# Accuracy Improvement in Close Range Photogrammetry 

by
K. Torlegård

## SCHRIFTENREIHE

Wissenschaftlicher Studiengang Vermessungswesen Hochschule der Bundeswehr München

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# ACCURACY IMPROVEMENT <br> IN <br> CLOSE RANGE PHOTOGRAMMETRY 

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## PREFACE

The following study is part of a report from a six months' research stay as Visiting Professor for Engineening Photogrammetry at the Hochschule der Bundeswehr München. This study may be seen as a summary of the state of a rt in highly a c curate close range photogrammetry with emphasis on detection, localization and elimina tion of blunders in observations. Several hitherto unsolved problems have been defined, and some new research plans have been outlined. It is my hope that the survey of available methods to achieve high accuracies in close range photogrammetry given in this report will be of value for engineers and surveyors using close range photogrammetry as a measuring tool, and that it will be a stimulance for scientists and researchers to set out new goals for the development of this mensuration technique.

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## 0 A BSTRACT

The state of art in highly accurate a nalytical photogrammetry is briefly overviewed. The concept of accuracy is described asprecision, model fidelity, and reliability, where precision is related to random errors and their propagation, model fidelity to systematic discrepancies between reality and mathematical model, and reliability to detection, localization and elimination of blunders. The effect of blunders in least squares adjustments is outlined, and a procedure in order to localize blunders after adjustment is suggested. The possibility to localize a nd eliminate blunders in all phases of the photogrammetric process is discussed in detail, partic ula rly for close-range a pplications. The principle of redundancy to achieve reliability is demonstrated, leading to the multi-concept in photogrammetry. It can be applied to all steps: coordinate readings, fiducial marks, targets, frames (new method of overlapping), stations, control points, object geometry, computer program options. Simulta neous adjustment of observations, of image coordinates, observations of exterior orientation elements, geometric conditions in object space, and approximations for the unknowns are reviewed. Several hitherto unsolved problems and open questions have been sta ted.

## 1 INTRODUC TIO N

Analogue methodsfor photogrammetric measurements have been practiced fora long time. Cameras, instruments and methods have reached a standardization that makes it possible top plan projects and predict costs and accuracies with narrow limits. This is certa inly the case for aerial photogrammetry as applied to topographic mapping. Wide angle cameras with a 0.15 m camera constant are used for taking near vertical photographs with 60 \% overlap within a nd 20 \% between the strips. Line maps with contours and orthophotos are the typical products. As an intemediate step the a erial triangulation provides the necessary control points in a manner that even emphasizes the sta ndardization.

In non topographic photogrammetry, however, the situation has been somewhat different. The measuring tasks have been very varied, a nd the requirements and specific ations so different from one a pplication to a nother that it has been diffic ult to develop standardized a nalogue methods for the restitution of photographs. A certa in standa rdiza tion has been reached in a rchitectural photogrammetry with the nomal case stereo photography often using stereometric cameras, and restitution in nomal case stereoplotters yielding as result a line drawing on vertic al or horizontal projection planes. This equipment and technique has been used also for other purposes, sometimes suc cessfully, but very often certa in aspects in the requirements could not be fulfilled, e.g. type of output, a c c ura cy or completeness. The customer or end-user has sometimes diffic ulties in defining his requirements, a nd the photogrammetrist then recommends his well-established methods even though both parties have the feeling that something else would be more appropriate.

During the last decade a nalytical methods have demonstrated its capability as a very flexible tool for solving the most varied measuring problems in almost any field where geometric quantities are needed. The limitations due to the necessary standardization in the analogue methods are not relevant any longer, especially concerning the choice of cameras, their location and direction in relation to the object, type of object space control, overlap and number of photos, accuracy, computerized evaluation and presentation of results. This flexibility has opened new vistas for the applic ation of close range photogrammetry. At the same time, however, planning the projects and predictive accuracies have become more uncertain, since the well-known standard procedureshave been abandoned. The automatic checking on the ray intersection condition in the a nalogue stereomodel by the operator's stereo vision is now lost when mono or stereo cooperators a re used for measurements. On-line computation or use of analytical plotters eliminate this drawback. Object space control based on straight lines, horizontal of vertical planes, angles, etc. that easily were ta ken to advantage in the analogue stereomodel have now to be included in the adjustment computations by a series of program routines and extension of the basic mathematical formulations. The a na logue approach is typical on-line in the sense that the result is readily a vailable for checking, correction, deletion and amendment. In the a nalytical approach, online computer programs have to be developed. To implement this, an analytical stereo
plotter is needed, and that is an expensive development and instrumentation. The photogrammetrist has now to develop his occupational skill and experience, and to leam how to use the new possibilities that are opened up by analytic al methods. This comprises, a mong other things, extensions of mathematical models, correction of systematic errors, detection, loc alization and elimination of blunders, prediction of precision a nd reliability, planning photogrammetric projects, statistical methods fortesting hypotheses in the evaluation process.

## 2 MATHEMATICAL MODEL

### 2.1 THE BUNDLE APPROACH

Notations, coordinate systems and rotations follow the conventions recommended by the Intemational Society for Photogrammetry 1960 which are characterized by
a) a right-handed $X Y Z$ system in object space with $Z$ positive upwardsand $X$ positive mainly in base direction (if this is appropriate in the close range case)
b) a free choice of the origin of the $X Y Z$ coordinate system
c) a right-handed coordinate system $x y z$ in the image space, with positive directionsas in the same sense as in the $X Y Z$ system
d) the choice of the $X$-axis as primary axis and the $Y$-axis as secondary axis; rotations a round rotated axes
e) treating aspositive: clockwise rotations about the positive direction of the $X-Y$ - and $Z$-axes in conformity with the conventions of a right-handed system

Notations:

| $x_{k} y_{k}$ | comparator coordinates |
| :--- | :--- |
| $x^{\prime} y^{\prime} / x^{\prime \prime} y^{\prime \prime}$ | plane image coordinates in left/right photo |
| $c$ | camera constant (calibrated focal length) |
| $x_{1} y_{1} z_{1} / x_{2} y_{2} z_{2}$ | three dimensional ima ge coordina tes in left/ <br> right photo with axes pa rallel to $X Y Z$ |


| $x y z$ | model coordinates |
| :---: | :---: |
| bx by bz / BX BY BZ | base components in model/object |
| $\omega_{i} \varphi_{i} \kappa_{i}$ | rotations a round the $x_{i} y_{i} z_{i}$ a xes for photo $i$ |
| $\xi \eta \alpha$ | rotations of the model system $x y z$ around the $X Y Z$ axes |
| $X_{O} Y_{O} Z_{O}$ | origin of the model system'in the $X Y Z$ system |
| $X_{g} Y_{g} Z_{g}$ | known control point coord inates |
| $O_{1} O_{2} \ldots . O_{i} \ldots$ | projection centres |

## See also Fig. 2.1

The basic relation is the collinearity condition for the imaging ray from the object point through the perspective centre to the image point, which is expressed by

$$
\left[\begin{array}{ccc}
X & -X_{o} \\
Y & -Y_{o} \\
Z & -Z_{o}
\end{array}\right]=\lambda\left[\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right]=\lambda R\left[\begin{array}{lll}
x^{\prime} & -x^{\prime} \\
y^{\prime} & -y^{\prime} o \\
& -c
\end{array}\right] .
$$

Here, $\lambda$ is a scalarfactor, and $R$ is an orthonormal matrix with, e.g. the elements

$$
\begin{aligned}
& r_{11}=\cos \varphi \cos \kappa \\
& r_{12}=-\cos \varphi \sin \kappa \\
& r_{13}=\sin \varphi \\
& r_{21}=\cos \omega \sin \kappa+\sin \omega \sin \varphi \cos \kappa \\
& r_{22}=\cos \omega \cos \kappa-\sin \omega \sin \varphi \sin \kappa \\
& r_{23}=-\sin \omega \cos \varphi \\
& r_{31}=\sin \omega \sin \kappa-\cos \omega \sin \varphi \cos \kappa \\
& r_{32}=\sin \omega \cos \kappa+\cos \omega \sin \varphi \sin \kappa \\
& r_{33}=\cos \omega \cos \varphi
\end{aligned}
$$



Fig. 2.1

Axes, rotations and notations recommended by ISP 1960

As $c \neq 0$, we divide the first two equations by the third, thus obta ining image coordinates as functions of the orientation elements and the object coordinates

$$
\begin{aligned}
& x^{\prime}=x^{\prime}{ }_{o}-c \frac{T_{x}}{N} \\
& y^{\prime}=y_{o}^{\prime}-c \frac{T_{y}}{N}
\end{aligned}
$$

Where

$$
\begin{array}{rr}
T_{x} \\
T_{y} \\
N & =R^{T} \cdot Y-Y_{O} \\
Z-Z_{O}
\end{array}
$$

For the determination of orientation elements from image coordinates $x^{\prime} y^{\prime}$ and known points $X_{g} Y_{g} Z_{g}$, we linearize the expressions either by a Taylor series expansion or by numeric al differentiation from approximate values for the unknowns. The system of linear equations will, as a rule, be overdetermined and can be written in the form

$$
A X=L+V
$$

### 2.2 CONVERGENCE CRITERIA FOR IERATIONS

The least squares solution of $X$ representscorections to the approximations, and the procedure has to be iterated until convergence in order to give the final values of the orientation elements. The iteration process can be terminated on different criteria, e.g. the following.

Assume $A$ to have the dimensions ( $n, p, n>p$ ) and full rank, rank $(A)=p$.
The least squares estimate of $X$ will be

$$
\hat{X}=\left(A^{T} A\right)^{-1} A^{T} L .
$$

The simultaneous magnitude of the unknowns is described by the quadratic form

$$
L^{T} A\left(A^{T} A\right)^{-1} A^{T} L .
$$

This is the reduction of $L^{T} L$ to $V^{T} V$ by the adjustment, and

$$
L^{T} L=V^{T} V+L^{T} A\left(A^{T} A\right)^{-1} A^{T} L,
$$

where the three terms have the degrees of freedom $n, n-p$ and $p$, respectively. The two latter ones are independent. We form the test-varia te

$$
t=\frac{L^{T} A\left(A^{T} A\right)^{-1} A^{T} L}{p} \cdot \frac{n-p}{V^{T} V}
$$

If $t<t_{O}$ the iterations may be stopped. Because of rounding errors, $t$ will never be zero, and $t_{0}$ maybe chosen, e.g., in the interval [0.001, 0.1].

The effect of the omitted higher order tems is smaller than that of the unknowns, and thus they have very little influence on $V^{T} V$. This test must not be regarded as an F-test used for testing the hypothesis that two variances are equal, because here we require one quadratic form to be zero as a measure forconvergence, but we still can make the convergence criterion dependent on the measuring precision. We simply do not want to perform unnecessarily many iterations in relation to the precision of observations.

Experience has shown that for relative orientation and three dimensional conformal transformation a value of $t_{O}=0.001$ is suitable. This means that the lineareffect of the unknowns is in the image scale of the magnitude $0.001 \cdot s$ to $0.1 \cdot s$, where $s$ is the standard error of unit weight in the adjustment

$$
s=\sqrt{\frac{V^{T} V}{n-p}} .
$$

The convergence test is thus related to the estimated va riance, which has the advantage that it works for varioustypes of observations, various cameras, various project designs and iteration procedures. This is important for close range applications with their varied conditions. At the same time, however, it has the drawback that the estimated variance itself is a random variable, a nd for a small number of redundant observations, the iteration may be stopped too early or too late compared to what is needed for convergence.

### 2.3 LENS DISTORTION AND ADDITIONAL PARAMEIERS

For some applications the basic expressions a re sufficient, but in other cases the mathematic al model shows systematic errors when it is compared to physical reality. We then have to extend the model to include parameters for these systematic errors in the expressions. Radial distortion is effectively covered by the following expressions

$$
\begin{aligned}
& d x^{\prime}=x^{\prime} \cdot\left\{a_{3}\left(r^{2}-r_{0}^{2}\right)+a_{5}\left(r^{4}-r_{0}^{4}\right)+a_{7}\left(r^{6}-r_{0}^{6}\right)+\ldots \ldots\right\} \\
& d y^{\prime}=y^{\prime} \cdot\left\{a_{3}\left(r^{2}-r_{0}^{2}\right)+a_{5}\left(r^{4}-r_{0}^{4}\right)+a_{7}\left(r^{6}-r_{0}^{6}\right)+\ldots \ldots\right\} .
\end{aligned}
$$

The number of parameters depend on the type of radial distortion and how well we want to model it.

