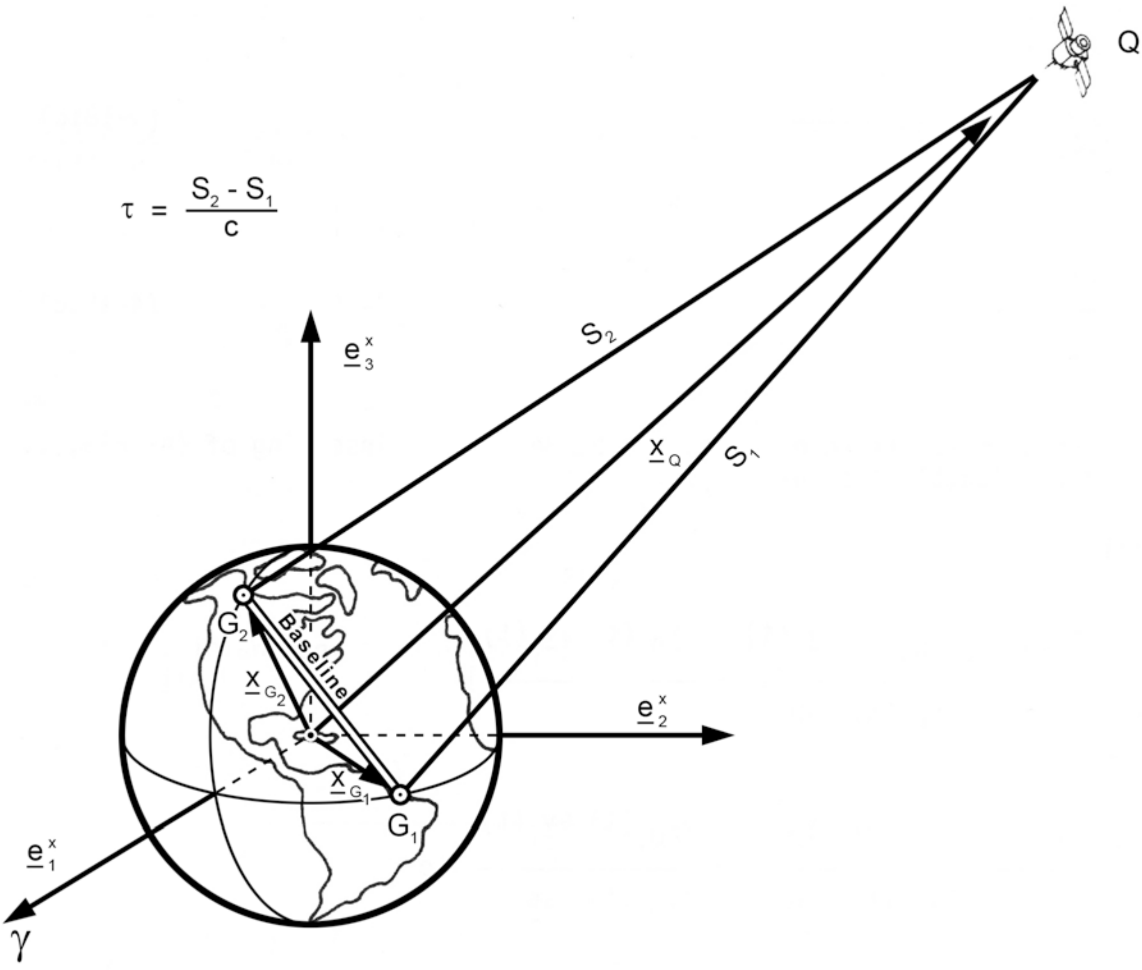
$$- \frac{f}{c}\left[\frac{\partial \dot{s}(t)}{\partial \underline{x}_{Q_2}(t)}\,\frac{\partial \underline{x}_{Q_2}(t)}{\partial \underline{v}_{Q_2}(t)} + \frac{\partial \dot{s}(t)}{\partial \underline{\dot{x}}_{Q_2}(t)}\,\frac{\partial \underline{\dot{x}}_{Q_2}(t)}{\partial \underline{v}_{Q_2}(t)}\right]\int_{t_0}^{t} \underline{Y}_{Q_2}(t)\,\delta\underline{g}_{Q_2}(t)\,dt +$$

$$+ \frac{f}{c}\left[\frac{\partial \dot{s}(t)}{\partial \underline{x}_{Q_2}(t)}\,\frac{\partial \underline{x}_{Q_1}(t)}{\partial \underline{v}_{Q_1}(t)} + \frac{\partial \dot{s}(t)}{\partial \underline{\dot{x}}_{Q_2}(t)}\,\frac{\partial \underline{\dot{x}}_{Q_1}(t)}{\partial \underline{v}_{Q_1}(t)}\right]\int_{t_0}^{t} \underline{Y}_{Q_1}(t)\,\delta\underline{g}_{Q_1}(t)\,dt \qquad (4\text{-}182)$$

Type S5.1: τ ... Interferometric time delays

In satellite interferometry the observations are taken, in general, at two ground stations G_1 , G_2 observing one satellite Q . Observables can be the time delay τ or the Doppler difference $\dot{\tau}$ by using a comparison between the phase of the transmitted satellite signal with that one of a reference signal generated by a frequency standard in the receiver at the ground stations.

Extensive use of this mode and variations of it is made using the satellites of the ***Global Positioning System (GPS).*** Since the observation equations with regard to GPS are outlined in detail in ***Hein and Eissfeller (1986)*** the main principle is only discussed in this context.



τ	.....	time delay
S_1, S_2	.....	ranges
c	.....	velocity of light
$\underline{x}_{G_1}$	.....	position vector of ground station (G_1)
$\underline{x}_{G_2}$	.....	position vector of ground station (G_2)
$\underline{x}_Q$	.....	position vector of satellite (Q)

Fig. 7: Interferometric time delays (Type S5.1)

The basic relation for interferometric time delays is given by

$$\tau(t) = \frac{s_2(t) - s_1(t)}{c} \tag{4-183}$$

where

$$s_2(t) = \left| \underline{x}_Q(t) - \underline{x}_{G_2}(t) \right| \tag{4-184a}$$

$$s_1(t) = \left| \underline{x}_Q(t) - \underline{x}_{G_1}(t) \right| \tag{4-184b}$$

By t we denote here the time when the satellite signal was transmitted, e.g., it is the time which is used as upper integration limit of the orbit integration. In principle, we have to consider in (4-184a,b) the time offset the signal needs for the distance satellite to ground station, e.g.

$$\underline{x}_{G_1} = \underline{x}_{G_1}(t_1) = \underline{x}_{G_1}\left(t + \frac{s_1}{c}\right) \tag{4-184c}$$

$$\underline{x}_{G_2} = \underline{x}_{G_2}(t_2) = \underline{x}_{G_2}\left(t + \frac{s_2}{c}\right) \tag{4-184d}$$

Using the common linearization principle we get

$$\tau(t) = \tau^o(t) + \delta\tau(t) \tag{4-185a}$$

$$\tau^o(t) = \frac{\left| \underline{x}_Q^o(t) - \underline{x}_{G_2}^o(t) \right| - \left| \underline{x}_Q^o(t) - \underline{x}_{G_1}^o(t) \right|}{c} \tag{4-185b}$$

$$\delta\tau(t) = \frac{\delta s_2(t) - \delta s_1(t)}{c} \tag{4-185c}$$

Using $\delta s_2(t)$ and $\delta s_1(t)$ according to (4-157) we get from (4-185c)

$$\delta\tau(t) = \left[\frac{\partial s_2(t)}{\partial \underline{x}_Q} - \frac{\partial s_1(t)}{\partial \underline{x}_Q} \right] \delta\underline{x}_Q(t) + \frac{\partial s_2(t)}{\partial \underline{x}_{G_2}} \delta\underline{x}_{G_2}(t) - \frac{\partial s_1(t)}{\partial \underline{x}_{G_1}} \delta\underline{x}_{G_1}(t) \tag{4-186}$$

where the gradients are given by

$$\frac{\partial s_1}{\partial \underline{x}_Q} = \frac{\left(\underline{x}_Q - \underline{x}_{G_1} \right)^T}{s_1} \tag{4-187a}$$

$$\frac{\partial s_1}{\partial \underline{x}_{G_1}} = - \frac{\left(\underline{x}_Q - \underline{x}_{G_1}\right)^T}{s_1} = - \frac{\partial s_1}{\partial \underline{x}_{G_1}} \qquad (4\text{-}187b)$$

$$\frac{\partial s_2}{\partial \underline{x}_Q} = \frac{\left(\underline{x}_Q - \underline{x}_{G_2}\right)^T}{s_2} \qquad (4\text{-}187c)$$

$$\frac{\partial s_2}{\partial \underline{x}_{G_2}} = - \frac{\left(\underline{x}_Q - \underline{x}_{G_2}\right)^T}{s_2} = - \frac{\partial s_2}{\partial \underline{x}_{G_2}} \qquad (4\text{-}187d)$$