The decentering distortion based on the thin-prism model is described by

$$
\begin{aligned}
& d x^{\prime}=\left\{\left(r^{2}+2 x^{\prime 2}\right) P_{1}+2 x^{\prime} y^{\prime} P_{2}\right\}\left\{1+P_{3} r^{2}+P_{4} r^{4}+\ldots \ldots\right\} \\
& d y^{\prime}=\left\{2 x^{\prime} y^{\prime} P_{1}+\left(r^{2}+2 y^{\prime 2}\right) P_{2}\right\}\left\{1+P_{3} r^{2}+P_{4} r^{4}+\ldots .\right\},
\end{aligned}
$$

where $r^{2}=\left(x^{\prime}-x_{0}^{\prime}\right)^{2}+\left(y^{\prime}-y_{0}^{\prime}\right)^{2}$ and $r_{0}$
is a constant for which the radial distortion is zero. The distortion may be known from calibrations and introduced as corrections to the image coordinates before adjustment, or it can be unknown and used as additional parameters to be detemined in the adjustment.

In aerial photogrammetry the atmospheric refraction, as a rule, is taken into consideration, but this is not very often necessary in the close range case. Here, however, other systematic effects have to be considered, e.g. refraction in filters and port windows, departures from emulsion flatness, regular deformations of the image during photographic processing. Again we have two possibilities: calibration or additional parameters in the adjustment.

In the calibration case the systematic emors are determined in some representative points covering the image area. In the successive measurements the new image coordinates are corrected for systematic error by interpolation from the calibration points. Different interpolation methods can be used. Some popularones in photogrammetry and surveying are

1. Polynomial approximations
2. Spline interpolations
3. Linear prediction with covariance functions
4. Multiquadratic interpolation.

See e.g. Hein - Lenze (1979), Shut (1976) and Rauhala (1974).

In the case of additional parameters in the adjustment one hasto a void over-parametrization and linear dependencies to already introduced variables. Ebner (1976) and G rün (1978a) show the effic iency of using orthogonal parameters, and statistical testing for the selection of the a ppropriate parameter set. J acobsen (1980) has a nother set of additional parameters and he combines statistic al tests with a test sta tistic that mea sures the rema ining systematic effect after adjustment. The set of additional parameters has in these cases been designed to fit the geometric conditions of aerial photogrammetry, and if the close range case differs from this it is not longer certa in whether the selected set is optimal.

### 2.4 ACCURACY OF CONTROL POINTS

In analogue photogrammetry the double point resection in space is solved in two steps, relative and absolute orientations. The given object control is, in the second step, regarded as free from notic eable emror, but experienced stereo-operators often use their knowledge about the control, its accuracy, the targeting, the type of surveying behind the given coordinates, etc., in such a way that the residuals in the model after adjustment are not distributed in as it would be the case from a numerical least squares solution. The operator has taken into consideration information on the accuracy of the control that the numerical absolute orientation did not know when the orientation parameters were calculated from the discrepancies between the control point coordinates in the model and object system. This sort of observational skill, experience and feeling for weakness and strength in the control can be transferred to the mathematical model by a formulation which allows for corrections also to the given control. The control coordinates are treated as unknowns, and in order to compensate for the rank deficiency, fictic ious direct observations on the control point coord inates are introduced with a weight that corresponds to the accuracy of the control in relation to the other equations of the system.

### 2.5 INTERPRETATION OF RESULTS

The mathematical model hasto be developed to such a degree that it can be used forprediction without bias and with known precision. We thus want our model to have stochastic properties such that
a) $A X=E(L)=\lambda$
b) $L$ normally distributed according to $N\left(\lambda, \sigma^{2} Q_{L L}\right)$.

This gives us the a priori weight matrix $P=Q_{L L}^{-1}$.
Very often, one assumes that $Q_{L L}=I$, i.e. the observations are independent and of equal weight $P=Q_{L L}^{-1}=I$. In othercases $Q_{L L}=D$ is a diagonal matrix, which means that the observations have different weights but they are still independent or uncorrelated:

It is a main task forscientists to develope models having the above quoted properties. Another task is to give a physical explanation to the parameters $X$ of the model, e.g. such as orienta tions elements, radial and tangential distortion parameters. If additional parameters such as purely mathematical parameters $a_{i j}$ and $b_{i j}$ in a general regression formulation yield the above properties to the model, the scientist should find the physical explanation to this in order to increase our knowledge. Such knowledge can stimulate to new instrument designs, and improve new possibilities to correct the model, based on methods and measurements that was not earlier applied in photogrammetry.

## 3 MEASURES OF PRECISION , FIDELITY, AND RELIABILITY

### 3.1 TERMINOLOGY

According to Baarda, a ccuracy comprises two parts, viz. precision and relia bility (Ba arda, 1977a). Here, we a mend the theory with the concept of model fidelity (German: Modelltreue; French: fidelité de modèle; Swedish: modellriktighet). Precision is expressed as standard error computed by the law of error propagation through functions of the observations, and based on the assumptions, on their stochastic nature. The fidelity of the model is the goodness of fit of the mathematical model to the observations, and the model should be general enough to the valid undervarying experimental conditions. Relia bility is the possibility to detect, localize and correct orexclude blunders from the adjustment, and still having redundancy enough to check for blunders in the rema ining system.

Ha llert (1967) used to discriminate between a c cura cy and precision. In his sense precision was the intemal closeness of data from repeated observations. Accuracy on the other hand was the closeness of observations and functions of these to given data with much higher accuracy than the observations, orcloseness to mathematical conditions between the observations. E.G.: standard deviations from a series of repeated settings on $y$-paralla xes in a point of a stereomodel are, according to Hallert, measures of precision, while the sta ndard error of unit weight after a least squaresadjustment of the relative orientation from observations on $y$-parallaxes having unit weight is a measure of accuracy.

Here we will try to use the definitions in such a way that precision is related to the stochastic behaviour of variables and functions of observations, model fidelity being related to the absence of systematic errors of the mathematical model, and reliability being the possibility of detection, localization and elimination of blunders.

Accuracy is the tem covering all the three quoted concepts. In recent publications by other a uthors (e.g. Ba arda, Grün, Förstner) on reliability of photogrammetric and geodetic observations, the concept reliability covers blunders as well as systematic errors. Here we thus introduce the new concept of fidelity. It should be noted here that the tem relia bility has a nother mea ning in mathematic al statistics. There, "relia bility has been formulated as the science of predicting, estimating, or optimizing the probability of survival, the mean life, or, more generally, the life distribution of components or systems" (Mann et al., 1974, see Preface). It is also started by Mann that the concept of reliability should be understood to mean 'the probability of a device (or item or organism) performing its (or his or hers) defined purpose a dequately for a specified period of time, under the operating conditions encounted" (Mann et al., 1974, chapt. 1.1). With statistic al methods times to fa ilure of devices or items a re studied. In the theory of errors in geodesy and photogrammetry statistical methods are used for detection and localization of blunders in a set of obsenvations. There are common features between the two sciences in the use of statistics, e.g. the importance of order statistics.

### 3.2 DEFINITIONS

We have the model

$$
A X=\lambda
$$

We make a series of observations on $\lambda$, denoted by $L$

$$
\begin{aligned}
E(L)=\lambda, & E\left(\left(L_{i}-\lambda_{i}\right)\left(L_{i}-\lambda_{i}\right)\right)=\sigma^{2} \\
& E\left(\left(L_{i}-\lambda_{i}\right)\left(L_{j}-\lambda_{j}\right)\right)=\operatorname{Cov} \lambda_{i} \lambda_{j}
\end{aligned}
$$

Then we can introduce the true errors $e$

$$
A X=L+e, E(e)=0, E\left(e_{i} e_{j}\right)=\operatorname{Cov} e_{i} e_{j}
$$

The individual true errors $e_{i}$ are unknown, and with redundant observations we have to introduce residuals $V$ to obta in a consistent system,

$$
A X=L+V \quad ; \quad P=\frac{1}{\sigma_{O}^{2}} \cdot\left(\operatorname{Cov} e_{i} e_{j}\right)^{-1}
$$

which we solve by the methods of least squares, and we get the estimates

$$
\begin{aligned}
& \hat{X}=\left(A^{T} P A\right)^{-1} A^{T} P L \\
& \widehat{V}=\left(A\left(A^{T} P A\right)^{-1} A^{T} P-I\right) L
\end{aligned}
$$

$$
\hat{s}_{O}^{2}=\widehat{V}^{T} P \widehat{V} / r \text {, where } r \text { is the number of redundant observations. }
$$

The general law of error propagation then gives the following estimation of the standard emror of a function $U=F L$ of the observations, viz.

$$
\operatorname{Cov}\left(U U^{T}\right)=\hat{s}_{O}^{2} F P^{-1} F^{T} .
$$

Let, e.g.
and

$$
U=\hat{X} \text { then } F=\left(A^{T} P A\right)^{-1} A^{T} P
$$

or

$$
\operatorname{Cov}\left(\hat{X} \hat{X}^{T}\right)=\hat{s}_{O}^{2}\left(A^{T} P A\right)^{-1}=\hat{s}_{O}^{2} Q ;
$$

$$
U=\widehat{V} \text {, then } \operatorname{Cov}\left(\widehat{V} \widehat{V}^{T}\right)=\widehat{s}_{O}^{2}\left(P^{-1}-A^{O}\right), A^{O}=A\left(A^{T} P A\right)^{-1} A^{T} .
$$

For $P=I$ we get

$$
Q_{D O}=I-A^{O} .
$$

Now, we mean with precision all measures of variation based on $\hat{s}_{O}^{2}$, e.g. standard emrors of unknowns $\hat{x}_{i}$ :

$$
\hat{s}_{\hat{x}_{i}}=\hat{s}_{O} \sqrt{q_{i i}},
$$

or the standard emor of any other function of the observations. A general measure of precision of an adjustment (a project, a design of an experiment) is
a) $\operatorname{tr}(Q) / p$ which is the average variance of the estimated unknowns measured in units of $\sigma^{2}$, and
b) the $\hat{s}_{O}$ which is an estimate of $\sigma$.

By model fidelity we mean the absence of bias of the estimates, and a lack of fidelity should measure the departure from the mathematical expectations in the model, which is the same as the systematic error $D$ of the model

$$
D=A X-E(L) .
$$

This means that our assumptions are not fulfilled. To check this, we have to design certa in experiments to test if $D=0$. This can be done with so-called controlled experiments, with testfields, etc.

By reliability we mean the possibility to detect, localize and eliminate blunders from the observations. If we suspect blunders in a group of observations, and if they form a submatrix

$$
\left(P_{2}^{-1}-A_{2}^{O}\right) \text { in }\left(P^{-1}-A^{O}\right) \text {, then this submatrix must be non-singular. }
$$

As $\left(P^{-1}-A^{O}\right)$ is idempotent with rank $=r$, not all blunders can be detected. The system can be said to be reliable if as many as ( $b_{0}<r$ ) blunders in any combination can be detected.

General measures of the reliability can be
a) $\frac{\operatorname{tr}\left(I-A^{0}\right)}{n}=\frac{r}{n}$
or
b) $\left(\prod_{i}^{n}\left(I-A^{O}\right)_{i i}\right)^{1 / n} \leq \frac{r}{n}$
or
c) $0 \leq \sum_{i}^{n}\left[\left(I-A^{O}\right)_{i i}-\frac{r}{n}\right]^{\frac{2}{r}} \leq \frac{r}{n}$.

The trace $\operatorname{tr}\left(I-A^{O}\right)$ is often referred to as a measure of the global reliability. The individual diagonal elements $q_{V_{i} V_{i}}$ are measures of the local reliability.

The individual $q_{V_{i} V_{i}}$ measures the contribution of the observation $l_{i}$ to the global reliability.

Baarda (1976) has introduced the concepts of intemal and extemal reliability. (See also Förstner (1979) and Grün (1979b)). If all observations are equally well controlled (all $q_{V_{i} V_{i}}$ taking the same value), then the intemal relia bility is high. If nondetected blunders influence the estimated variables to a very small extent in the adjustment, the extemal relia bility is high.