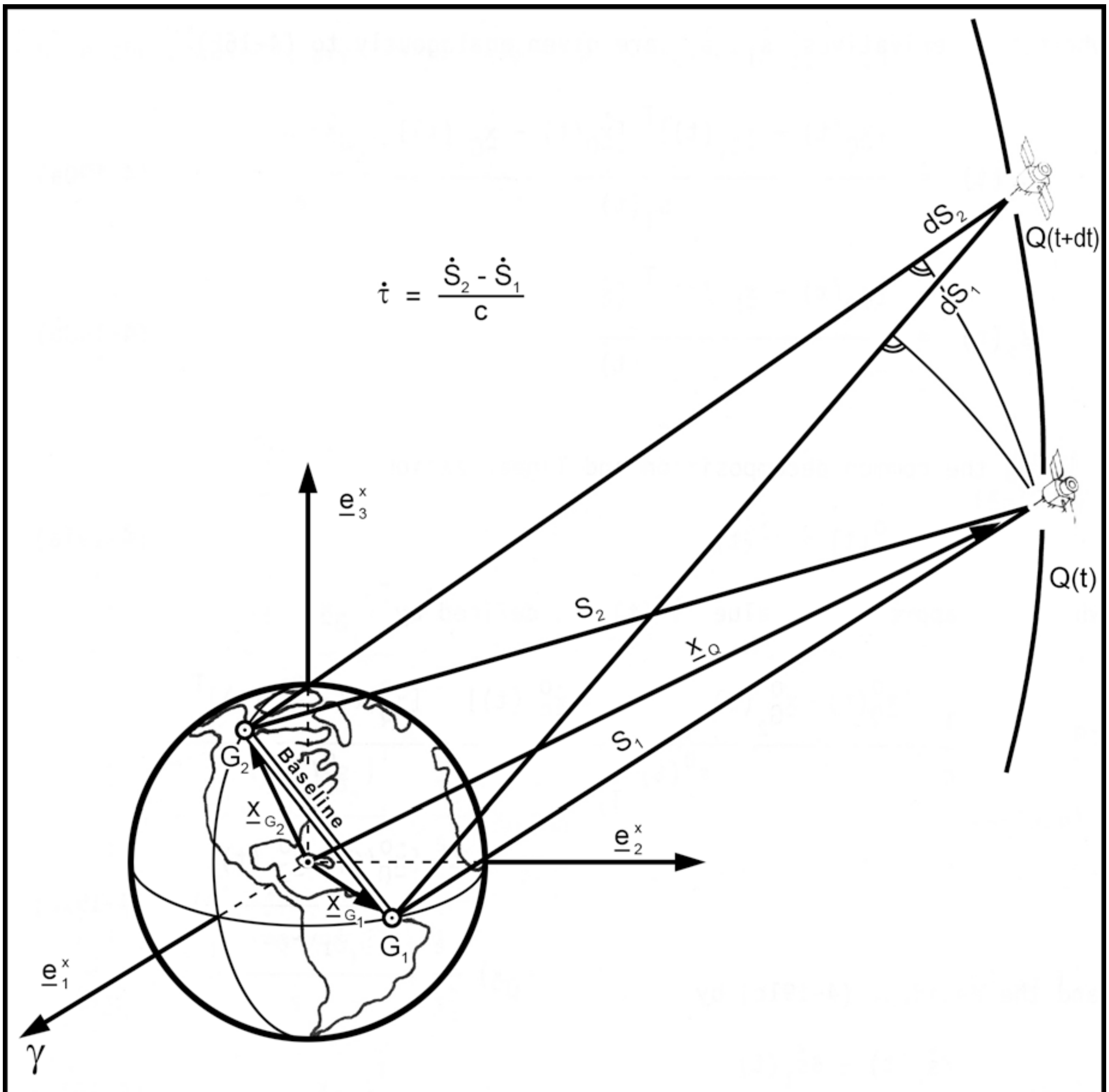
Inserting (4-187a,...,d) and (4-142a,b) into (4-186) yields the final observation equation for interferometric time delays.

$$\begin{aligned}
\delta\tau(t) = \frac{1}{c} \Bigg\{ & \left[\frac{\partial s_2(t)}{\partial \underline{x}_Q(t)} - \frac{\partial s_1(t)}{\partial \underline{x}_Q(t)} \right] \frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \frac{\partial \underline{v}(t)}{\partial \underline{p}} + \\
& + \frac{\partial s_2(t)}{\partial \underline{x}_{G_2}(t)} \frac{\partial \underline{x}_{G_2}(t)}{\partial \underline{p}} - \frac{\partial s_1(t)}{\partial \underline{x}_{G_1}(t)} \frac{\partial \underline{x}_{G_1}(t)}{\partial \underline{p}} \Bigg\} \delta\underline{p} + \\
& + \frac{1}{c} \left[\frac{\partial s_2(t)}{\partial \underline{x}_{G_2}(t)} \underline{R}(t)\, \delta\underline{y}_{G_2} - \frac{\partial s_1(t)}{\partial \underline{x}_{G_1}(t)} \underline{R}(t)\, \delta\underline{y}_{G_1} \right] + \\
& + \frac{1}{c} \left[\frac{\partial s_2(t)}{\partial \underline{x}_Q(t)} - \frac{\partial s_1(t)}{\partial \underline{x}_Q(t)} \right] \frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \int_{t_0}^{t} \underline{Y}(t)\, \delta\underline{g}(t)\, dt
\end{aligned} \qquad (4\text{-}188)$$

Type S5.2: $\dot{\tau}$ ***... Differenced interferometric time delays (Doppler differences)***

The non-linear observation equation for Doppler differences is derived by differentiation of (4-183) with respect to time,

$$\dot{\tau}(t) = \frac{\dot{s}_2(t) - \dot{s}_1(t)}{c} \qquad (4\text{-}189)$$



$\dot{\tau}$ ..... differencedtime delay

$\dot{S}_1, \dot{S}_2$ ..... range-rates

c ..... velocity of light

$\underline{x}_{G_1}, \underline{x}_{G_2}$ ..... position vectors of ground stations (G_1, G_2)

$\underline{x}_Q$ ..... position vector of satellite (Q)

Fig. 8: Differenced interferometric time delays (Type S5.2)

where the derivatives $\dot{s}_1$, $\dot{s}_2$ are given analogously to (4-162)

$$\dot{s}_1(t) = \frac{\left[\underline{x}_Q(t) - \underline{x}_{G_1}(t)\right]^T \left[\underline{\dot{x}}_Q(t) - \underline{\dot{x}}_{G_1}(t)\right]}{s_1(t)} \tag{4-190a}$$

$$\dot{s}_2(t) = \frac{\left[\underline{x}_Q(t) - \underline{x}_{G_2}(t)\right]^T \left[\underline{\dot{x}}_Q(t) - \underline{\dot{x}}_{G_2}(t)\right]}{s_2(t)} \tag{4-190b}$$

Using the common decomposition and linearization we get

$$\dot{\tau}(t) = \dot{\tau}^o(t) + \delta\dot{\tau}(t) \tag{4-191a}$$

where the approximate value $\dot{\tau}^o(t)$ is defined by

$$\dot{\tau}(t) = \frac{1}{c}\left\{\frac{\left[\underline{x}^o_Q(t) - \underline{x}^o_{G_2}(t)\right]^T \left[\underline{\dot{x}}^o_Q(t) - \underline{\dot{x}}^o_{G_2}(t)\right]}{s^o_2(t)} - \frac{\left[\underline{x}^o_Q(t) - \underline{x}^o_{G_1}(t)\right]^T}{s^o_1(t)} \cdot \frac{\left[\underline{\dot{x}}^o_Q(t) - \underline{\dot{x}}^o_{G_1}(t)\right]}{s^o_1(t)}\right\} \tag{4-191b}$$

and the residual (4-191c) by

$$\delta\dot{\tau}(t) = \frac{\delta\dot{s}_2(t) - \delta\dot{s}_1(t)}{c} \tag{4-191c}$$

Inserting $\delta\dot{s}_1$, $\delta\dot{s}_2$ analogousloy to (4-180b) into (4-191c) results in

$$\delta\dot{\tau}(t) = \frac{1}{c}\left\{\left[\frac{\partial\dot{s}_2(t)}{\partial\underline{x}_Q} - \frac{\partial\dot{s}_1(t)}{\partial\underline{x}_Q}\right]\delta\underline{x}_Q(t) + \left[\frac{\partial\dot{s}_2(t)}{\partial\underline{\dot{x}}_Q} - \frac{\partial\dot{s}_1(t)}{\partial\underline{\dot{x}}_Q}\right]\delta\underline{\dot{x}}_Q(t) + \right.$$
$$\left. + \frac{\partial\dot{s}_2(t)}{\partial\underline{x}_{G_2}}\delta\underline{x}_{G_2}(t) + \frac{\partial\dot{s}_2(t)}{\partial\underline{\dot{x}}_{G_2}}\delta\underline{\dot{x}}_{G_2}(t) - \frac{\partial\dot{s}_1(t)}{\partial\underline{x}_{G_1}}\delta\underline{x}_{G_1}(t) - \frac{\partial\dot{s}_1(t)}{\partial\underline{\dot{x}}_{G_1}}\delta\underline{\dot{x}}_{G_1}(t)\right\} \tag{4-192}$$

where the gradients are defined analogously to (4-181a,...,d)

$$\frac{\partial \dot{s}_2}{\partial \underline{x}_Q} = \frac{\left(\underline{\dot{x}}_Q - \underline{\dot{x}}_{G_2}\right)^T}{s_2} - \frac{\dot{s}_2}{s_2^2}\left(\underline{x}_Q - \underline{x}_{G_2}\right)^T \qquad (4\text{-}193a)$$

$$\frac{\partial \dot{s}_1}{\partial \underline{x}_Q} = \frac{\left(\underline{\dot{x}}_Q - \underline{\dot{x}}_{G_1}\right)^T}{s_1} - \frac{\dot{s}_1}{s_1^2}\left(\underline{x}_Q - \underline{x}_{G_1}\right)^T \qquad (4\text{-}193b)$$

$$\frac{\partial \dot{s}_2}{\partial \underline{\dot{x}}_Q} = \frac{\left(\underline{x}_Q - \underline{x}_{G_2}\right)^T}{s_2} \qquad (4\text{-}193c)$$

$$\frac{\partial \dot{s}_1}{\partial \underline{\dot{x}}_Q} = \frac{\left(\underline{x}_Q - \underline{x}_{G_1}\right)^T}{s_1} \qquad (4\text{-}193d)$$

$$\frac{\partial \dot{s}_2}{\partial \underline{x}_{G_2}} = - \frac{\left(\underline{\dot{x}}_Q - \underline{\dot{x}}_{G_2}\right)^T}{s_2} + \frac{\dot{s}_2}{s_2^2}\left(\underline{x}_Q - \underline{x}_{G_2}\right)^T \qquad (4\text{-}193e)$$

$$\frac{\partial \dot{s}_1}{\partial \underline{x}_{G_1}} = - \frac{\left(\underline{\dot{x}}_Q - \underline{\dot{x}}_{G_1}\right)^T}{s_1} + \frac{\dot{s}_1}{s_1^2}\left(\underline{x}_Q - \underline{x}_{G_1}\right)^T \qquad (4\text{-}193f)$$

$$\frac{\partial \dot{s}_2}{\partial \underline{\dot{x}}_{G_1}} = - \frac{\left(\underline{x}_Q - \underline{x}_{G_1}\right)^T}{s_1} \qquad (4\text{-}193g)$$

$$\frac{\partial \dot{s}_2}{\partial \underline{\dot{x}}_{G_2}} = - \frac{\left(\underline{x}_Q - \underline{x}_{G_2}\right)^T}{s_2} \qquad (4\text{-}193h)$$

Inserting (4-193a,...,h) and (4-142a,...,d) into (4-192) we derive the final observation equation for Doppler differences $\dot{\tau}$

$$
\begin{aligned}
\delta\dot{\tau}(t) &= \\
&= \frac{1}{c}\left\{ \frac{\partial \dot{s}_2(t)}{\partial \underline{x}_Q(t)} \left[\frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \frac{\partial \underline{v}(t)}{\partial \underline{p}} - \frac{\partial \underline{x}_{G_2}(t)}{\partial \underline{p}} \right] - \frac{\partial \dot{s}_2(t)}{\partial \underline{x}_Q(t)} \left[\frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \frac{\partial \underline{v}(t)}{\partial \underline{p}} - \frac{\partial \underline{x}_{G_2}(t)}{\partial \underline{p}} \right] + \right. \\
&\quad \left. + \frac{\partial \dot{s}_2(t)}{\partial \underline{\dot{x}}_Q(t)} \left[\frac{\partial \underline{\dot{x}}_Q(t)}{\partial \underline{v}(t)} \frac{\partial \underline{v}(t)}{\partial \underline{p}} - \frac{\partial \underline{\dot{x}}_{G_2}(t)}{\partial \underline{p}} \right] - \frac{\partial \dot{s}_1(t)}{\partial \underline{\dot{x}}_Q(t)} \left[\frac{\partial \underline{\dot{x}}_Q(t)}{\partial \underline{v}(t)} \frac{\partial \underline{v}(t)}{\partial \underline{p}} - \frac{\partial \underline{\dot{x}}_{G_1}(t)}{\partial \underline{p}} \right] \right\} \delta\underline{p} + \\
&+ \frac{1}{c} \left[\frac{\partial \dot{s}_2(t)}{\partial \underline{x}_{G_2}} \underline{R}(t) + \frac{\partial \dot{s}_2(t)}{\partial \underline{\dot{x}}_{G_2}} \underline{\dot{R}}(t) \right] \delta\underline{y}_{G_2} - \\
&- \frac{1}{c} \left[\frac{\partial \dot{s}_1(t)}{\partial \underline{x}_{G_1}} \underline{R}(t) + \frac{\partial \dot{s}_1(t)}{\partial \underline{\dot{x}}_{G_1}} \underline{\dot{R}}(t) \right] \delta\underline{y}_{G_1} + \\
&+ \frac{1}{c} \left\{ \left[\frac{\partial \dot{s}_2(t)}{\partial \underline{x}_Q(t)} - \frac{\partial \dot{s}_1(t)}{\partial \underline{x}_Q(t)} \right] \frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} + \left[\frac{\partial \dot{s}_2(t)}{\partial \underline{\dot{x}}_Q(t)} - \frac{\partial \dot{s}_1(t)}{\partial \underline{\dot{x}}_Q(t)} \right] \frac{\partial \underline{\dot{x}}_Q(t)}{\partial \underline{v}(t)} \right\} \int_{t_0}^{t} \underline{Y}(t)\, \delta\underline{g}(t)\, dt
\end{aligned}
\tag{4-194}
$$

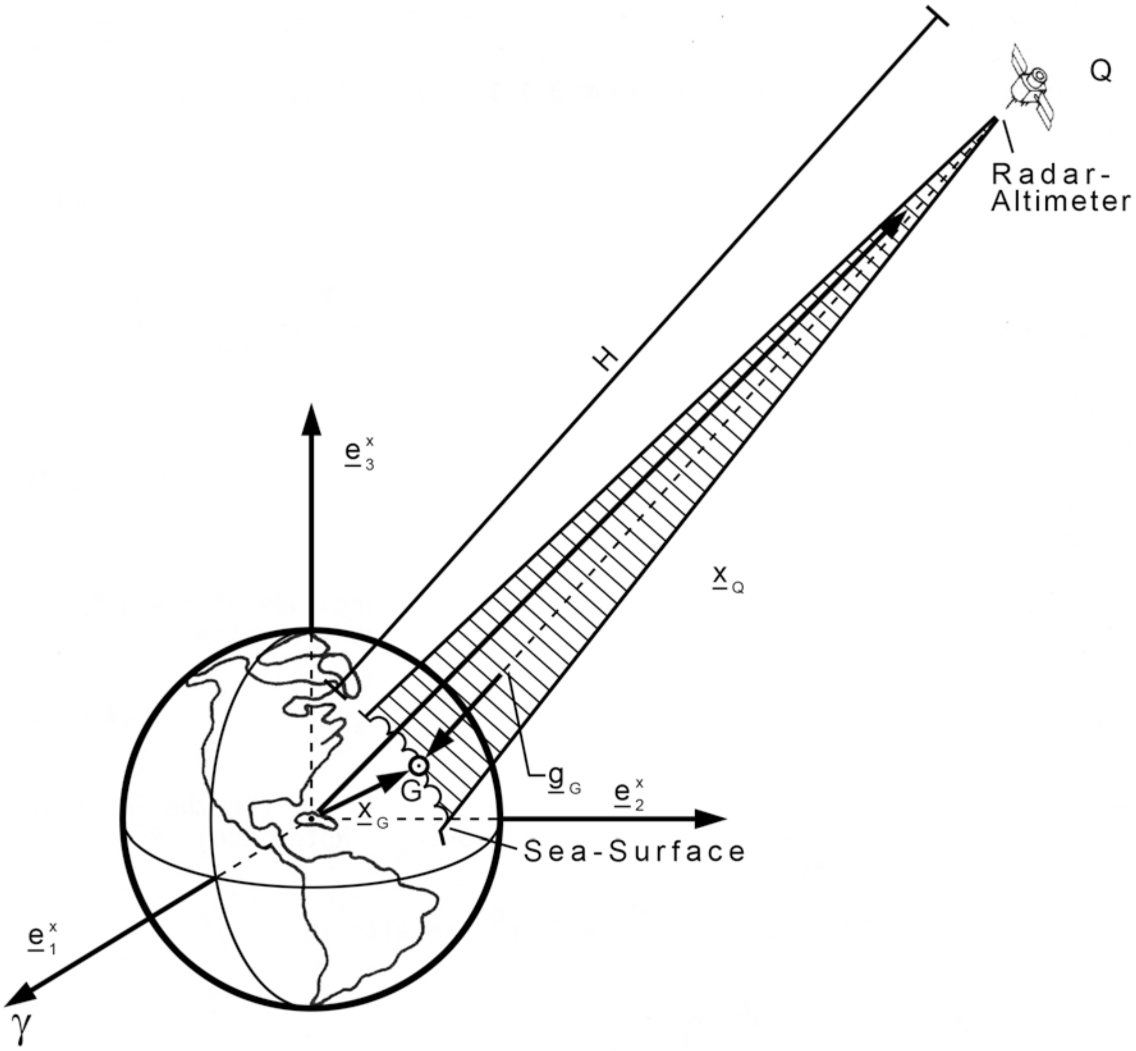
Type S6: H ... *Altimeter measurements*

Assuming that the radar altimeter instrument in the satellite Q is directed parallel to the gravity vector $\underline{g}$ in Q, then the basic observable is the distance satellite Q to the geoid neglecting atmospherical and other instrumental effects as well as sea-surface topography, the deviation of the equipotential surface $W = W_0$ from the sea-surface. In this context we will not discuss the implications due to these effects. The interesting reader finds an extension of the main principle outlined here and a proposal for a vertical datum solution in ***Hein and Eissfeller (1985)***.

We begin with the basic decomposition

$$H(t) = H^o(t) + \delta H(t) \tag{4-195}$$

where $H^o(t)$ is an approximate value and $\delta H(t)$ the corresponding linear variation. The derivation of $\delta H(t)$ is basically done in two steps.
In the first the straight-line condition $\overline{QG}$ with respect to the unit vector $\underline{n}_G$ in direction of the vector in G (G is a point on the geoid $W = W_0$ here) is used. The second step consists then in the linearization of the gravity potential W_0 of the geoid.



H ..... radar-altimeter reading

x_G ..... position vector of sub-satellite point on sea-surface (G)

x_Q ..... position vector of satellite (Q)

g_G ..... gravity vector at sea-surface

Fig. 9: Altimeter measurements (Type S6)

Step 1.