### 3.3 EXPERIMENTAL DESIG N

Requirements and specific ations for experiments and projects can be formulated in several ways and can have several parameters. Some of these parameters are contradictory, others are not. For the contradictory parameters an optimizing function has to be found. Very often the formulations are verbal, and as such they are of little help to the scientist orengineer for the design of the model. New formulations have then to be derived. As parameters in the new formulation we can have

$$
\begin{array}{lrl}
\text {--precision: } & s_{O}^{2} \cdot Q & \text { and derivations thereof can be used; } \\
s_{O}^{2} \rightarrow \min & \text { can be obtained from including a dditional pa ra meters; } \\
\operatorname{tr} Q \rightarrow \min & \begin{array}{l}
\text { more observations perphoto, more photos, more photo } \\
\\
\\
\\
\\
\\
\text { stations in locations such that new pairs of rays do not lie in } \\
\text { already established epipolar pla nes; }
\end{array}
\end{array}
$$

precision of unknowns and precision of functions of unknowns can be traced back to the above items, viz. $s_{O}^{2}$ and $Q$.
-- fidelity: no systematic errors in the model, i.e. $A X=E(L)$, or in functions of the estimate $\hat{X}$;
this can be tested by controlled experiments. Departures between discrepancies in check points and the corresponding precision is a measure of the remaining systematic errors. Fidelity can also be checked by inspection of the residuals. The standardized residuals shall be independent and follow a normal distribution. It is more diffic ult to design experiments to guarantee just precision.
-- reliability: the design matrix $A$ should be such that each observation is at least double checked, so that if one observation is deleted due to a blunder, there still is a possibility to detect (but perhaps not to localize) blunders in the remaining observations. The consequence will be that the redundancy should be rather high, e.g.
$r=2 p$ or which is the same $n=3 p$ and the relative redundancy $r / n$ thus $2 / 3$ in the planning stage. The design matrix $A$ influence the matrix $\left(I-A^{O}\right)$, and the aim should be to have the same magnitude on all diagonal elements of $\left(I-A^{O}\right)$.
-- economy: this can be expressed in terms of risks for the producer (photogrammetrist) or the consumer (client, end-user), or in cost-functions which consider the cost for the measuring operations and costs for damages caused by emors in the results due to lacking precision, fidelity and relia bility. Here decision theory is of interest.

## 4 BLUNDERS IN PHOTOGRAMMETRIC DATA

### 4.1 BLUNDERS IN ADJ USTMENT

Up to now detection, localization and elimination of blunders from photogrammetric data has prima rily been based upon the operational skill of the photogrammetrist. Methodshave been designed to avoid blunders ratherthan accepting their presence in observations, which logically lead to blunder detection procedures. Theoretic al studies and practic al examples for blunder detection, localization and elimination in photogrammetric data have been published by a few authors during the last decade, e.g.

Förstner (1976), (1978) and (1979), G rün (1978a), (1978b) and (1979b), Bo uloucos (1979), Molena r (1976) a nd (1978), Stefa novic (1978a) a nd (1978b), J a cobsen (1980).

All these studies are based on a so-called "data-snooping" a c cording to the theories developed by Baarda (1965), (1967), (1968) and (1976) for geodetic observations, just to mention a few important papers from the rich production by this a uthor. Pelzer (1979) has demonstrated the method on some clearand simple examples in surveying. Recently a new theoretical approach to blunderdetection in surveying and geodesy has been presented by Heindl and Reinhart (1979) ba sed on linear programming and tolerance for the residuals. This linear formulation obviously revea ls the blunders easier tha $n$ the residuals after a least squares adjustment. The computational effort, however, seems to be larger with linear programming.

With matrix notations as in Bjerha mmar (1973) the effect of blunders in least squares adjustment can be summarized as follows:

Functional model $\quad A X=L+e$

$$
\begin{aligned}
& E(L)=\lambda \\
& E(e)=0, \operatorname{Cov}\left(e e^{T}\right)=\sigma^{2} \cdot I
\end{aligned}
$$

Matrix $A$ hasdimensions $(n, p)$ and $\operatorname{rank}(A)=p, r=n-p$.

We now have blunders(denoted $\nabla$ nabla) in the observations so that

$$
A X=L+e+\nabla
$$

The number of blunders must be less than $r=n-p$.

The problem is that we do not know in which observations these blundersoccur. Thus, we proceed in the standard way of least squa res adjustment by writing

$$
\begin{aligned}
& A X=L+V \\
& (V=e+\nabla+A(\hat{X}-X))
\end{aligned}
$$

from which we estimate

$$
\hat{X}=\left(A^{T} A\right)^{-1} A^{T} L
$$

As we have blunders in the observations the expectation of $X$ is

$$
E(X)=\left(A^{T} A\right)^{-1} A^{T} E(L+\nabla)=X+\left(A^{T} A\right)^{-1} A^{T} V
$$

We also have the influence from the blunders on the residuals

$$
\widehat{V}=\left(A\left(A^{T} A\right)^{-1} A^{T}-I\right)(L+\nabla), \quad A^{O}=A\left(A^{T} A\right)^{-1} A^{T}
$$

so that the blunders are distributed among the residuals by the operator $A^{O}-I$. The weight coeffic ient ma trix of the residuals is given by

$$
Q_{\tilde{D} \tilde{V}}=I-A^{0} .
$$

If $\sigma^{2}$ is unknown, it is estimated by the estimator

$$
s_{O}^{2}=\widehat{V}^{T} \widehat{V} / r,
$$

but since we have blunders,

$$
\widehat{V}^{T} \widehat{V}=L^{T}\left(I-A^{O}\right) L+\nabla^{T}\left(I-A^{O}\right) \nabla+2 \nabla^{T}\left(I-A^{O}\right) L .
$$

Thus, the bias is given by the two last terms, which is the sum of the squared blunders times the corresponding diagonal elements of $\left(I-A^{O}\right)$, plus twice the sum of the product of the blunders and the correct observation, multiplied by the same elements. The sum is taken over the observations that contain blunders, the problem still being that we do not know where the blunders are.

Suppose for a moment that we have blunders in $b$ of the $n$ observations, forming group 2; the observations in group 1 are assumed without blunders.

$$
\begin{aligned}
& A_{1} X=L_{1}+V_{1} \\
& A_{2} X=L_{2}+V_{2}+\nabla \\
& Q_{1}=\left(A_{1}^{T} A_{1}\right)^{-1} \\
& Q \quad=\left(A_{1}^{T} A_{1}+A_{2}^{T} A_{2}\right)^{-1}
\end{aligned}
$$

Rank $\left(A_{1}\right)=p, \operatorname{rank}\left(A_{2}\right)=b, n-p=r, r \geq b$.
The estimate of $X$ will be

$$
\hat{X}=Q A_{1}^{T} L_{1}+Q A_{2}^{T} L_{2}+Q A_{2}^{T} \nabla
$$

where

$$
Q=\left(A^{T} A\right)^{-1} \quad A=\left[\begin{array}{l}
A_{1} \\
A_{2}
\end{array}\right]
$$

The residuals will be

$$
\begin{aligned}
& \widehat{V}_{1}=\left(A_{1} Q A_{1}^{T}-I\right) L_{1}+A_{1} Q A_{2}^{T} L_{2}+A_{1} Q A_{2}^{T} \nabla \\
& \widehat{V}_{2}=A_{2} Q A_{1}^{T} L_{1}+\left(A_{21} Q A_{2}^{T}-I\right) L_{2}+\left(A_{2} Q A_{2}^{T}-I\right) \nabla .
\end{aligned}
$$

Now we are interested to see what the result would be if we used only group 1 for the estimation. As we already have computed inverse $Q=\left(A^{T} A\right)^{-1}$ and estimate $\bar{X}$, we use the method of sequential adjustment (see Mikhail (1976), chapt. 13 and app 9) to find the inverse $Q_{1}=\left(A_{1}^{T} A_{1}\right)^{-1}$, the estimate $\widehat{X}_{1}$, and the residuals $\bar{V}_{1}$ from the adjustment of group 1 .

The new inverse after deletion of group 2 will be

$$
Q_{1}=Q\left\{I+A_{2}^{T}\left(I-A_{2} Q A_{2}^{T}\right)^{-1} A_{2} Q\right\} .
$$

Aswe expect blunders to occur only in a limited number of observations, the inversion of the matrix ( $I-A_{2} Q A_{2}^{T}$ ) will be rathereasy to be calculated, as the dimensions are small. However, the inverse must exist, a problem to that we will retum to later. The new estimate of $X$ will be

$$
\bar{X}_{1}=\hat{X}+Q A_{2}^{T}\left(I-A_{2} Q A_{2}^{T}\right)^{-1} \widehat{V}_{2} .
$$

The new residuals of group 1 will be

$$
\bar{V}_{1}=A_{1} \bar{X}_{1}-L_{1} .
$$

The old residuals were

$$
\widehat{V}_{1}=A_{1} \widehat{X}-L_{1}
$$

and thus the new onescan be calculated from the old ones by

$$
\bar{V}_{1}=\widehat{V}_{1}+A_{1}\left(\bar{X}_{1}-\widehat{X}\right)=\widehat{V}_{1}+A_{1} Q A_{2}^{T}\left(I-A_{2} Q A_{2}^{T}\right)^{-1} \widehat{V}_{2} .
$$

It can be shown that the sum of the squared residuals will be reduced from $\widehat{V}^{T} \widehat{V}$ to $\bar{V}_{1}^{T} \bar{V}_{1}$ after excluding group 2 from the adjustment, viz.

$$
\bar{V}_{1}^{T} \bar{V}_{1}=\widehat{V}^{T} \widehat{V}-\widehat{V}_{2}^{T}\left\{I+A_{2} Q A_{2}^{T}\left(I+A_{2} Q_{1} A_{2}^{T}\right)\right\} \widehat{V}_{2} .
$$

The new matrix of weight coefficients, $Q_{\bar{V} \bar{V}}$, for the residuals $\bar{V}_{1}$ can be obtained directly from the old matrix $Q_{D V}$ :

$$
Q_{\bar{V}_{1} \bar{V}_{1}}=I-A_{1} Q A_{1}^{T}-A_{1} Q A_{2}^{T}\left(I-A_{2} Q A_{2}^{T}\right)^{-1} A_{2} Q A_{1}^{T}
$$

where

$$
\begin{gathered}
I-A_{1} Q A_{1}^{T}=Q_{V_{1} V_{1}} \\
I-A_{2} Q A_{2}^{T}=Q_{\widehat{V}_{2} V_{2}} \\
A_{1} Q A_{2}^{T}=Q_{\overparen{V}_{1} V_{2}} \\
A_{2} Q A_{1}^{T}=Q_{V_{2} V_{1}}
\end{gathered}
$$

| $Q_{\widehat{V}_{1} V_{1}}$ | $Q_{\widehat{V}_{1} V_{2}}$ |
| :--- | :--- |
|  |  |
| $Q_{\widehat{V}_{2} V_{1}}$ | $Q_{\widehat{V}_{2} V_{2}}$ |

which are all submatrices of the old matrix $Q_{\hat{V} \hat{V}}$.
The new discrepancies for group 2 using the new estimate of $X$ from group 1 is $\bar{V}_{2}=A_{2} \bar{X}_{1}-L_{2}$. This can directly be obtained from the old $\widehat{V}_{2}$ and the matrix $\left(I-A_{2} Q A_{2}^{T}\right)^{-1}$ as $V=\left(I-A_{2} Q A_{2}^{T}\right)^{-1} \widehat{V}_{2}$. Or of group 2 has only one observation $\bar{v}_{2 i}=\hat{v}_{2 i} / q_{2 v_{i} v_{i}}$.

Data-snooping according to Baarda is based on a test, where the test statistic $w_{i}=v_{i} / s_{O} \sqrt{q_{v_{i} v_{i}}}$ is compared to a critical value that is found after fixing the risk and powerfor a simultaneoustest of the hypothesis $H_{0}$ that there are no blunders in the observations, against the group of altemative hypotheses $H_{i}$ that there is a blunder in observation number $i$. In this way, the blunders are localized to a limited number of observations.

The error propagation from group 1 to group 2 is given by

$$
\operatorname{Cov} \bar{V}_{2}=s_{O}\left(I+A_{2} Q_{1} A_{2}^{T}\right) .
$$

It can be shown that $I+A_{2} Q_{1} A_{2}^{T}=\left(I-A_{2} Q A_{2}^{T}\right)^{-1}$.

Like Baarda, most authors have based the test on the assumption that there is not more than one blunder in the data. Stefanovic (1978a) has suggested a strategy fordividing the observations in two groups, one with observations that are free from blunders and the other with observations contaminated by mistakes, blunders a nd outliers. He presupposes known critical values for the sum of squared residuals belonging to each group, and these critical valuesare based on a priori known variances for the observations that are dealt with.

## 4.2 <br> A SUGGESTED TEST STATISTIC

In the following we will to try to find a method to group the observations without any a priori information on the variances. This is also often the most realistic starting point when dealing with close range photogrammetry, where the conditions detemining the variances may vary considerably and thus are diffic ult to predict.

The new discrepancies $\bar{V}_{2}$ between the observations $L_{2}$ of group 2 and their prediction $A_{2} \bar{X}_{1}$ based on the observations in group 1 also contain the blunders, and we now want to describe their magnitude. We form the test statistic

$$
F^{\max }=\frac{\left(\bar{V}_{2}+\nabla\right)^{T}\left(I+A_{2} Q_{1} A_{2}^{T}\right)^{-1}\left(\bar{V}_{2}+\nabla\right)}{\bar{V}_{1}^{T}\left(I-A_{1} Q_{1} A_{1}^{T}\right)^{-1} \bar{V}_{1}} .
$$

If the blunder $\nabla \neq 0$, the nominator inc reases with inc reasing $\nabla$. Thus, if $F^{\max }>F_{c r i t}^{\max }$, where $F_{c r i t}^{\max }$ is a critical value for the test, we regard the obsenvations in group 2 as being contaminated by blunders. The $F^{\max }$ is a ration of two quadratic forms, yet in this case they are not independent because of two reasons: firstly, they originate from the same adjustment, in which case it is necessary that $A_{1} Q A_{2}^{T}=0$ forthe two quadratic forms to be independent (see Graybill (1961), Theorem 4.21); secondly, we select the observations in such a way that we have the largest standardized residuals in the nominator, and the smallest in the denominator. If these two restrictions would not be at hand, we easily could find the critical values from the F-distribution (Snedecor's variance ratio). As this is not possible, we try to derive the distribution of $F^{m a x}$ under the conditions mentioned. An algebraic denivation is extremely diffic ult, and the possible altemative method seems to be computer simulation. However, also here the lineardependence through $A_{1} Q A_{2}^{T}$ may be difficult to formulate in a general way. The grouping of the residuals, on the other hand, is rather straight forward.

### 4.3 THE $Q_{V V}$ - MATRIX

The idempotent properties of $Q_{V V}$ mean that

$$
\begin{aligned}
& Q_{V V} \cdot Q_{V V}=Q_{V V} \\
& q_{i i}=\sum_{j}^{n} q_{i j}^{2}=\sum_{k}^{n} q_{k i}^{2} \\
& 0 \leq q_{i j} \leq 1 .
\end{aligned}
$$

Thus, $q_{i i}$ is always the a bsolute largest element in its row or column. If $q_{i i}=1$ all other elements are zero. If $q_{i i}=0$ all other elements of row $i$ are 0 which happens when no other equations in $A$ check the observation $l_{i}$ belonging to $q_{i i}$. A blunderin observation $l_{i}$ influences the unknowns $X$ directly. This, e.g., is the case for the image coord inates in the epipolar pla nes of a single stereopair. This becomes directly evident when the epipolar plane is parallel to one image coordinate axis. The partial correlation coefficients of the residuals $V_{2}$ can be calculated from the submatrix ( $I-A_{2} Q A_{2}^{T}$ ) by dividing rows and columns by the square roots of the diagonal elements

$$
r_{i j}=q_{v_{i} v_{j}} / \sqrt{q_{v_{i} v_{i}} \cdot q_{v_{j} v_{j}}}
$$

If $r_{i j}=1$, the observations $i$ and $j$ fully compensate each other. An error in one of the observations will contribute to the residuals in $i$ and $j$ with the same a mount when we chose

$$
\sqrt{q_{v_{i} v_{i}}} \text { and } \sqrt{q_{v_{j} v_{j}}} \text {, respectively, as units. }
$$

The localization of a blunder and its designation to one of the observations $i$ or $j$ is not possible. It can be either of the observations. The inverse of the submatrix ( $I-A_{2} Q A_{2}^{T}$ ) conta ining observations $i$ and $j$ does not exist as its deteminant is zero. Thus, the sequential adjustment routine is not a pplic able for both observations $i$ and $j$. A least one of them necessarily lies in $A_{1}$, in order to mainta in the rank $\left(A_{1}\right)=p$. The observations $i$ and $j$ are the only ones that check each other. Such design matrices $A$ yielding these effects should be a voided in order to mainta in the reliability of the system. To be able to localize blunders we must have "double" redundancy of all observations. If we want to be able to localize $b$ blunders in an adjustment of $n$ observations with $r$ degrees of freedom $(r>b)$, this requirement can be transferred to the condition that all $\binom{n}{b}$ combinations of observationsleading to group 2 must give a non-singular submatrix ( $I-A_{2} Q A_{2}^{T}$ ) of the total matrix $Q_{v v}$, so thatr the inverse of the submatrix exists. For direct measurements of the same unknown (mean of repeated settings) this means that the diagonal elements of $Q_{v v}$ should be larger than $2 / 3$. Single redundancy yields $q_{v v}=1 / 2$. Further, the correlation coeffic ients between observations must not be equal to $\pm 1$, their magnitude should preferably be smaller than 0.5 .

### 4.4 DISTRIBUTION FUNCTION OF $F^{\max }$

Methods for the exact derivation of distribution functionscan be found in the more advanced textbooks on mathematical statistics, e.g. Kendall - Stuart (1969), chapt. 11. In the same book we also find the distribution of the $r$ - th orderstatistic, (formula 11.34). Further derivations for order statistics are found in Mann et al (1974), chapt. 3.5 and 3.8, and Rohatgi (1976) chapt's 4.5 and 13.6.
(1) The parent distribution of $x_{i}$ is $f(x)=\frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2}$
(2) The distribution function of $y_{i}=x_{i}^{2}$ is

$$
\begin{array}{cc}
f(y)=\frac{1}{\sqrt{2 \pi y}} e^{-y / 2}, & y \geq 0 \\
0, & y<0
\end{array}
$$

and the sum $z$ of two $y_{i}, z=y_{1}+y_{2}$ (point error) is distributed

$$
\begin{array}{cc}
f(z)=\frac{1}{2} e^{-z / 2}, & z \geq 0 \\
0, & z<0
\end{array}
$$

(3) The distribution function of the $r$-th order statistic $Z_{(r)}$ is

$$
F\left(Z_{(r)}\right)=\frac{n!}{(r-1)!(n-r)!}\left[\int_{0}^{Z_{r}} f(Z) d z\right]^{r-1}\left[1-\int_{0}^{Z_{r}} f(Z) d z\right]^{n-r} \cdot f
$$

(4) The joint distribution of the $r$ smallest $Z_{i}$ is

$$
f_{\text {min }}=f\left(Z_{(1)} \ldots Z_{(r)}\right)=\frac{n!}{(n-r)!}\left[1-\int_{0}^{Z} f(Z) d z\right]^{n-r} \prod_{j=1}^{r} f\left(Z_{i}\right) .
$$

(5) The joint distribution of the $b$ largest $Z_{i}$ is

$$
f_{\max }=f\left(R_{(n-b)} \ldots Z_{(n)}\right)=\frac{r(n)}{r(n+1)}\left[\int_{0}^{Z_{(n-b)}} f(Z) d z\right]^{n-b}
$$

$$
\int_{y_{(n-b)}}^{y_{(n-b+1)}} f(Z) d z \cdot \int_{y_{(n-b+1)}}^{y_{(n-b+2)}} f(Z) d z \ldots \int_{Z_{(n-1)}}^{z_{(n)}} f(Z) d z \cdot b \cdot f(Z) .
$$

(6) The distribution of $F_{(b, r)}^{\max }=u$ can then be found from

$$
H(u)=F_{1}(u v) f_{2}(v) d v
$$

where

$$
\begin{aligned}
& F_{1}=\int_{0}^{\sum_{n-b}^{n} y_{(i)}} f_{\max } d y, \quad f_{2}=f_{\min } \\
& u=\sum_{n-b}^{n} y_{(i)} / \sum_{1}^{r} y_{i} \quad v=\sum_{1}^{r} y_{(i)} .
\end{aligned}
$$

In (3) we have squares of integrals, a nd this leads to very complicated calculations. The calculations are, however, relatively simple if the parent frequency function is of the exponential type, viz.

$$
f(x)=\frac{1}{a} e^{-\frac{x}{a}} .
$$

Thus, if we define the squared standardized point residual for the observed image point as

$$
z=\frac{v_{x}^{2}}{\sigma_{o}^{2} q_{v_{x} v_{x}}}+\frac{v_{y}^{2}}{\sigma_{o}^{2} q_{v_{y} v_{y}}},
$$

this is a sum of two random variables being $N(0,1)$ distributed, and this sum is distributed according to $\chi^{2}$ with two degrees of freedom,

$$
\begin{array}{ll}
f(z)=\frac{1}{2} e^{-z / 2} & z \geq 0 \\
F(z)=1-e^{-z / 2} & z \geq 0,
\end{array}
$$

which is an exponential distribution.

Thus, working with the squared standardized residual would open the possibility to derive the exact distribution functions for those test statistic sthat we need for testing hypotheses on blunders a nd outliers.

Then, there is the problem of "Studentization", which arises when we do not know $\sigma^{2}$, but have to estimate the varia nce from observations. This has been disc ussed by Pope (1975). (See also Anscombe (1960)). In this case we use the same observations
to determine both $V$ and $\sigma^{2}$ (intemal Studentization), then nominator and denominator were mutually dependent, and we a rive at a so-called $\tau$ (tau)-distribution for $\frac{V}{s_{o} \sqrt{q_{V V}}}$. From this we have to derive the distribution of our va riate $z$.

In (6) we a lso have to consider the lineardependence of the two quadratic forms originating from the same adjustment. We know that, if nominator and denominatorare based on two independent samples from the same distribution, the $F$ - ration will be distributed as Snedocor's $F$ with $b$ and $n-p-b$ degrees of freedom. But, first, the nominator contains the $b$ largest values of the sample, and secondly, the independence of nominator and denominator requires that

$$
\left(I-A_{2} Q A_{2}^{T}\right)\left(I-A Q A^{T}\right)\left(I-A_{1} Q A_{1}^{T}\right)=0 .
$$

(Graybill (1961), Theorem 4.21).
The matrices ( $I-A_{2} Q A_{2}^{T}$ ) and ( $I-A_{1} Q A_{1}^{T}$ ) are two hyperdiagonal submatrices of ( $I-A Q A^{T}$ ),

$$
\left[\begin{array}{cc}
I-A_{1} Q A_{1}^{T} & -A_{1} Q A_{2}^{T} \\
-A_{2} Q A_{1}^{T} & I-A_{2} Q A_{2}^{T}
\end{array}\right]
$$

The quadratic forms are calculated overall observations, and thus we have to multiply the three matrices

$$
\left[\begin{array}{cc}
I-A_{1} Q A_{1}^{T} & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{cc}
I-A_{1} Q A_{1}^{T} & -A_{1} Q A_{2}^{T} \\
-A_{2} Q A_{1}^{T} & I-A_{2} Q A_{2}^{T}
\end{array}\right]\left[\begin{array}{lc}
0 & 0 \\
0 & I-A_{2} Q A_{2}^{T}
\end{array}\right]
$$

which obviously is equal to zero only if

$$
\left(I-A_{1} Q A_{1}^{T}\right)\left(-A_{1} Q A_{2}^{T}\right)\left(I-A_{2} Q A_{2}^{T}\right)=0 .
$$

This is the case if and only if $A_{1} Q A_{2}^{T}=0$ which happens when the design matrix $A$ is such that the observations of group 1 and 2 determine two different groups $A$ and $B$ of unknowns, i.e. we could have adjusted the two groupsseparately instead.

Since the exact distribution function seems to be very diffic ult to derive by algebraic means, there remains the possibility to calculate the distribution by means of computer simulation. Some hints can be found in Mann et al (1974), chapt. 7.2. The distribution function has to be simulated fordifferent sizes of group 1 and group 2, and for each combination of sizes some 10000 samples have to be simulated. The interesting percentage values (e.g. $5 \%, 1 \%, 0.1 \%$ ) should be printed in tables. There is a partic ular problem to be solved, namely the simulation of the lineardependence between nominatorand denominator through off-diagonal elements of the $Q_{v v}$ matrix.

Further, simulations of the non-central distribution are then needed for the calculation of the power of the test. Thus, computer simulation is not a trivial ta sk either.