In the inertial system (see paragraph 3.1.1) the following relation holds

$$\underline{x}_G = \underline{x}_Q + H\,\underline{n}_G \tag{4-196}$$

As mentioned above, $\underline{n}_G$ is the unit vector in the altimeter foot point coinciding with the gravity vector at the geoid in the inertial system. In the earth-fixed reference frame $\underline{n}_G$ has the form ***(Hein, 1982a, p. 37)***

$$\underline{n}_G^* = -\begin{bmatrix} \cos\Phi\cos\Lambda \\ \cos\Phi\sin\Lambda \\ \sin\Phi \end{bmatrix} \tag{4-197}$$

where Φ and Λ are the astronomical latitude and longitude of the altimeter foot point. According to (3-5a) we have the transformation

$$\underline{n}_G = \underline{R}(t)\,\underline{n}_G^* \tag{4-198}$$

between $\underline{n}_G^*$ (in the earth-fixed reference frame) and $\underline{n}_G$ (in the inertial system) where $\underline{R}(t)$ is the rotation matrix (3-5c). Note, that $\underline{R}(t)$ is a function of earth rotation $\underline{p}_4$ (4-11d).

Linearizing of (4-196) at $\underline{x}_Q^o$, $\underline{x}_G^o$ and $\underline{p}^o$ results in

$$\delta\underline{x}_G = \delta\underline{x}_Q + \delta H\,\underline{n}_G^o + H^o\,\delta\underline{n}_G \tag{4-199}$$

where the variation $\delta\underline{n}_G$ is given by

$$\delta\underline{n}_G = \frac{\partial\underline{n}_G}{\partial\underline{p}}\,\delta\underline{p} + \underline{R}\,\delta\underline{n}_G^* \; . \tag{4-200}$$

Since $\underline{n}_G$ depends only on the subvector $\underline{p}_4$ of $\underline{p}$ we get for the Jacobi matrix $\partial\underline{n}_G/\partial\underline{p}$

$$\frac{\partial\underline{n}_G}{\partial\underline{p}} = \left[\; \underline{0}\;,\;\underline{0}\;,\;\underline{0}\;,\; \frac{\partial\underline{n}_G}{\partial\underline{p}_4}\;,\;\underline{0} \;\right] \tag{4-201a}$$

where

$$\frac{\partial\underline{n}_G}{\partial\underline{p}_4} = \left[\; \frac{\partial\underline{R}}{\partial\zeta}\,\underline{n}_G^*\;,\; \frac{\partial\underline{R}}{\partial\eta}\,\underline{n}_G^*\;,\; \frac{\partial\underline{R}}{\partial\Theta}\,\underline{n}_G^* \;\right] \tag{4-201b}$$

The partial derivatives of the rotation matrix $\underline{R}$ with respect to the earth rotation parameters are given already by (4-136a,...,c).

Using the approximate values φ^* , λ^* (***normal*** latitude and ***normal*** longitude) for Φ , Λ (***see Hein, 1982a, p. 39,40***) and the corresponding decomposition

$$\Phi = \varphi^* + \delta\Phi \tag{4-202a}$$

$$\Lambda = \lambda^* + \delta\Lambda \tag{4-202b}$$

we get for the variation $\delta\underline{n}_G^* = \underline{n}_G^* - \underline{n}_G^{*o}$

$$\delta\underline{n}_G^* = \underline{n}_\Phi\,\delta\Phi + \underline{n}_\Lambda\,\delta\Lambda \tag{4-203}$$

with

$$\underline{n}_\Phi = \begin{bmatrix} \cos\lambda^* \sin\varphi^* \\ \sin\lambda^* \sin\varphi^* \\ -\cos\varphi^* \end{bmatrix} \tag{4-204a}$$

$$\underline{n}_\Lambda = \begin{bmatrix} \sin\lambda^* \cos\varphi^* \\ -\cos\lambda^* \cos\varphi^* \\ 0 \end{bmatrix} \tag{4-204b}$$

$\underline{n}_\Phi$ and $\underline{n}_\Lambda$ was derived by partial differentiation of (4-197) at (φ^*, λ^*) . The approximate vector $\underline{n}_G^{*o}$ is analogously to (4-197) given by

$$\underline{n}_G^{*o} = -\begin{bmatrix} \cos^0\varphi^* \cos^o\lambda^* \\ \cos^0\varphi^* \sin^o\lambda^* \\ \sin^o\varphi^* \end{bmatrix} \tag{4-205a}$$

and the normal coordinates φ^* , λ^* ***(Hein, 1982a, p. 39, 40)*** by

$$\varphi^* = \arctan \frac{\left(\frac{\partial U^*}{\partial y_3}\right)}{\left[\left(\frac{\partial U^*}{\partial y_1}\right)^2 + \left(\frac{\partial U^*}{\partial y_2}\right)^2\right]^{0.5}} \tag{4-205b}$$

$$\lambda^* = \arctan \frac{\left(\frac{\partial U^*}{\partial y_2}\right)}{\left(\frac{\partial U^*}{\partial y_1}\right)} \tag{4-205c}$$

The star (*) in the formulas above indicates that the quantities refer to the earth-fixed reference frame, where the normal gravity potential is given here by

$$U^* = U_0 + U_1 + Z \tag{4-205d}$$

with $U_0 = kM/r$ and the centrifugal part $Z = 0.5\,\omega^2 r^2 \cos^2\varphi$ (for U_1 see eq. (3-16)). For convenience we use for $\partial U/\partial \underline{y}^*$ the expression

$$\frac{\partial U^*}{\partial \underline{y}} = \left[\frac{\partial U^*}{\partial r} , \frac{\partial U^*}{\partial \varphi} , \frac{\partial U^*}{\partial \lambda} \right] \frac{\partial r}{\partial \underline{y}} \tag{4-205e}$$

since U is usually given in a spherical harmonic series. For the relevant expressions in (4-205e) see (3-15) and (3-17a,...,c). We have further

$$\frac{\partial U_0}{\partial \underline{r}} = \left[-\frac{kM}{r^2} , 0 , 0 \right] \tag{4-205f}$$

$$\frac{\partial Z}{\partial \underline{r}} = \left[\omega^2 r \cos^2\varphi , -0.5\,\omega^2 r^2 \sin 2\varphi , 0 \right] \tag{4-205g}$$

and $\delta\Phi$, $\delta\Lambda$ in (4-203) can be found analogously to ***Hein (1982a, p. 39,40)***

$$\delta\Phi = \underline{\varphi}^T \delta\underline{y}_G + \underline{r}_\Phi^T \underline{t}_G \tag{4-206a}$$

$$\delta\Lambda = \underline{\lambda}^T \delta\underline{y}_G + \underline{r}_\Lambda^T \underline{t}_G \tag{4-206b}$$

where

$$\underline{\varphi}^T = \frac{1}{j} \left[\sin\varphi^* \cos\lambda^* , \sin\varphi^* \sin\lambda^* , -\cos\varphi^* \right] \frac{\partial \underline{j}}{\partial \underline{y}} \tag{4-207a}$$

$$\underline{\lambda}^T = \frac{1}{j \cos\varphi^*} \left[\sin\lambda^* , -\cos\lambda^* , 0 \right] \frac{\partial \underline{j}}{\partial \underline{y}} \tag{4-207b}$$

with

$$\underline{j}^T = \frac{\partial U^*}{\partial \underline{y}} \tag{4-207c}$$

$$j = \left(\underline{j}^T \underline{j} \right)^{0.5} \tag{4-207d}$$

$$\frac{\partial \underline{j}}{\partial \underline{y}} = \operatorname{grad}_y \operatorname{grad}_y^T U^* = \begin{bmatrix} \frac{\partial^2 U^*}{\partial y_1^2} & \frac{\partial^2 U^*}{\partial y_2 \partial y_1} & \frac{\partial^2 U^*}{\partial y_3 \partial y_1} \\ \frac{\partial^2 U^*}{\partial y_1 \partial y_2} & \frac{\partial^2 U^*}{\partial y_2^2} & \frac{\partial^2 U^*}{\partial y_3 \partial y_2} \\ \frac{\partial^2 U^*}{\partial y_1 \partial y_3} & \frac{\partial^2 U^*}{\partial y_2 \partial y_3} & \frac{\partial^2 U^*}{\partial y_3^2} \end{bmatrix} \tag{4-207e}$$

$$\underline{r}_\varphi^T = \frac{1}{j} \left[\sin\varphi^* \cos\lambda^* \ , \ \sin\varphi^* \sin\lambda^* \ , \ -\cos\varphi^* \right] \frac{\partial \underline{\tilde{r}}}{\partial \underline{y}} \tag{4-207f}$$

$$\underline{r}_\lambda^T = \frac{1}{j \cos\varphi^*} \left[\sin\lambda^* \ , \ -\cos\lambda^* \ , \ 0 \right] \frac{\partial \underline{\tilde{r}}}{\partial \underline{y}} \tag{4-207g}$$

$$\underline{t}_G = \left[\frac{\partial T}{\partial r} \ , \ \frac{1}{r} \frac{\partial T}{\partial \varphi} \ , \ \frac{1}{r \cos\varphi} \frac{\partial T}{\partial \lambda} \right]^T \tag{4-207h}$$

$$\frac{\partial \underline{\tilde{r}}}{\partial \underline{y}} = \begin{bmatrix} \cos\varphi \cos\lambda & \cos\varphi \sin\lambda & \sin\varphi \\ -\sin\varphi \cos\lambda & -\sin\varphi \sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \end{bmatrix} \tag{4-207i}$$

Inserting $\delta\Phi$, $\delta\Lambda$ (4-206a,b) in (4-203), we get for the variation $\delta \underline{n}_G^*$

$$\delta \underline{n}_G^* = \underline{N}_1 \delta \underline{y}_G + \underline{N}_2 \underline{t}_G \tag{4-208}$$

where

$$\underline{N}_1 = \underline{n}_\Phi \underline{\varphi}^T + \underline{n}_\Lambda \underline{\lambda}^T \tag{4-208a}$$

$$\underline{N}_2 = \underline{n}_\Phi \underline{r}_\Phi^T + \underline{n}_\Lambda \underline{r}_\Lambda^T \tag{4-208b}$$

Note that the products in (4-208a,b) are vector products (dyadic products). Thus, for $\delta \underline{n}_G$ (4-200) we get the linear form in the vertical system

$$\delta \underline{n}_G = \frac{\partial \underline{n}_G}{\partial \underline{p}} \delta \underline{p} + \underline{R} \, \underline{N}_1 \delta \underline{y}_G + \underline{R} \, \underline{N}_2 \underline{t}_G \tag{4-209}$$

Step 2.