### 4.5 BLUNDER LOCALZATION

Let us now return to the unsolved problem of dividing the observations into two groups. Let ustry the following procedure.
(0) Consider all observations in group 1. Set $b_{O}<r$ as the maximum number of blunders to be tested. Make an initial adjustment. Go to (2).
(1) Make a sequential adjustment of all observations in group 1.
(2) Compute the sta nda rdized residuals of group 1

$$
w_{i}=v_{i} / s_{O} \sqrt{q_{v_{i} v_{i}}}
$$

(3) Select the observation in group 1 with $\max \left|w_{i}\right|$ a nd refer it to group 2 . Check if there is a nother residual
$v_{j}$ having $q_{v_{i} v_{j}} / \sqrt{q_{v_{i} v_{i}} \cdot q_{v_{j} v_{j}}}= \pm 1$, if so, they are $100 \%$ correlated and blunder location is not possible. The blunder can be either of the observations. The inverse of the submatrix corresponding to these residuals in $Q_{v v}$ does not exist. If this happens or if the number of blunders have reached $b_{O}$ the procedure is stopped. Otherwise proceed to (4).
(4) Test the variance ratio between group 2 and 1 . If the hypothesis $H_{O}: \operatorname{Var} 1=$ $\operatorname{Var} 2$ is accepted it might be so that there are more blunders that influence $\operatorname{Var} 1$ so that the test is biased and therefore go to (3) a nd select the next absolutely la rgest $\left|w_{i}\right|$.
(5) If $H_{O}$ is rejected the observations in group 2 are regarded to conta in blunders. Estima te them :

$$
V=\left(I-A_{2} Q A_{2}^{T}\right)^{-1} \widehat{V}_{2}=\left(I+A_{2} Q_{1} A_{2}^{T}\right) \widehat{V}_{2}
$$

Try to explain the blunder and correct the observation a c cordingly. Go to (6). If it is not possible to explain and correct the blunder, refer the observation to group 3 which is outside the procedure. Go to (2).
(6) Test the new discrepancies after correction of the blunders to see if the observation can now be recalled to group 1 . If so go to (1).
(7) If the correction was not suc cessful, refer the observation to group 3. Go to (1).

Another method forgrouping the observations in blunder-free and mista kes is the following.
(1) Make an initial adjustment, all observations regarded as belonging to group 1, set $F_{O}=0, j=0$.

(2) Compute the standardized residuals $w_{i}$ of group 1 , set $j=j+1$.
(3) Refer the absolutely largest $\left|w_{i}\right|$ to group 2.
(4) Compute $\bar{X}_{1}, Q_{1}, \bar{V}_{1}, \bar{V}_{2}, \bar{s}_{O}^{2}$ by sequential adjustment.
(5) Calculate $\quad F_{j}=\frac{\bar{V}_{2}^{T}\left(I-A_{2} Q A_{2}^{T}\right)^{-1} \bar{V}_{2}}{\bar{V}_{1}^{T}\left(I-A_{1} Q_{1} A_{1}^{T}\right)^{-1} \bar{V}_{1}}$
(6) If $F_{i} \geq F_{(j, r) p \%}^{\max }$ then go to (2).
(7) If $F_{j}<F_{(j, r) p \%}^{\max }$ then recall the last (the $j:$ th) observation to group 1, and the observations in group 2 are now regarded as blunders.
(8) Stop blunder loc alization.

These methods are suggested by intuition and they have to be tested in theoretical and empiric al studies. The effectiveness of the localization can preferably be studied by computer simulation.

## 5 STOCHASTIC PROPERTIES OF IMAGE COORDINATES

### 5.1 TRUE ERRORS AND RESIDUALS

In most adjustment it is a ssumed that the observations are normally distributed with mean zero and a common variance $\sigma^{2}$, and that the observations are independent:

$$
\begin{array}{ll}
A X=L+e, & E(e)=0, \\
& E\left(e_{i} e_{j}^{T}\right)=\sigma^{2} I
\end{array}
$$

The equations are given the same weight $P=I^{-1}=I$.
Not very seldom, the equations are given different weights $P$, based on the assumption that the observations are independent and have different variances $\sigma_{i}^{2}$ :

$$
E\left(e_{i} e_{j}^{T}\right)=\sigma^{2} D, \quad P=D^{-1}
$$

It is very uncommon that the observations in the collinearity model (chap. 2) are given the stochastic properties saying that the observations a re correlated (or even dependent) and have different variances

$$
\begin{aligned}
& E\left(e_{i} e_{j}^{T}\right)=\sigma^{2} D^{1 / 2} C D^{1 / 2}=\operatorname{Cov} \\
& P=\frac{1}{\sigma^{2}} \operatorname{Cov}^{-1}=D^{1 / 2} C^{-1} D^{-1 / 2}
\end{aligned}
$$

$C$ is a correlation matrix with $c_{i i}=1$ in the main diagonal, and the off-diagonal elements are $-1 \leq c_{i j} \leq 1$.
$D^{1 / 2}$ is a diagonal matrix with elements equal to the square root of the elements in $D$.
$D^{-1 / 2}$ is the inverse of $D^{1 / 2}$.
The information in the weight matrix $P$ is known a priori, i.e. before the adjustment; a priori weights of the observations are derived from the a priori variances and covariances. The individual true errors $e$ of the observations $L$, however, are not known generally, and the adjustment is based on the equations

$$
A X=L+V .
$$

The properties of $V$ sometimes have been studied in order to check the assumption on the stochastic properties of $e$.

Very often, there have been produced histograms showing the distribution of $v_{i}$. (see e.g. Hallert et al (1964), (1967)). The $\chi^{2}$-test has been applied to show the normal distribution. Vector diagrams have been plotted to show dependence or independence of the residuals. Sometimes a dependence has been described as function of the distance between pairs of points in the image, the model or the block; Torlegå rd (1967), Kupfer (1973), Ackermann (1978). Correlations between observations in different photos having the same location in the image have been studied by Ackermann (1978). Bachmann, Ha wa wini (1978) have studied the correlation between observations as a function of time. Hallert (1967), Morén (1967) and Brown (1969) have studied the variation of the magnitude of the residuals as a function of the radial distance from the principal point.

In a lmost all of these investigations the residuals are used directly as they appear. The residuals $V$ have been used asobservations on the true errors $e$. This is an approximation that can be justified sometimes, but far from always. We know that the residuals are functions of the observations and the design matrix $A$

$$
V=\left(A^{O}-I\right) L .
$$

If we happened to know the true error $e$, they will be mixed in the residuals through the relation

$$
V_{e}=\left(A^{O}-I\right) e,
$$

but then the adjustment is an unnecessary operation, because the unknowns can directly be determined from the consistent system of equations $A X=L+e$ with $n=p$ equations, the only condition being rank $(A)=p$.
However, the dimensions of $\left(A^{O}-I\right)$ are $(n, n)$, and the rank $\left(A^{O}-I\right)=r=$ $n-p$, so it is, in principle, impossible to find the true errors from the residuals. But what we can do is to derive the covariance matrix for the residuals, and use the standardized residuals $w_{i}$ for our further studies.

$$
\begin{aligned}
\operatorname{Cov} V & =\sigma^{2}\left(I-A^{O}\right)=\sigma^{2} Q_{v v} \\
w_{i} & =v_{i} / \sigma \sqrt{q_{v_{i} v_{i}}} .
\end{aligned}
$$

This, a nyhow, is betterthan using the residuals directly, especially if the $q_{v_{i} v_{i}}$ vary considerably. If all residuals have the same precision (i.e. the same $q_{v_{i} v_{i}}$ ), the sta nda rdiza tion is not very important for tests on distribution and correlations.

The variation of the $q_{v_{i} v_{i}}$ terms depends very much on the homogeneity and symmetry of the geometry behind the observations. Asan example, we can use the affine coordinate transformation on fiducial marks.

## Ex 5.1

The four mid-side marks common in Zeiss c a meras yield

where the $q_{v_{i} v_{i}}$ all are equal. The relative is

$$
1 / 4=(4-3) / 4=0.25
$$

## Ex 5.2

The four comer marks (J ena UMK, Wild RC-series) yield

$$
\underbrace{}_{4}{ }^{3}
$$

where all the $q_{v_{i} v_{i}}$ are equal, too. The relative redundancy is again $1 / 4=(4-3) / 4=0.25$.

## Ex 5.3

By contrast we have the unsymmetric geometry of the fiducials in the Wild P31 and P32 cameras, where we get

The relative redundancy is $2612 / 1306 / 5=(5-3) / 5=0.4$.
The minimum local relative redundancy is $147 / 1306=0.11$, the maximum is 1032 / $1306=0.79$.

In this case it is very important to standardize the residuals before comparison or blunder loc a lization, as the ratio between the largest and the smallest $q_{v_{i} v_{i}}$ is as large as 1032 / $147=7.0$.

The main diagonal is $\left[\begin{array}{lllll}0.79 & 0.29 & 0.29 & 0.11 & 0.50\end{array}\right]$.
The correlations between the residuals are given by the matnix
$\left[\begin{array}{ccccc}1 & -.44 & -.44 & -.40 & -.40 \\ & 1 & 1 & -.65 & -.65 \\ & & 1 & -.65 & -.65 \\ & & & 1 & 1 \\ & & & & 1\end{array}\right]$
and from this we can see that points 2 and 3 compensate each othertotally, as do points 4 and 5. Thus, we only have reliable control forblunders in point 1, which has no total correlation with a ny other point.

## Ex 5.4

For the Hasselblad camera with the 25 - point-reseau we get



| 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: |
| 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 |
| 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 |

Here, the va riation between the largest and the smallest $q_{v_{i} v_{i}}$ is $48 / 40=1.2$. The redundancy is $44 / 50=(25-3) / 25=0.88$. Minimum local redundancy is 0.80 (comer point) and maximum local redundancy is 0.96 (centerpoint).

As the residuals in this case have approximately the same standard error it does not seem to be very important to standardize the residuals before comparison, a nalysis, orerror detection. In the case of block adjustment of a erial tria ngulation the standard errors of the residuals a re rather similar in magnitude, too. This has been disc ussed by Grün (1979b), who suggests a simplified method for data-snooping. By taking into consideration the variation of the sta ndard errors of the residuals he can avoid the laborious calculation of the matrix $I-A^{O}$.

For close range applications of photogrammetry, the geometry, as a rule, is such that the standard errors of the residuals vary considerably, a nd thus the standardization of the residuals is necessary. Furthemore, various a pplic a tions have very different geometries, and both these circumstances indicate the importance of deriving matrix $I-A^{O}$ in order to have an effective method when a nalyzing residuals.

Now, if we study the correlations between residua ls after a $n$ adjustment, we still cannot be sure that we know what we are doing. The correlations between the residuals theoretically can have three different causes:
(1) A priori correlation between the true errors through the matrix $C$.
(2) Blunders ( $\nabla$ ) that have not been detected, localized a nd deleted from the observations in the adjustment cause spurious correlation. Such blunders are spread a round in the residuals by the operator

$$
V_{b}=\left(A^{O}-I\right) \nabla
$$

and if the off-diagonal elements are $\neq 0$, which is the case when we have redundant observations in the adjustment to determine unknowns $X$ through indirect observations $L$, then we get spurious correlation.
(3) Systematic errors of the mathematical model (lacking fidelity) such that $E(e) \neq 0$ (or expressed as $A X \neq E(L)$, which is the same). The effect of the systematic emors in the observations are also spread a round in the same way as are blunders.