The gravity potential W_0 at the geoid in G is linearized by

$$W_0 = U_G^* + T_G^* = \text{const} \qquad (4\text{-}210)$$

where $U_G^* = U_G^*\left(\underline{p}, \underline{y}_G\right) = U_G^*\left(\underline{p}_2, \underline{\dot{p}}_4, \underline{y}_G\right)$. For $\underline{p}_2$ and $\underline{\dot{p}}_4$ se (4-11c,e). Using the approximate values $\underline{p}_2^o$, $\underline{\dot{p}}_4^o$, $\underline{y}_G^o$ we get by the Taylor series of (4-210)

$$W_0 = U_G^{o*} + \frac{\partial U_G^*}{\partial \underline{p}} \delta \underline{p} + \frac{\partial U_G^*}{\partial \underline{y}_G} \delta \underline{y}_G + T_G \qquad (4\text{-}211)$$

For the determination of $\partial U_G^* / \partial \underline{y}_G$ see (4-205e). $\left(\partial U_G^* / \partial \underline{p}\right)$ is given in consideration of (4-10) and (4-11a,...,d)

$$\frac{\partial U_G^*}{\partial \underline{p}_1} = \underline{0}^T \qquad (4\text{-}212a)$$

$$\frac{\partial U_G^*}{\partial \underline{p}_2} = \left[\frac{\partial U^*}{\partial c_{20}} , \frac{\partial U^*}{\partial c_{21}} , \ldots , \frac{\partial U^*}{\partial c_{nn}} , \frac{\partial U^*}{\partial s_{20}} , \frac{\partial U^*}{\partial s_{21}} , \ldots , \frac{\partial U^*}{\partial s_{nn}} \right]_G \qquad (4\text{-}212b)$$

$$\frac{\partial U_G^*}{\partial \underline{p}_3} = \underline{0}^T \qquad (4\text{-}212c)$$

$$\frac{\partial U_G^*}{\partial \underline{p}_4} = \underline{0}^T \qquad (4\text{-}212d)$$

$$\frac{\partial U_G^*}{\partial \underline{\dot{p}}_4} = \left[0 , 0 , \omega r^2 \cos^2\varphi \right] \qquad (\omega \doteq \Theta) \qquad (4\text{-}212e)$$

For the elements of the vector (4-212b) we find

$$\frac{\partial U^*}{\partial c_{\mu\nu}} = kM \frac{R^{\mu}}{r^{\mu+1}} P_{\mu\nu}(\sin\varphi) \cos\nu\lambda \qquad (4\text{-}213a)$$

$$\frac{\partial U^*}{\partial s_{\mu\nu}} = kM \frac{R^{\mu}}{r^{\mu+1}} P_{\mu\nu}(\sin\varphi) \sin\nu\lambda \tag{4-213b}$$

In order to get $\delta H(t)$ we insert (4-209) into (4-199). Thus, we get

$$\delta \underline{x}_G = \delta \underline{x}_Q + \delta H \, \underline{n}_G^o + H^o \frac{\partial \underline{n}_G}{\partial \underline{p}} \delta \underline{p} + H^o \underline{R} \, \underline{N}_1 \, \delta \underline{y}_G + H^o \underline{R} \, \underline{N}_2 \, \underline{t}_G \tag{4-214}$$

Using further for $\delta \underline{x}_G$ on the left hand side ofe (4-214) the expression (4-134a), results in

$$\frac{\partial \underline{x}_G}{\partial \underline{p}} \delta \underline{p} + \underline{R} \, \delta \underline{y}_G = \delta \underline{x}_Q + \delta H \, \underline{n}_G^o + H^o \frac{\partial \underline{n}_G}{\partial \underline{p}} \delta \underline{p} + H^o \underline{R} \, \underline{N}_1 \, \delta \underline{y}_G + H^o \underline{R} \, \underline{N}_2 \, \underline{t}_G \tag{4-215}$$

Solving (4-215) with respect to $\delta \underline{y}_G$ by left-multiplication with $\underline{R}^T(\underline{R}^T \underline{R}) = \underline{I}$ yields

$$\delta \underline{y}_G = \underline{N}_3 \, \underline{R}^T \delta \underline{x}_Q + \delta H \, \underline{N}_3 \, \underline{R}^T \, \underline{n}_G^o + \underline{N}_3 \, \underline{R}^T \left(H^o \frac{\partial \underline{n}_G}{\partial \underline{p}} - \frac{\partial \underline{x}_G}{\partial \underline{p}} \right) \delta \underline{p} + H^o \underline{N}_3 \, \underline{N}_2 \, \underline{t}_G \tag{4-216a}$$

where

$$\underline{N}_3 = \left(\underline{I} - H^o \underline{N}_1 \right)^{-1} \tag{4-216b}$$

The final form for δH can be derived by rearranging (4-211) and inserting (4-217) into (4-216a,b)

$$W_0 - U_G^{o*} - T_G - \frac{\partial U_G^*}{\partial \underline{p}} \delta \underline{p} = \frac{\partial U_G^*}{\partial \underline{y}_G} \delta \underline{y}_G \tag{4-217}$$

$$\begin{aligned} \tilde{H} \, \delta H = {} & W_0 - U_G^{o*} - \frac{\partial U_G^*}{\partial \underline{y}_G} \underline{N}_3 \, \underline{R}^T \delta \underline{x}_Q + \\ & + \left(\frac{\partial U_G^*}{\partial \underline{y}_G} \underline{N}_3 \, \underline{R}^T \frac{\partial \underline{x}_G}{\partial \underline{p}} - H^o \frac{\partial U_G^*}{\partial \underline{y}_G} \underline{N}_3 \, \underline{R}^T \frac{\partial \underline{n}_G}{\partial \underline{p}} - \frac{\partial U_G^*}{\partial \underline{p}} \right) \delta \underline{p} - \\ & - H^o \frac{\partial U_G^*}{\partial \underline{y}_G} \underline{N}_3 \, \underline{N}_2 \, \underline{t}_G - T_G \end{aligned} \tag{4-218a}$$

where

$$\tilde{H} = \frac{\partial U_G^*}{\partial \underline{y}_G} \underline{N}_3 \underline{R}^T \underline{n}_G^o \tag{4-218b}$$

Furthermore we insert $\delta \underline{x}_Q$ (4-142a) into (4-218a) and consider the time-dependence of the relevant quantities.

$$\begin{aligned}\delta H(t) = \frac{1}{\tilde{H}(t)} \Big\{ & W_0 - U_G^{o*}(t) + \\ & + \Big[\frac{\partial U_G^*(t)}{\partial \underline{y}_G(t)} \underline{N}_3(t) \underline{R}^T(t) \frac{\partial \underline{x}_G(t)}{\partial \underline{p}} - H^o(t) \frac{\partial U_G^*(t)}{\partial \underline{y}_G(t)} \underline{N}_3(t) \underline{R}^T(t) \frac{\partial \underline{n}_G(t)}{\partial \underline{p}} - \\ & - \frac{\partial U_G^*(t)}{\partial \underline{y}_G(t)} \underline{N}_3(t) \underline{R}^T(t) \frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \frac{\partial \underline{v}(t)}{\partial \underline{p}} - \frac{\partial U_G^*(t)}{\partial \underline{y}_G(t)} \Big] \delta \underline{p} - \\ & - \frac{\partial U_G^*(t)}{\partial \underline{y}_G(t)} \underline{N}_3(t) \underline{R}^T(t) \frac{\partial \underline{x}_Q(t)}{\partial \underline{v}(t)} \int_{t_0}^{t} \underline{y}(t) \underline{R}(t) \frac{\partial \tilde{\underline{r}}^T(t)}{\partial \underline{y}_Q(t)} \underline{t}_Q(t)\, dt - \\ & - H^o(t) \frac{\partial U_G^*(t)}{\partial \underline{y}_G^*(t)} \underline{N}_3(t) \underline{N}_2(t) \underline{t}_G(t) - T_G(t) \Big\}\end{aligned} \tag{4-219}$$

Note that all terms of sum in (4-219) are scalars, since $\left(\partial U_G^*/\partial \underline{p}\right)$ are row vectors.

For the modification of the integral kernel in (4-219) the expression (4-142e) was used

The approximate value $H^o(t)$ can be computed from the expression

$$H^o(t) = \left| \underline{x}_Q^o(t) - \underline{x}_G^o(t) \right| \tag{4-220}$$

where $\underline{x}_Q^o(t)$ is, in general, known from the orbit integration.

The position vector $\underline{x}_G^o(t)$ of the altimeter foot point has to be computed iteratively by using the cross-point between the straight line of the altimeter beam and the corresponding geometry belonging to U^* . This means, that in principle the following two expressions (four scalar equations) have to be solved with respect to $\underline{x}_G^o$, H^o .

$$\underline{x}_G^o - \underline{x}_Q^o = H^o \, \underline{n}_G^o\left(\underline{x}_G^o\right) \tag{4-221a}$$

$$W_0 = U_G^*\left(\underline{x}_G^o\right) = \text{const.} \tag{4-221b}$$

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The following rotation matrices refer to a right-handed Cartesian coordinate system. Positive (counter clockwise) rotations are treated here.

(i) Rotation around the x_1-axis

$$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{A-1}$$

(ii) Rotation around the x_2-axis

$$\underline{R}_2(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{A-2}$$

(iii) Rotation around the x_3-axis

$$\underline{R}_3(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A-3}$$

For every rotation matrix $\underline{R}_i(\alpha)$, $i = \{1, 2, 3\}$ holds

$$\underline{R}_i(\alpha)\,\underline{R}_i^T(\alpha) = \underline{R}_i^T(\alpha)\,\underline{R}_i(\alpha) = \underline{I} \tag{A-4a}$$

$$\Leftrightarrow \underline{R}_i^T(\alpha) = \underline{R}_i^{-1}(\alpha)$$

$$\underline{R}_i^T(\alpha) = \underline{R}_i(-\alpha) \tag{A-4b}$$

According to ***Stumpff (1974, p. 567 f.)*** the following propositions hold for canonical transformations:

Let H be the Hamilton function of the unperturbed problem (2-8),

$$H(\underline{x},\underline{y}) = 0.5\left(\underline{y}^T \underline{y}\right) - \frac{\kappa^2}{\left(\underline{x}^T \underline{x}\right)^{0.5}} \tag{B-1}$$

where

$$\underline{x} = [x_1, x_2, x_3]^T \tag{B-1a}$$

$$\underline{y} = [y_1, y_2, y_3]^T \tag{B-1b}$$

$$\underline{y} = \dot{\underline{x}} = [\dot{x}_1, \dot{x}_2, \dot{x}_3]^T \tag{B-1c}$$

$$\kappa^2 = kM \tag{B-1d}$$

Proposition 1. The equations of motion of the mechanical systems described by (B-1) are given then by (B-2a,b), which represent a canonical system of differential equations.

$$\dot{\underline{x}} = \frac{\partial H^T}{\partial \underline{y}} \tag{B-2a}$$

$$\dot{\underline{y}} = -\frac{\partial H^T}{\partial \underline{x}} \tag{B-2b}$$

$$\frac{\partial H}{\partial \underline{y}} = \left[\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \frac{\partial H}{\partial y_3}\right]^T \tag{B-2c}$$

$$\frac{\partial H}{\partial \underline{x}} = \left[\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \frac{\partial H}{\partial x_3}\right]^T \tag{B-2d}$$

Thus, for the Hamilton function (B-1) we have explicitly:

$$\frac{\partial H}{\partial y_i} = y_i \tag{B-3a}$$

$$\frac{\partial H}{\partial x_i} = \kappa^2 \frac{x_i}{\left(\underline{x}^T \underline{x}\right)^{1.5}} \tag{B-3b}$$

Using $y_1 := \dot{x}_1$ and $\dot{y}_1 = \ddot{x}_1$ the ***Kepler problem*** is derived from (B-2a,...,d) and (B-3a,b).

$$\underline{\ddot{x}} = -\kappa^2 \frac{\underline{x}}{(\underline{x}^T \underline{x})^{1.5}} \qquad \text{(B-3c)}$$

Proposition 2. Let $\underline{q} = [q_1, q_2, q_3]^T$ and $\underline{p} = [p_1, p_2, p_3]^T$ a set of six new coordinates the equations of motion have to be transformed, e.g.

$$\underline{x} = \underline{x}(\underline{q}, \underline{p}) \qquad \text{(B-4a)}$$

$$\underline{y} = \underline{y}(\underline{q}, \underline{p}) \qquad \text{(B-4b)}$$

The transformation (B-4a,b) is a ***canonical*** one, if the differential equation system (B-2a,b) is of the form (B-5a,b) with ***unchanged*** Hamilton function H .

$$\underline{\dot{q}} = \frac{\partial H^T}{\partial \underline{p}} \qquad \text{(B-5a)}$$

$$\underline{\dot{p}} = -\frac{\partial H^T}{\partial \underline{q}} \qquad \text{(B-5b)}$$

$$\frac{\partial H}{\partial \underline{p}} = \left[\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \frac{\partial H}{\partial p_3}\right]^T \qquad \text{(B-5c)}$$

$$\frac{\partial H}{\partial \underline{q}} = \left[\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \frac{\partial H}{\partial q_3}\right]^T \qquad \text{(B-5d)}$$

$$\underline{\dot{q}} = [\dot{q}_1, \dot{q}_2, \dot{q}_3]^T \qquad \text{(B-5e)}$$

$$\underline{\dot{p}} = [\dot{p}_1, \dot{p}_2, \dot{p}_3]^T \qquad \text{(B-5f)}$$

The new coordinates $\underline{q}$, $\underline{p}$ are ***canonical*** coordinates.

The canonical transformation described in proposition 2 above has the following properties (a) and (b).

Let $\underline{J}$ be the Jacobi matrix of the coordinate transformation (B-4a,b).

$$\underline{J} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} & \frac{\partial y_1}{\partial q_1} & \frac{\partial y_2}{\partial q_1} & \frac{\partial y_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} & \frac{\partial y_1}{\partial q_2} & \frac{\partial y_2}{\partial q_2} & \frac{\partial y_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} & \frac{\partial y_1}{\partial q_3} & \frac{\partial y_2}{\partial q_3} & \frac{\partial y_3}{\partial q_3} \\ \frac{\partial x_1}{\partial p_1} & \frac{\partial x_2}{\partial p_1} & \frac{\partial x_3}{\partial p_1} & \frac{\partial y_1}{\partial p_1} & \frac{\partial y_2}{\partial p_1} & \frac{\partial y_3}{\partial p_1} \\ \frac{\partial x_1}{\partial p_2} & \frac{\partial x_2}{\partial p_2} & \frac{\partial x_3}{\partial p_2} & \frac{\partial y_1}{\partial p_2} & \frac{\partial y_2}{\partial p_2} & \frac{\partial y_3}{\partial p_2} \\ \frac{\partial x_1}{\partial p_3} & \frac{\partial x_2}{\partial p_3} & \frac{\partial x_3}{\partial p_3} & \frac{\partial y_1}{\partial p_3} & \frac{\partial y_2}{\partial p_3} & \frac{\partial y_3}{\partial p_3} \end{bmatrix} \qquad \text{(B-6)}$$

Inserting the block matrices

$$\frac{\partial \underline{x}}{\partial \underline{q}} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{bmatrix} ; \quad \frac{\partial \underline{x}}{\partial \underline{p}} = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \frac{\partial x_1}{\partial p_3} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \frac{\partial x_2}{\partial p_3} \\ \frac{\partial x_3}{\partial p_1} & \frac{\partial x_3}{\partial p_2} & \frac{\partial x_3}{\partial p_3} \end{bmatrix} \qquad \text{(B-7a,b)}$$

$$\frac{\partial \underline{y}}{\partial \underline{q}} = \begin{bmatrix} \frac{\partial y_1}{\partial q_1} & \frac{\partial y_1}{\partial q_2} & \frac{\partial y_1}{\partial q_3} \\ \frac{\partial y_2}{\partial q_1} & \frac{\partial y_2}{\partial q_2} & \frac{\partial y_2}{\partial q_3} \\ \frac{\partial y_3}{\partial q_1} & \frac{\partial y_3}{\partial q_2} & \frac{\partial y_3}{\partial q_3} \end{bmatrix} ; \quad \frac{\partial \underline{y}}{\partial \underline{p}} = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} & \frac{\partial y_1}{\partial p_3} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} & \frac{\partial y_2}{\partial p_3} \\ \frac{\partial y_3}{\partial p_1} & \frac{\partial y_3}{\partial p_2} & \frac{\partial y_3}{\partial p_3} \end{bmatrix} \qquad \text{(B-7c,d)}$$

in (B-6) we get the abbreviations

$$\underline{J} = \begin{bmatrix} \frac{\partial \underline{x}}{\partial \underline{q}} & \frac{\partial \underline{x}}{\partial \underline{p}} \\ \\ \frac{\partial \underline{y}}{\partial \underline{q}} & \frac{\partial \underline{y}}{\partial \underline{p}} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial \underline{x}^T}{\partial \underline{q}} & \frac{\partial \underline{x}^T}{\partial \underline{p}} \\ \\ \frac{\partial \underline{y}^T}{\partial \underline{q}} & \frac{\partial \underline{y}^T}{\partial \underline{p}} \end{bmatrix} \tag{B-8}$$

$$\underline{J}^T = \begin{bmatrix} \frac{\partial \underline{x}}{\partial \underline{q}} & \frac{\partial \underline{x}}{\partial \underline{p}} \\ \\ \frac{\partial \underline{y}}{\partial \underline{q}} & \frac{\partial \underline{y}}{\partial \underline{p}} \end{bmatrix} \tag{B-9}$$

Property (a)

For the determinant D of $\underline{J}$ and $\underline{J}^T$ of the canonical transformation (B-5a,...,f) holds

$$D = \det \underline{J} = \det \underline{J}^T = 1 \tag{B-10}$$

Property (b)

For the sub-determinants of D , resulting by deleting the corresponding row and column of a certain matrix element, the following rules of computation holds

$$D_{x_k q_i} = \left| \frac{\partial x_k}{\partial q_i} \right| = \frac{\partial y_k}{\partial p_i} \tag{B-11a}$$

$$D_{x_k p_i} = \left| \frac{\partial x_k}{\partial p_i} \right| = -\frac{\partial y_k}{\partial q_i} \tag{B-11b}$$

$$D_{y_k q_i} = \left| \frac{\partial y_k}{\partial q_i} \right| = -\frac{\partial x_k}{\partial p_i} \tag{B-11c}$$

$$D_{y_k p_i} = \left| \frac{\partial y_k}{\partial p_i} \right| = \frac{\partial x_k}{\partial q_i} \tag{B-11d}$$

The partial derivatives between the vertical lines in the mid of (B-11a,...,d) characterize the corresponding matrix elements the sub-determinants refer to.

Thus, the introduction of canonical transformation has according to (B-11a,...,d) the advantage, that the sub-determinants of $\underline{J}$ can be calculated by partial derivates and that non singularities will occur referring to $\det \underline{J} \neq 0$ (B-10).