### 5.2 EMPIRIC AL STUDIES

Now, what type of experiment could possibly a nswer the question on the stoc hastic nature of the true errors $e_{i}$ in the observations $l_{i}$. Well, the causes forcorrelation quoted under (2) a nd (3) a bove have to be eliminated. Let us a ssume that we have been suc cessful in the elimination of blunders using the methods described in chap. 4.5, and by a careful procedure, oc cupational skill, a nd other means avail-
able to avoid blunders. Then, we have to eliminate the effect of the systematic errors of the model, which traditionally can be done by repetition of the experiment under the same exterior conditions, so that the variation in the observations are not influenced by changes of the systematic errors. This, of course is impossible for a erial photography and block triangulations, but for close range photogrammetry in laboratory environment we can repeat our experiment after a short time interval, a nd under very similar exterior environmental conditions. Let us assume one experiment including photography, photographic processing, measurement and adjustment with $n$ observations $l_{i}$ from which we obtain the $n$ residuals $v_{i}$, where $i=1 \ldots n$. The outcome of the experiment can be regarded as one observation point in $n$-dimensional observation space. This experiment is now repeated $m$ times, where $m$ islarge number, say 100 or 200. This gives us $m$ sets of observations $l_{i}$ and $m$ sets of residuals $v_{i}^{k}$ where $k=1 \ldots m$; or $m$ points in $n$-dimensional observation space. Thus, we have one sample of size $m$ on the variables $\lambda_{i}$, the coordinate axes in the $n$ dimensional observation space, and from this sample we are able to estimate the means $\bar{\mu}_{i}=\sum_{k=1}^{m} l_{i}^{k} \quad$ and the variances $\quad \sigma_{i}^{2}=\sum_{k=1}^{m}\left(l_{i}^{k}-\bar{\mu}_{i}\right)^{2} /(m-1)$.
In this way we obtain $n$ means $\bar{\mu}_{i}$ and variances $\bar{\sigma}_{i}^{2}$. Then, we can test the hypothesis that they all have the same mean $\mu_{0}=0$ and the same variance $\sigma_{0}^{2}$. Furthermore, we can also estimate the correlation between the observations by using the estimator $r_{i j}$, giving the estimate $\bar{r}_{i j}$ :

$$
\bar{r}_{i j}=\sum_{k=1}^{m}\left(l_{i}^{k}-\bar{\mu}_{i}\right)\left(l_{j}^{k}-\bar{\mu}_{j}\right) / \sqrt{\bar{\sigma}_{i}^{2} \cdot \bar{\sigma}_{j}^{2}} .
$$

Then the significance of $\bar{r}_{i j}$ forall pairs $i, j$ of pointscan be tested. It may even be possible to suggest a function for $\bar{r}_{i j}$ and $\bar{\sigma}_{i}^{2}$ depending on the position in the photo, for $\bar{r}_{i j}$ depending on distance between points $i, j$ in the photo, time interval between the measurements of $l_{i}$ and $l_{j}$, or something else. To find such a function it is necessary to use a ratherlarge number $n^{k}$ of observations $l_{i}$ in each experiment $k$. Similar experiments have been suggested by Pope (1975) in order to study the stochastic properties of observations.

What has been done up to now, are studies with just one experiment $k$ with a large number of observations $l_{i}$, thus $m=1$ and $n=100 \ldots 10,000$. If conclusions from these studies can be made conceming the properties of true emors, this is very much dependent on whether or not systematic errors of the mathematical model have been corrected, and whether or not all blunders have been eliminated.

Brown (1969) has shown in such a study that there are, for a certain aerial camera, functions for $\sigma^{2}$ with the position in the photo as argument. He derived two functions, one for the radial and a nother for the tangential emor of image coordinates with the radius from the principal point as independent variable. Studies by Hallert (1967) and Morén (1967) suggest just one function for the standard error of the image coordinates as a function of radius. Brown's approach seems to be very reasonable as the physical reality and other knowledge also suggest different functions in radial and tangential directions. The resolution and MTF of lenses are separated in this way, and the random variation of the flatness of the image surface caried by the emulsion has an influence only in the radial direction. But, on the other hand, there are also physical matters that suggest the variation to be described in cartesian coordinate variables rather than
polar, and such a matter is the random variation of image position depending on the photographic processing. Brown (1969) has shown the fact that independent polarvariancescauses the cartesian image coordinates to be correlated.

The error propagation is obtained from

$$
\left[\begin{array}{cc}
\sigma_{x}^{2} & \sigma_{x y} \\
\sigma_{y x} & \sigma_{y}^{2}
\end{array}\right]=\frac{1}{r^{2}}\left[\begin{array}{cc}
y & -x \\
x & y
\end{array}\right]\left[\begin{array}{cc}
\sigma_{r}^{2} & 0 \\
0 & \sigma_{t}^{2}
\end{array}\right]\left[\begin{array}{cc}
y & x \\
-x & y
\end{array}\right]
$$

which gives the correlation coeffic ient

$$
\rho_{x y}=\left(\sigma_{r}^{2}-\sigma_{t}^{2}\right) /\left(\sigma_{r}^{2}+\sigma_{t}^{2}\right) .
$$

This correlation inc reases if there are other physical causes that make the $x$ and $y$ coordinates on the same point to be correlated. The covariances of the contributing components will be added. It should be noted that these studies emphasize the variances of image coordinates of single points. Brown has not given any information on the covariances between different points. It is of course possible that the radialand tangential error show correlation between different points and that such correlation can be dependent on the distance between the points and their radii from the principal point.

Ackermann (1978) has shown that the correlation between different points can be considerably reduced after introduction of additional parameters in the mathematical model. This is an evidence for the presence of systematic error in the model. Thus, his study is rather a motivation for the use of additional parameters in the adjustment, or a more sophisticated coordinate refinement procedure prior to the adjustment, than it is a determination of the stochastic behavior of the true image errors.

### 5.3 CARTESIAN OR SPHERICAL AND CYUNDRICAL COORDINATES

Analytical photogrammetry is almost entirely based on cartesian coordinate systems. Such a system is also the most convenient one ascomparators and plotters provide rectangular coordinates. The photos are flat, and as such easily measured with an $x$ - $y$-digitizer. But there are also instruments to measure polar coordinates availa ble for photogrammetry, e.g. the DBA "bug"comparator, and the now obsolete picture theodolites. The latter instrument really measures the angles in the bundle of rays that is reconstructed from the photo, and thus these instruments are based on a design that is closer to the nature of photogrammetric recordings, namely the bundles of rays and the angles between the rays in the bundle.

It is of course no principal diffic ulty to convert all the cartesian-based formulae for a nalytic al photogrammetry into spherical or cylindrical form. The question is what we would gain by doing so. Following the evaluation procedure in photogrammetry we find:

-     - The mensuration if images is made in comparators that gives $x, y$. Conversion to angles is necessary. No gain.
-     - In both cases, relative orientation is based on series expansions, and the same a mount of computation is needed. No gain.
-     - Absolute orientation seems to be more easily performed on cartesian model coordinates since the absolute coordinates most often are given in such a system. Here it seems to be a drawback to work with other than cartesian model coordinates.
-     - Block triangulation based on bundle angles can have the advantage that available adjustment techniques for geodetic space net works can offersome advanta ges, especially for simultaneous adjustment of geodetic and photogrammetric observations. This might be an advantage, but modem aerial triangulation methods and adjustments are now very effic ient and it must be doubted if the angularapproach can offera better method in general.
-     - Theory of errors might be the best field for a gain to be expected, e.g. for the understanding and correction of systematic and random errors that preferably are divided into radial and tangential components (see, e.g., Brown (1969)). For studies of emror propagation in a stereomodel it seems to be easier to describe the orientation of errorellipsoids in cylindrical ratherthan rectangular coordinates because of the fact that point intersections lie in epipolar planes. This makes the error components more independent of each other.


## 6 THE MULTI-CONCEPT IN CLOSE RANGE PHOTO G RAMMETRY

### 6.1 HIGH ACCURACY PROCEDURES

The analyticalmethods provide new possibilities in orderto increase the accuracy in photogrammetry. Accuracy here comprises precision, fidelity and relia bility. The price for an increased accuracy hasto be paid with better instruments, careful project planning, more observations (i.e. larger measuring times), and rigorous adjustment. The multi-photogrammetry concept is more related to planning, observation and adjustment than to instruments. The multi-concept is the idea of check and control in all phases, repetition and replication, and redundancy in the determination of every unknown variable. The means to reach the utmost accuracy in close range photogrammetry (and it may be applied to photogrammetry in general in most cases) can be summarized as follows

- High quality pictures for the mea surements. The better one can identify object features to be determined, the better will be the accuracy. The importance of photographic quality is not always recognized by photogrammetrists who mostly emphasize geometric problems. Not only high geometric quality of image coordinates obtained through sophisticated camera calibration, but also photographic experience is important (emulsions, filters, illumination, exposure, processing etc.).
- High precision comparators for image coord inate readings. Calibration of the measuring instrument reveals systematic emors, that have to be corrected for.
- Multi-readings of all points on each plate in order to detect blunders in numbering, identification and recording of coordinates. Averaging reduces the variance due to reading the images.
- Multi-fiducials in order to give redundancy in the transformation of coordinates from the comparator system to the camera system, and to control the departures of film deformation from the commonly applied conformal or a ffine fiducial transformation a ssumptions.
- Multi-targets to signa lize object points in the photograph. The regula rity in the pattem of such targets reduces the risk of measuring other images than the signalized points. Local irregularities of the imaging system (atmosphere, lens, emulsion, comparator, operator, etc.) will be averaged out.
- Multi-frames on each station can be arranged in several ways, e.g.

1. repeated photography with the same orientation
2. repeated photography with rotation a round the camera axis
3. repeated photography with camera axis directions changed, yielding partly overlapping bundles from the same station. Imperfections of the imaging system will be averaged out and the precision of the resulting bundle will be increased.

- Multi-stations for point intersections in object space are necessary to provide a tool for blunder detection in the final adjustment. The stations should be positioned as to give optimal intersection angles at the object points, and each point should be determined from at least four stations in order to give redundancy also after deletion of one ray due to a blunder in the observations.
- Multi-control means that the transformation into the object space coordinate system hasto be based on redundant information not only in the form of given coordinates of control points, but also in the form of geometric object space conditions (lines, planes, angles, distances, parallelism, etc.) a nd direct observations on the orientation elements.
- Multi-purpose program with rigorous adjustment is based best on the bundle approach. Forclose range applications the number of bundles and the number of object points are often limited such that least squares adjustment can contain all unknowns in a direct solution, also providing the error propagation to unknownsand functions of observations and unknowns. The adjustment process must take into consideration all sorts of observations, a nd therefore, a priori information on the weights of the observations have to be known and considered. Additional parameters are necessary to design a mathematical model that has high fidelity to the real geometry of the images. Rigorous adjustment based on a general approach does not impose a ny restrictions on the choice of cameras, location of photo stations, camera orientations, type of object space control, etc. The term "multipurpose program"also covers the possibility to use the program both for the calibration of camerasand for the determination of object geometry.


### 6.2 MULTI - READINGS

Hottier (1976) has made a very comprehensive study of the gain in accuracy that can be obtained by
a) repeating the settings,
b) using multi-targets, a nd
c) talking the average of more picture frames on each station.

He found that the gain in accuracy by repetition of the settings is at most $30 \%$ (decrease of RMS in check points) for up to 7 settings on single targets in single frames. He also states that the effect of repeated settings is smaller than the effect of multi-targets and multi-frames, and for an optimal gain of accuracy in relation to the measuring work, he recommends one setting, two targets, two frames, or one setting, one target, three frames.

With computer-assisted data capture there is the possibility to control the setting precision interactively. The requirements on the precision can be formulated in different ways such as maximum range, maximum standard deviation of the mean, rejection of out-liers a fter a djustment, etc ., see e.g. Dorrer (1977).

Hottier worked under "controlled" conditions with a test field, and it can be assumed that blunders were immediately detected, localized and eliminated. For practicalclose range photogrammetric projects, however, the procedure must be such that blunders are detected, localized and eliminated without a priori knowledge on the object.

Repeated readings of image points is a well established method to detect a nd localize blunders. Mista kes in points identific a tion and numbering is often detected immediately. The order of reading the points when repeating is commonly reverse to the order of the first set. If the maximum range criterion is a pplied, the points with large differences will be measured a third time to see which of the readings on the points is being likely to be an out-lier. If this third reading also seems to be an out-lier the read-
ings are repeated until any two of them fall within the predetemined tolerances. These two are then regarded as representative for the point. Intuitively we can say that, if the tolerance is too wide in relation to the setting precision, out-liers will conta minate the result; and if it is too na rrow, the measuring work will be enlarged without yielding a ny better results, because we a re deleting good observations until it happens that a ny two of them are close enough to each other.

If the tolerance is extremely small, it has the effect that a lot of good observations are thrown away before two observations fall within the tolerance interval. Thus, it is rather important to have tolerance limits that a re reasonable. The following example will demonstrate the effect of different tolerance ranges. Suppose the true error of our observations is nomally distributed a coording to $N(0, \sigma / \sqrt{2})$. The differences between repeated observations are then distributed according to $N(0, \sigma)$. Using the acceptance interval $\pm 3 \sigma$ for the differences leads to rejecting $0.3 \%$ of the good observations. The acceptance interval of $\pm 1 \sigma$ leads to a rejection rate of $31.7 \%$. The power of these is shown in Fig. 6.1 where we find that a power of $80 \%$ leads to acceptance of erroneous observations if the blunder is smaller than $2.9 \sigma$ and $2.3 \sigma$, respectively. The standard deviation of the mean will be smaller than $\sigma / 2$, beca use we have truncated the distribution of the differencesat $\pm 3 \sigma$ and $\pm 1 \sigma$, respectively.