(a) *Jacobi matrices*

Definition 1

Let $\underline{y} = [y_1, y_2, ..., y_n]^T$ and $\underline{x} = [x_1, x_2, ..., x_m]^T$ be two coordinate vectors of dimension n and m, respectively. With respect to the transformation $y_i = y_i(x_1, x_2, ..., x_m)$ where $i = \{1, ..., n\}$, the following scheme of first order partial derivatives is called the ***Jacobi matrix*** $\underline{J}$ ***of the coordinate transformation.***

$$\underline{J} = \frac{\partial \underline{y}}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots\cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots\cdots & \frac{\partial y_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots\cdots & \frac{\partial y_n}{\partial x_m} \end{bmatrix} \tag{C-1}$$

$$\underline{o}(\underline{J}) = (n, m) \tag{C-2}$$

Definition 2

In case $n = m$, the coordinate transformation is called ***regular***, if

$$\det \underline{J} \neq 0 \tag{C-3a}$$

and ***singular***, if

$$\det \underline{J} = 0 \tag{C-3b}$$

Definition 3

The determinant $\det \underline{J} = \det \underline{J}^T$ is called the functional determinant of the corresponding coordinate transformation.

(b) ***Chain rule for Jacobi matrices***

Given the vector $\underline{f} = \underline{f}\left(\underline{g}\left(\underline{h}(\underline{x})\right)\right)$ where

$$\underline{f} = [f_1, f_2, f_3, \ldots, f_l]^T \quad \text{with} \quad o\left(\underline{f}\right) = (l,1) \tag{C-4a}$$

$$\underline{g} = [g_1, g_2, g_3, \ldots, g_m]^T \quad \text{with} \quad o\left(\underline{g}\right) = (m,1) \tag{C-4b}$$

$$\underline{h} = [h_1, h_2, h_3, \ldots, h_n]^T \quad \text{with} \quad o\left(\underline{h}\right) = (n,1) \tag{C-4c}$$

$$\underline{x} = [x_1, x_2, x_3, \ldots, x_o]^T \quad \text{with} \quad o\left(\underline{x}\right) = (o,1) \tag{C-4d}$$

Then, the Jacobi matrix of $\underline{f}$ with respect to the coordinates $\underline{x}$ is defined by

$$\frac{\partial \underline{f}}{\partial \underline{x}} = \frac{\partial \underline{f}}{\partial \underline{g}} \frac{\partial \underline{g}}{\partial \underline{h}} \frac{\partial \underline{h}}{\partial \underline{x}} \tag{C-5}$$

where the dimensions are

$$o\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right) = (l,o) \tag{C-6a}$$

$$o\left(\frac{\partial \underline{f}}{\partial \underline{g}}\right) = (l,m) \tag{C-6b}$$

$$o\left(\frac{\partial \underline{g}}{\partial \underline{h}}\right) = (m,n) \tag{C-6c}$$

$$o\left(\frac{\partial \underline{h}}{\partial \underline{x}}\right) = (n,o) \tag{C-6d}$$

Proof. For the element $\partial f_i/\partial x_j$ located in row i and column j of $\partial \underline{f}/\partial \underline{x}$, we get by the chain rule of differentiation

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_i}{\partial g_1}\left(\frac{\partial g_1}{\partial h_1}\frac{\partial h_1}{\partial x_j} + \frac{\partial g_1}{\partial h_2}\frac{\partial h_2}{\partial x_j} + \ldots\ldots + \frac{\partial g_1}{\partial h_n}\frac{\partial h_n}{\partial x_j}\right) +$$

$$+ \frac{\partial f_i}{\partial g_2}\left(\frac{\partial g_2}{\partial h_1}\frac{\partial h_1}{\partial x_j} + \frac{\partial g_2}{\partial h_2}\frac{\partial h_2}{\partial x_j} + \ldots\ldots + \frac{\partial g_2}{\partial h_n}\frac{\partial h_n}{\partial x_j}\right) +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \frac{\partial f_i}{\partial g_m}\left(\frac{\partial g_m}{\partial h_1}\frac{\partial h_1}{\partial x_j} + \frac{\partial g_m}{\partial h_2}\frac{\partial h_2}{\partial x_j} + \ldots\ldots + \frac{\partial g_m}{\partial h_n}\frac{\partial h_n}{\partial x_j}\right) \qquad \text{(C-7)}$$

Rewriting (C-7) in vector/matrix notation, we may write

$$\frac{\partial f_i}{\partial x_j} = \left[\frac{\partial f_i}{\partial g_1}, \frac{\partial f_i}{\partial g_2}, \ldots, \frac{\partial f_i}{\partial g_m}\right] \begin{bmatrix} \frac{\partial g_1}{\partial h_1} & \frac{\partial g_1}{\partial h_2} & \cdots & \frac{\partial g_1}{\partial h_n} \\ \frac{\partial g_2}{\partial h_1} & \frac{\partial g_2}{\partial h_2} & \cdots & \frac{\partial g_2}{\partial h_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_m}{\partial h_1} & \frac{\partial g_m}{\partial h_2} & \cdots & \frac{\partial g_m}{\partial h_n} \end{bmatrix} \begin{bmatrix} \frac{\partial h_1}{\partial x_j} \\ \frac{\partial h_2}{\partial x_j} \\ \vdots \\ \frac{\partial h_n}{\partial x_j} \end{bmatrix} \qquad \text{(C-8)}$$

In (C-8) the column vector right of the Jacobi matrix $\partial \underline{g}/\partial \underline{h}$ corresponds just to the vector of column j of the Jacobi matrix $\partial \underline{h}/\partial \underline{x}$, and the row vector left of $\partial \underline{g}/\partial \underline{h}$ corresponds just to vector of row i of the Jacobi matrix $\partial \underline{f}/\partial \underline{g}$.

Rearranging the elements $\partial f_i/\partial x_j$ for $i = \{1,\ldots,l\}$ and $j = \{1,\ldots,o\}$ corresponding to (C-.1), results in (C-5).

(c) ***Gradients***

Definition

Let y be a scalar and $\underline{x} = [x_1, x_2, \ldots, x_n]^T$ be a vector of dimension n . With respect to the scalar function $y = y(x_1, x_2, \ldots, x_n) = y(\underline{x})$ the vector $\text{grad}_x y$ is called the ***gradient of*** y with respect to the coordinates $\underline{x}$.

$$\text{grad}_x^T y(\underline{x}) = \frac{\partial y}{\partial \underline{x}} = \left[\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n}\right] \qquad \text{(C-9)}$$

Thus, every row i of (C-1) can be considered as gradient of the corresponding coordinate y_i .

(d) ***Differentiation of matrix-vector equations***

Given be the linear equation

$$\underline{y}\left(\underline{p}\right) = \underline{A}\left(\underline{p}\right)\ \underline{x}\left(\underline{p}\right) \tag{C-10}$$

defined by

$$\underline{p} = \left[p_1, p_2, \ldots, p_l\right]^T \qquad \text{with} \quad o\left(\underline{p}\right) = (l,1) \tag{C-11a}$$

$$\underline{y} = \left[y_1, y_2, \ldots, y_n\right]^T \qquad \text{with} \quad o\left(\underline{y}\right) = (n,1) \tag{C-11b}$$

$$\underline{x} = \left[x_1, x_2, \ldots, x_m\right]^T \qquad \text{with} \quad o\left(\underline{x}\right) = (m,1) \tag{C-11c}$$

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix} \qquad \text{with} \quad o\left(\underline{A}\right) = (n,m) \tag{C-11d}$$

then the Jacobi matrix $\partial\underline{y}/\partial\underline{p}$ has the form

$$\frac{\partial\underline{y}}{\partial\underline{p}} = \left[\frac{\partial\underline{A}}{\partial p_1}\underline{x}\ ,\ \frac{\partial\underline{A}}{\partial p_2}\underline{x}\ ,\ \frac{\partial\underline{A}}{\partial p_3}\underline{x}\ ,\ \ldots\ ,\ \frac{\partial\underline{A}}{\partial p_l}\underline{x}\right] + \underline{A}\,\frac{\partial\underline{x}}{\partial\underline{p}} \tag{C-12}$$

Proof. Consider the vector of column k of $\partial\underline{y}/\partial\underline{p}$. Thus, we get

$$\frac{\partial\underline{y}}{\partial p_k} = \frac{\partial}{\partial p_k}\left[\underline{A}\left(\underline{p}\right)\ \underline{x}\left(\underline{p}\right)\right] = \frac{\partial\underline{A}}{\partial p_k}\underline{x} + \underline{A}\,\frac{\partial\underline{x}}{\partial p_k} \tag{C-13}$$

Adding the column vectors $\partial\underline{y}/\partial p_k$ to form the matrix $\partial\underline{y}/\partial\underline{p}$, we get using (C-13)

$$\frac{\partial \underline{y}}{\partial \underline{p}} = \left[\frac{\partial \underline{y}}{\partial p_1} , \frac{\partial \underline{y}}{\partial p_2} , \ldots , \frac{\partial \underline{y}}{\partial p_l} \right] =$$

$$= \left[\frac{\partial \underline{A}}{\partial p_1} \underline{x} + \underline{A} \frac{\partial \underline{x}}{\partial p_1} , \frac{\partial \underline{A}}{\partial p_2} \underline{x} + \underline{A} \frac{\partial \underline{x}}{\partial p_2} , \ldots\ldots , \frac{\partial \underline{A}}{\partial p_l} \underline{x} + \underline{A} \frac{\partial \underline{x}}{\partial p_l} \right] =$$

$$= \left[\frac{\partial \underline{A}}{\partial p_1} \underline{x} , \frac{\partial \underline{A}}{\partial p_2} \underline{x} , \ldots\ldots , \frac{\partial \underline{A}}{\partial p_l} \underline{x} \right] + \left[\underline{A} \frac{\partial \underline{x}}{\partial p_1} , \underline{A} \frac{\partial \underline{x}}{\partial p_2} , \ldots\ldots , \underline{A} \frac{\partial \underline{x}}{\partial p_l} \right] =$$

$$= \left[\frac{\partial \underline{A}}{\partial p_1} \underline{x} , \frac{\partial \underline{A}}{\partial p_2} \underline{x} , \ldots\ldots , \frac{\partial \underline{A}}{\partial p_l} \underline{x} \right] + \left[\underline{A} \frac{\partial \underline{x}}{\partial p_1} , \frac{\partial \underline{x}}{\partial p_2} , \ldots\ldots , \frac{\partial \underline{x}}{\partial p_l} \right] =$$

$$= \left[\frac{\partial \underline{A}}{\partial p_1} \underline{x} , \frac{\partial \underline{A}}{\partial p_2} \underline{x} , \ldots\ldots , \frac{\partial \underline{A}}{\partial p_l} \underline{x} \right] + \underline{A} \frac{\partial \underline{x}}{\partial \underline{p}} \qquad \text{(C-14)}$$