Fig. 6.1

Power functions for testing outliers

## Ex 6.1

Twenty points have been observed twice each. The true errors of the first setting are unknown to the obsenver, but he hasdifferences between the double readings. Points having differences exceeding the tolerances are repeated until two observations are accepted. The values in the following table 6.1 represent a simulated sample with single observations $N(0,0.71)$ and the differences $N(0,1)$. The sample means and the sample variances are given at the end of the table 6.1.

Table 6.1:

True errors of simulated sample and successive differences applying the $1 \sigma$-rule.

| Point no | True error 1st setting |  | Differencesapplying the $1 \sigma$-rule |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |  |  | Further |  | Rem |
|  |  | $e_{y}$ | $d_{x}$ | $d_{y}$ |  | $d_{y}$ |  | $d_{y}$ |  |  |  |
| 1 | -0.952 | 0.884 | 0.464 | 0.060 |  |  |  |  |  |  |  |
| 2 | 0.136 | -0.141 | 0.137 | -2.526 | $-0.555$ | -0.513 |  |  |  |  |  |
| 3 | -0.850 | -0.204 | 2.455 | -0.531 | 0.046 | -0.525 |  |  |  |  |  |
| 4 | 0.279 | 1.280 | -0.323 | -0.194 |  |  |  |  |  |  |  |
| 5 | -0.739 | 0.974 | -0.068 | 0.543 |  |  |  |  |  |  |  |
| 6 | 0.596 | 0.413 | 0.290 | -1.558 | 0.321 | 0.595 |  |  |  |  |  |
| 7 | 0.666 | 0.860 | -0.288 | 0.187 |  |  |  |  |  |  |  |
| 8 | 0.739 | 0.518 | 1.298 | -1.190 | 2.945 | 0.881 | $-1.005$ | -0.044 | 0.712 | 0.203 | 6th |
| 9 | 0.022 | 0.284 | 0.241 | 0.022 |  |  |  |  |  |  |  |
| 10 | 0.546 | 0.160 | -0.957 | 0.525 |  |  |  |  |  |  |  |
| 11 | 0.445 | 0.265 | 1.486 | 1.022 | 1.974 | -0.934 | 0.007 | -0.162 |  |  |  |
| 12 | -0.380 | -1.372 | $-0.354$ | -0.472 |  |  |  |  |  |  |  |
| 13 | 0.553 | 0.175 | -0.634 | 1.279 | -0.258 | 1.579 |  |  | 0.376 | 0.300 | 3 rd |
| 14 | 0.042 | -0.347 | 0.697 | 3.521 | 0.412 | 0.161 |  |  |  |  |  |
| 15 | 0.353 | 0.470 | 0.926 | 0.571 |  |  |  |  |  |  |  |
| 16 | -0.298 | -0.095 | 1.375 | -1.851 | 0.439 | $-1.885$ |  |  | -0.926 | -0.034 | 3 rd |
| 17 | 1.206 | -0.103 | 0.785 | 0.194 |  |  |  |  |  |  |  |
| 18 | 0.823 | -0.352 | -0.963 | 1.192 | -0.035 | 0.371 |  |  |  |  |  |
| 19 | 0.625 | 0.323 | -0.853 | -0.501 |  |  |  |  |  |  |  |
| 20 | -0.211 | 0.752 | -1.865 | -0.278 | 1.179 | $-1.501$ | 0.769 | -0.136 |  |  |  |
| sample | 0.085 | 0.237 | 0.192 | 0.001 |  |  |  |  |  |  |  |
| mean | 0.782 | 0.595 | 1.030 | 1.298 |  |  |  |  |  |  |  |
| st. dev. |  |  |  |  |  |  |  |  |  |  |  |
| ms | 0.766 | 0.623 | 1.023 | 1.265 |  |  |  |  |  |  |  |

Applying the $3 \sigma$-rule on $d_{x}$ and $d_{y}$, observation no. 14 would be repeated, giving 3rd -1st $d_{x}=1.394$ and $d_{y}=0.906$, and thus

$$
\text { 3rd-2nd } d_{x}=0.697 \text { and } d_{y}=2.615
$$

We obviously delete setting no. 2 on point 14.

Now, we estimate the precision of the setting from the differences that we have after applying the $1 \sigma$-rule and the $3 \sigma$-rule. But as we here have simulated data we know the true errors and then we can calculate the mean, standard deviation and root mean square value of the true errors after taking the average of the accepted two observations per point. The decrease of the root mean square (RMS) errorafter averaging is the measure of the gain in accuracy that we obtain by only applying the acceptance tolerance for repeated settings.

Table 6.2:

Precision estimated from differences between double readings.
The precision refers to the difference which is $N(0,1)$

|  | $\infty \sigma$ |  | $3 \sigma$ |  | $1 \sigma$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ |
| Mean | 0.192 | 0.001 | 0.227 | -0.130 | -0.024 | 0.074 |
| St. dev. | 1.030 | 1.298 | 1.060 | 1.028 | 0.560 | 0.370 |
| Rms | 1.023 | 1.265 | 1.058 | $1.0-0$ | 0.546 | 0.369 |

Table 6.3:
Accuracy from true errors of the average of two settings.
The accuracy refers to the average which is $N(0,0.5)$

|  | $\infty \sigma$ |  | $3 \sigma$ |  | $1 \sigma$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ |
| Mean | 0.276 | 0.264 | 0.326 | 0.239 | 0.235 | 0.295 |
| St. dev. | 0.701 | 0.786 | 0.744 | 0.755 | 0.637 | 0.801 |
| Rms | 0.737 | 0.811 | 0.795 | 0.773 | 0.664 | 0.834 |

Thus, in this sample the observable improvement in $x$ goes from 1.023 to 1.058 , resp. 0.546, but it corresponds to the real improvement from 0.737 to 0.795 , resp. 0.664, and in $y$ the observable improvement goes from 1.265 to 1.010, resp. 0.369, but it corresponds to the real improvement from 0.811 to 0.773 , resp. 0.834 . The conclusion must be that smaller tolerances do not yield a ny better results in relation to the true values of the observed quantities. A tolerance of $5 \sigma$ may be as good asone of $1 \sigma$ but without unnecessary work by avoiding rejection of good observations. In order to decide in which of the two repeated observations the blunderoccurs, a third, or more, reading is as a rule done, until two readings a re close enough. But the difficulty can also be overcome by using the original observations in an adjustment where observations in other points help to localize the blunder.

## Ex 6.2

Four fiducial marks are measured twice. In one of the fiducial marks the double readings differ considerably.

If the averages of the double measurements are treated asobservations in an affine coordinate transformation we obtain the following matrices

$$
\begin{aligned}
& A=\left[\begin{array}{rrr}
1 & a & b \\
1 & a & -b \\
1 & -a & b \\
1 & -a & -b
\end{array}\right] \\
& N=\left[\begin{array}{ccc}
4 & 0 & 0 \\
0 & 4 a^{2} & 0 \\
0 & 0 & 4 b^{2}
\end{array}\right] \\
& Q=\left[\begin{array}{ccc}
1 / 4 & 0 & 0 \\
0 & 1 / 4 a^{2} & 0 \\
0 & 0 & 1 / 4 b^{2} \\
Q_{v v} & =I-A^{o}=\frac{1}{4}\left[\begin{array}{rrrr}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1
\end{array}\right]
\end{array}\right]
\end{aligned}
$$

From the last matrix it is obvious that an error in any of the observations will be distributed with equal magnitudes in all residuals. The trace is 1 , the relative redundancy $1 / 4=0.25$, all diagonal elements 0.25 . The algebraic correlation between the residuals is +1 or -1 in all combinations and thus it is not possible to localize the blunder. If we eliminate one of the observations, the matrix $I-A^{O}$ will have only zero elements; there is no redundancy left.

The inverse $\left(I-A_{2} Q A^{T}\right)^{-1}$ does not exist for any two observations.

But, on the other hand, if we introduce the original eight observations in the adjustment of the affine transformation we get the following:

$$
A=\left[\begin{array}{rrr}
1 & a & b \\
1 & a & b \\
1 & a & -b \\
1 & a & -b \\
1 & -a & b \\
1 & -a & b \\
1 & -a & -b \\
1 & -a & -b
\end{array}\right]
$$

$$
\begin{aligned}
N & =8 \cdot\left[\begin{array}{rrc}
1 & 0 & 0 \\
0 & a^{2} & 0 \\
0 & 0 & b^{2}
\end{array}\right] \\
Q & =\frac{1}{8} \cdot\left[\begin{array}{rrrr}
1 & 0 & 0 \\
0 & 1 / a^{2} & 0 \\
0 & 0 & 1 / b^{2}
\end{array}\right] \\
I-A^{O} & =\frac{1}{8} \cdot\left[\begin{array}{rrrrrrrr}
5 & -3 & -1 & -1 & -1 & -1 & 1 & 1 \\
& 5 & -1 & -1 & -1 & -1 & 1 & 1 \\
& & 5 & -3 & 3 & 3 & -1 & -1 \\
& & & 5 & 1 & 1 & -1 & -1 \\
& & & & & -3 & -1 & -1 \\
\text { symmetric }
\end{array}\right.
\end{aligned}
$$

The trace is 5 , relative redundancy $5 / 8=0.625$, the diagonal elements are all 0.625 , the correlation between the double readings are $r_{d}=0.6$, while the correlation to the other readings is $r_{v}= \pm 0.2$. Now, it is easy to localize the blunders (after standardization by dividing the residuals by $s_{O}$ and the square root of the corresponding diagonal element of $I-A^{O}$ ), a s the major part of the blunder (0.625) stay at the position of its origin, and only a smaller part (0.375, resp. 0.125 ) is transferred to the other residuals. Let us say that we have a blunder in the last observation, a nd after deletion we get

$$
\begin{aligned}
& Q_{1}=\frac{1}{40} \cdot\left[\begin{array}{ccc}
6 & -1 / a & -1 / b \\
& 6 / a^{2} & 1 / a b \\
& & 6 / b^{2}
\end{array}\right] \\
& I-A^{O}=\frac{1}{40} \cdot\left[\begin{array}{rrrrrrr}
24 & -16 & -4 & -4 & -4 & -4 & 8 \\
& 24 & -4 & -4 & -4 & -4 & 8 \\
& & 24 & -16 & 4 & 4 & -8 \\
& & & 24 & 4 & 4 & -8 \\
& & & & 24 & -16 & -8 \\
& & & & & 24 & -8 \\
\text { symmetric } & & & & & 16
\end{array}\right]
\end{aligned}
$$

The trace is $160 / 40=4$, the relative redundancy is $4 / 7=0.571$, the residuals of the double observations have $q_{v v}=24 / 40=0.60$, while the single one has $q_{v v}=16 / 40=0.4$, all correlation coeffic ients a re a bsolutely smaller than 1 ( $0.67,0.41$ and 0.17 ), which makes it possible to localize a second blunder in the observations, even when it occurs in the last point.

The conclusion that can be drawn from this example is the following: In order to a void rejection of good observations and unnecessary repetition of observations the original measurements rather than their a verages should be used in the next a djustment step of the procedure. The localization of blunders is then done easier because the design matrix $A$ is changed. This, however, requires in practice that the computation can be done as soon as there are enough repetitions of observations. It is not necessary to measure all points twice before adjustment; the testing can start as soon as a solution is possible, and then succeeding observations are added to the adjustment by sequential tec hniques referred to in chapter 4.1. Analytic al plotters and comparators with online computa tional facilities offer the hard ware needed for such procedures.

### 6.3 MULTI - FIDUCIALS

In most cameras there are four fiducial marks. Using a conformal fiducial transformation of coordinates from the compa rator system to the camera system we have four unknowns, eight observations and four redund ant observations giving a relative redundancy of 0.5 , which is just on the limit of acceptability when thinking of blunders. But what sort of correlations exist between the residuals after the adjustment? Can we localize and eliminate blunders, and still have redundancy? The answercan be found in the matrix $I-A^{O}$.