Thus, (C-12) is derived.
Considering again (C-10), where now $\underline{p} = \underline{p}(\underline{u})$, we get using (C-5) and (C-14) the Jacobi matrix $\partial \underline{y} / \partial \underline{u}$.

$$\underline{Y}\left(\underline{p}(\underline{u})\right) = \underline{A}\left[\underline{p}(\underline{u})\right] \underline{x}\left[\underline{p}(\underline{u})\right] \qquad \text{(C-15)}$$

$$\frac{\partial \underline{y}}{\partial \underline{u}} = \frac{\partial \underline{y}}{\partial \underline{p}} \frac{\partial \underline{p}}{\partial \underline{u}} = \left[\left(\frac{\partial \underline{A}}{\partial p_1} \underline{x} , \frac{\partial \underline{A}}{\partial p_2} \underline{x} , \ldots , \frac{\partial \underline{A}}{\partial p_e} \underline{x} \right) + \underline{A} \frac{\partial \underline{x}}{\partial \underline{p}} \right] \frac{\partial \underline{p}}{\partial \underline{u}} \qquad \text{(C-16)}$$

In paragraph 3 the reference systems needed in satellite geodesy are discussed. Here the transformation used in 3.1.3 should be summarized. The basic formulas are taken from ***Nagel (1976)***.

In order to derive the transformation between the earth-fixed coordinate system and the inertial system the following rotation matrices are necessary. (For the $\underline{R}_i(\alpha)$ see appendix A.)

(i) Precession matrix $\underline{P}(t)$ ***(Nagel, 1976, p. 76)***

$$\underline{P}(t) = \underline{P}\big(\omega(t), \nu(t), \kappa(t)\big) = \underline{R}_3\big(-\omega(t)\big)\,\underline{R}_2\big(\nu(t)\big)\,\underline{R}_3\big(-\kappa(t)\big) \qquad \text{(D-1)}$$

$$\underline{P}^T\underline{P} = \underline{P}\,\underline{P}^T = \underline{I} \qquad \text{(D-1a)}$$

where $\omega(t)$, $\nu(t)$ and $\kappa(t)$ are the Eulerian precession angles.

(ii) Nutation matrix $\underline{N}(t)$ ***(Nagel, 1976, p. 85)***

$$\underline{N}(t) = \underline{N}\big(\Delta\varepsilon(t), \Delta\psi(t)\big) = \underline{R}_1\big(-\Delta\varepsilon(t)\big)\,\underline{R}_2\big(\Delta\psi(t)\sin\varepsilon(t)\big)\,\underline{R}_3\big(-\Delta\psi(t)\cos\varepsilon(t)\big) \qquad \text{(D-2)}$$

$$\underline{N}^T\underline{N} = \underline{N}\,\underline{N}^T = \underline{I} \qquad \text{(D-2a)}$$

where

$\Delta\varepsilon(t)$ ... nutation in obliquity

$\Delta\psi(t)$ ... nutation in longitude

$\varepsilon(t)$ ... obliquity of the ecliptic

(iii) Earth rotation matrix $\underline{R}_3\big(\Theta(t)\big)$ ***(Nagel, 1976, p. 119, 120)***

The argument of $\underline{R}_3$ is the time Θ of the astronomical meridian of Greenwich. We further have the relation

$$\underline{R}_3^T(\Theta)\,\underline{R}_3(\Theta) = \underline{R}_3(\Theta)\,\underline{R}_3^T(\Theta) = \underline{I} \qquad \text{(D-3)}$$

(iv) Polar motion matrix $\underline{S}(t)$ ***(Nagel, 1976, p. 107)***

$$\underline{S}(t) = \underline{S}\big(\xi(t), \eta(t)\big) = \underline{R}_1\big(\eta(t)\big)\,\underline{R}_2\big(\xi(t)\big) \qquad \text{(D-4)}$$

$$\underline{S}(t) = \begin{bmatrix} 1 & 0 & -\xi(t) \\ 0 & 1 & \eta(t) \\ \xi(t) & -\eta(t) & 1 \end{bmatrix}$$

The pole coordinates $\xi(t)$, $\eta(t)$ are defined as follows: ξ is the tangent at the astronomical meridian of Greenwich passing through the CIO pole with positive axis southwards, η is perpendicular to ξ and perpendicular to the direction geocenter - CIO pole.

The explicit representation of S(t) by (D-4) is derived by linearization of $\underline{R}_1$, $\underline{R}_2$ and multiplication. Because of the orthogonality relation between $\underline{R}_1$ and $\underline{R}_2$ we have

$$\underline{S}^T \underline{S} = \underline{S}\,\underline{S}^T = \underline{I} \qquad \text{(D-4a)}$$

The small angles $\xi(t)$ and $\eta(t)$ represent the position of the instandtaneous earth rotation axis with respect to CIO.

Using the matrices (i) to (iv) the transformation between the coordinates $\underline{x}$ of the inertial system and those of the earth-fixed system $\underline{y}$, see also (3-1) and (3-2), can be built-up.

We still have to define the following parameters:

T_0 ... is the reference epoch of the transformation parameters $\omega, \nu, \kappa, \Delta\varepsilon, \Delta\psi, \varepsilon, \Theta$,

$\underline{x}_0$ ... is the coordinate vector in the ***"mean"*** (→ only precession is considered in the transformation) astronomical system at time T_0 , and

$\underline{x}_t$ ... is the coordinate vector in the ***"true"*** (→precession and nutation is considered in the transformation) astronomical system at time t .

In general, the transformation parameters are referring to different time epochs T . It is, however, possible to transform them in such a way that they refer to T_0 .

(a) ***Transformation*** $\underline{y} \leftrightarrow \underline{x}_t$ ***(Nagel, 1976, p. 101, 119)***

$$\underline{x}_t = \underline{R}_3\left(-\Theta(t)\right)\, \underline{S}\left(\xi(t),\eta(t)\right)\, \underline{y} \qquad \text{(D-5a)}$$

$$\underline{y} = \underline{S}^T\left(\xi(t),\eta(t)\right)\, \underline{R}_3^T\left(-\Theta(t)\right)\, \underline{x}_t \qquad \text{(D-5b)}$$

(b) ***Transformation*** $\underline{x} \leftrightarrow \underline{x}_0$ ***(Nagel, 1976, p. 76, 85)***

$$\underline{x} = \underline{N}(t_0)\, \underline{P}(t_0)\, \underline{x}_0 \qquad \text{(D-6a)}$$

$$\underline{x}_0 = \underline{P}^T(t_0)\, \underline{N}^T(t_0)\, \underline{x} \qquad \text{(D-6b)}$$

(c) ***Transformation*** $\underline{x}_t \leftrightarrow \underline{x}_0$

According to (D-6a,b) we get $(t \rightarrow t_0)$

$$\underline{x}_t = \underline{N}(t)\ \underline{P}(t)\ \underline{x}_0 \tag{D-7a}$$

$$\underline{x}_0 = \underline{P}^T(t)\ \underline{N}^T(t)\ \underline{x}_t \tag{D-7b}$$

(d) ***Transformation*** $\underline{x} \leftrightarrow \underline{y}$

Using the transformations (a) to (c) we are now able to write down the transformation between the coordinates $\underline{x}$ of the inertial system and those of the earth-fixed reference frame $\underline{y}$. Equating (D-7b) and (D-6b) we have

$$\underline{P}^T(t_0)\ \underline{N}^T(t_0)\ \underline{x} = \underline{P}^T(t)\ \underline{N}^T(t)\ \underline{x}_t \tag{D-8a}$$

$$\underline{x}_t = \underline{N}(t)\ \underline{P}(t)\ \underline{P}^T(t_0)\ \underline{N}^T(t_0)\ \underline{x} \tag{D-8b}$$

Inserting further (D-8b) in (D-5b) results in the final relations.

$$\underline{x} = \underline{R}(t)\ \underline{y} \tag{D-9a}$$

$$\underline{y} = \underline{R}^T(t)\ \underline{x} \tag{D-9b}$$

where

$$\underline{R}(t) = \underline{N}(t_0)\ \underline{P}(t_0)\ \underline{P}^T(t)\ \underline{N}^T(t)\ \underline{R}_3\left(-\Theta(t)\right)\ \underline{S}(t) \tag{D-9c}$$

$$\underline{R}^T(t) = \underline{S}^T(t)\ \underline{R}_3^T\left(-\Theta(t)\right)\ \underline{N}(t)\ \underline{P}(t)\ \underline{P}^T(t_0)\ \underline{N}^T(t_0) \tag{D-9d}$$

$$\underline{R}^T(t)\ \underline{R}(t) = \underline{R}(t)\ \underline{R}^T(t) = \underline{I} \tag{D-9e}$$

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