## Ex 6.3

For the case of a conformal transformation on four comer fiducials we have

$$
A=\left[\begin{array}{rrrr}
1 & 0 & a & -b \\
0 & 1 & b & a \\
1 & 0 & a & b \\
0 & 1 & -b & a \\
1 & 0 & -a & -b \\
0 & 1 & b & -a \\
1 & 0 & -a & b \\
0 & 1 & -b & -a
\end{array}\right] \quad 3 a^{1}
$$

$$
I-A^{O}=\frac{1}{4 r^{2}} \cdot\left[\begin{array}{cccccccc}
2 r^{2} & 0 & -2 a^{2} & +2 a b & -2 b^{2} & -2 a b & 0 & 0 \\
& 2 r^{2} & -2 a b & -2 a^{2} & -2 a b & -2 b^{2} & 0 & 0 \\
& & 2 r^{2} & 0 & 0 & 0 & -2 b^{2} & +2 a b \\
& & & 2 r^{2} & 0 & 0 & -2 a b & -2 b^{2} \\
\\
& & & & 2 r^{2} & 0 & -2 a^{2} & -2 a b \\
\text { symmetric } & & & & 2 r^{2} & +2 a b & -2 a^{2} \\
& & & & & 2 r^{2} & 0 \\
& & & & & & 2 r^{2}
\end{array}\right]
$$

where $r^{2}=a^{2}+b^{2}$.
The trace is 4 , relative redundancy $4 / 8$, all $q_{v v}=0.5$ and the comelation coefficients $0, a^{2} / r^{2}, b^{2} / r^{2}$ or $\pm a b / r^{2}$ which, for

$$
a=b=r / \sqrt{2} \text { give } \rho=0 \text { or } \rho= \pm 0.5 .
$$

Blunder loc a lization is feasible, but does there remain redundancy after elimination ? Using the formula given in chapter 4.1 we find

$$
\begin{aligned}
& \left(I-A_{2} Q A_{2}^{T}\right)^{-1}=\left[\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right] \\
& A_{1} Q A_{2}^{T}=\frac{1}{4 r^{2}} \cdot\left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
-2 b^{2} & -2 a b \\
-2 a b & -2 b^{2} \\
-2 a^{2} & -2 a b \\
2 a b & -2 a^{2}
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
& A_{1} Q A_{2}^{T}\left(I-A_{2} Q A_{2}^{T}\right)^{-1} A_{2} Q A_{1}^{T}= \\
& =\frac{8}{16 r^{4}} \cdot\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 \\
& & r^{2} b^{2} & 0 & 0 & a b r^{2} \\
& & & r^{2} b^{2} & a b r^{2} & 0 \\
\\
& & & r^{2} a^{2} & 0 \\
\text { symmetric } & & & r^{2} a^{2}
\end{array}\right] \\
& Q_{1}=\frac{1}{2 r^{2}} \cdot\left[\begin{array}{cccccc}
r^{2} & 0 & -a^{2} & a b & -b^{2} & -a b \\
& r^{2} & -a b & a^{2} & a b & -b^{2} \\
& & a^{2} & 0 & 0 & a b \\
& & & a^{2} & -a b & 0 \\
& & & & b^{2} & 0 \\
\text { symmetric } & & & b^{2}
\end{array}\right]
\end{aligned}
$$

The trace is 2 , relative redundancy $2 / 6$, the $q_{v v}$ are 0.5 , $a^{2} / 2 r^{2}$ or $b^{2} / 2 r^{2}$ (for $a=b=r / \sqrt{2}$ the latterare 0.25 ), the correlation coefficients are $0, \pm 1, \pm a / r$ or $\pm b / r$ (for $a=b=r / \sqrt{2}$ the latter two are $\pm 0.71$ ),
and we still have redundancy, but not enough for blunder localization in points 2 and 3 , because here the correlation is $\pm 1$ between $v_{x 2}$ and $v_{y 3}$, resp. $v_{y 2}$ and $v_{x 3}$.

For the case of conformal tra nsformation on four midside fiducials we obtain analogously

$$
A=\left[\begin{array}{rrrr}
1 & 0 & a & 0 \\
0 & 1 & 0 & a \\
1 & 0 & 0 & -b \\
0 & 1 & b & 0 \\
1 & 0 & -a & 0 \\
0 & 1 & 0 & -a \\
1 & 0 & 0 & b \\
0 & 1 & -b & 0
\end{array}\right]
$$

$I-A^{O}=$


The trace is $=4$, relative redundancy $0.5, q_{v_{x} v_{x}}=\frac{1}{4}+\frac{a^{2}}{2 r^{2}}$
and $q_{v_{y} v_{y}}=\frac{1}{4}+\frac{b^{2}}{2 r^{2}}$, both equal $1 / 2$ for $a=b=r / \sqrt{2}$.

The correlation coefficients are 0 or $\pm 1 / 2$ for $a=b$, and blunder localization is possible in a ny point; but after elimination of one point further localization of blunders is not possible, as in the previous example.

It is very common to use an affine transformation for the fiducial marks, and here two more parameters are introduced as unknowns, the design matrix $A$ is correspondingly changed and the redundancy is dec reased to such an extent that blunder loc a lization is impossible on four fiducial marks with just one observation. Here, more fiducial marks are necessary, as e.g. in the Wild P31 and P32 with 5 or 17 , or in the Hasselblad MK 70 with 25 fiducial marks. The $Q_{v v}$-matrices for affine transformations have been presented and disc ussed in chapter 5.1 and 6.2.

In reseau cameras and in some othertypes, the fiducial marks have the shape of a cross. In this case one can preferably take four (or two times four) mea surements on the arms of the crosses and compute the intersection ratherthan repeat the settings in the centre, in order to increase the precision. An example of a measuring instruction for cross-shaped fiducial is given in a ppendix 1.

### 6.4 MULTI - TARGETS

The points to be measured are very often targeted before photography. In aerial triangulation pre-targeting of all points (known, unknown and tie) has proved to be an effic ient method to increase the accuracy, thereby avoiding errors in artificial point transfer and bad point definition when using natural transfer points. Multi-targeting has been used in aerial (e.g. Kupfer (1973) a nd Hvidegaard (1976)) as well as in close range photogrammetry (Hottier (1976)). Multi-targets can be designed in several ways depending on the relevant parameters in the specification. Such parameters are, e.g., the size and shape of the measuring mark in the comparator, the average image scale, the variation of the image scale within and between the pictures, the angle between the target plane and the image plane (convergency angle), method of attaching the targets to the points on the object (gluing, self-a dhesive, painting, nailing) number of targets per signalized point, environmental conditions, resolution viewing magnific ation, etc. Some examples of multi-targets a re shown in figs. 6.2-6.5.

The sevenfold target has the advantage that one can choose the targets such that the average of the observations coincides with the position of the central one for any number of targets from 1 to 7 . For 2,3 or 4 targets there are severalcombinations and the operator can choose the one that has the best pictorial quality. The cross-like target is suited for oblique photography with varying image scale. The fivecircle target is good for varying image scale, but it is limited to 2 or 3 targets per point. Hottier (1976) found that the maximum gain of accuracy, by increasing the number of targets per object point, is about $40 \%$, which is atta ined for $4-5$ targets. He recommendsasa practical optimum one setting, one target, and three frames, or one setting, two targets and two frames. If only one frame is available it seems to be a good compromise to have two settings on two targets or one setting on three orfour targets as judged from Hottier's results.


Fig. 6.2
The sevenfold target, suited for any number of targetsfrom 1 to 7


Fig. 6.3
The star-like target, suited foroblique photos
in varying scales


Fig. 6.4
The five-circle target, suited for va rying ima ge scales

A most effic ient means to inc rease the precision of the reconstructed bundle of rays is to take more pictures on each station and use their average in the following steps. Hottier (1976) has studied the effect of this technique for a close range normal case stereomodel, and he found that the maximum gain in precision was $40 \%$ for up to 5 frames per station (measured with one setting on single targets). For practical purposes recommends one setting pertarget, two (three) targets per point and two (three) frames perstation, which yields $40 \%$ (50\%) ga in of precision.

For the duplic ated frames Hottier used the same exterior orienta tion. The effect of the random errors is reduced through averaging, but systematic and constant errors rema in the same. Such systematic errors of our mathematical model of the imaging geometry can, e.g., be caused by imperfections of the lens, principal point error, bending of plates, film flattering and film deformation. These errors can be described in terms of radial and tangential components in the image plane. The tangential component, and that part of the radial component that is not rotationally symmetric can be detemined (oraveraged out) by rotating the camera around the camera axis between the pictures. The rotationally symmetric part of the radial component is not determinable. Here a quadratic image format is ideal, since coverage will be identical in the fourcardinal positions. Ta king the average of the bundles of rays from frames with $\kappa$-rotations 0 , $\pi / 2, \pi / 2,3 \pi / 2$ corresponds in some way to the station adjustment in geodetic triangulation.


Fig. 6.6
Multi-frames on the same station with the camera axes in the same direction but with different $\kappa$-rotations make it possible to eliminate tangential image errors and the irregular part of the radial image errors.

Each frame is measured and the observations are transformed to the camera system. After rotating the camera systems to one and the same reference position, the sets of image coordinates can be compared. Small discrepancies will always occur due to setting precision and error propagation from the fiducial transformation, systematic and random errors (in the camera-film performance, changes of the exterior orientation other than just the $\kappa$-rotation, and blunders. Therefore, for comparison of different sets of coordinates a differential perspective transformation is appropriate. If tangential distortion is considerable the observation equations have to be amended with tems covering this effect.
















$\therefore 8_{i}^{n} 0_{i}^{n} 8_{i}^{n} 0_{i}^{n} 0_{i}^{n} 8_{i}^{n} 8_{i}^{n} 0_{i}^{n} 8_{i}^{0} \quad$ O









$\therefore 8 \div 0 \times$
$N \times \underset{i}{ \pm} \underset{i}{\infty} \infty_{0}^{\infty}$

Table No 6.4
$\therefore 8.9$
$-\times \geqslant$


After the perspective transformation has been done we can study the residuals in orderto detect, localize and eliminate occuming blunders, and for that purpose we need the $Q_{v v}$-matrix, which as usual is $\left(I-A^{O}\right)$.

Assume that we have pictures taken with a camera having a format slightly larger than $2 / 3 \times 2 / 3$ of the principle distance $c$ (e.g. Hasselblad MK 70 with Biogon 60) and that the perspective transformation is based on 25 well distributed image pointslocated in grid intersections with the $x^{\prime}$ and $y^{\prime}$ coordinates equal to $0, \pm c / 6, \pm c / 3$. The $Q_{v v}$-matrix has $50 \times 50$ elements, but due to the point symmetry there is a good symmetry in the matrix as well and it is not necessary to calculate more than $1 / 8$ of the elements. The diagonal elements are shown in Table 6.4. The maximum absolute value of the correlation coeffic ients between the residuals is 0.210 , a nd it occur between $v_{x}$ of a corner point and $v_{x}$ of its closet neighbour having the same $x^{\prime}$ coordinate (and similarly for $v_{y}$ and $y^{\prime}$ ). It can be noted that the maximum value of $q_{v_{i} v_{i}}$ does not belong to the central point. The comer points take the smallest values asexpected. The trace of the matrix is 44 , which is the number of degrees of freedom in the adjustment. The mean value of the diagonal elements is 0.880 , and the range thus $0.762-0.944$. In this case it is easy to localize and eliminate blunders because the correlation coefficients are small, the inverse exist, a nd the diagonal elements are well over 0.5 .

In order to determine or eliminate the effect of the rotationally symmetric part of the radial component, the frames have to be taken with the camera axis pointing in different directions. This has been applied by Borchers (1965) in structural deformation measurements. He determined the bending of the plates from the radial components and the structural deformation from the tangential components of the image displacement vector. Taking nine frames as indicated in Fig. 6.7 we can determine both tangential and radial error components. For very high accuracy requirements, e.g. camera calibration, the "nine-frame-pattem" can be repeated forthe fourcardinal $\kappa$-rotations giving in total 36 frames of the same bundle of rays. This certa inly is too much to be practical, and the optimum combination might be found by an investigation devoted to that particular problem. A combination of $4 \kappa$-rotated central pictures, two $\omega$-rotated and two $\varphi$ - rotated would possibly yield an optimum. To facilitate a convenient photography one has to design and build a camera support with gimbals axes going through the exterior projection centre, so that the camera position is unchanged when the directions are varied. For ultra precise work the eccentricities might have to be considered in the computations. The systematic componentscan be formulated as additional parameters in the mathematic al model as it is done in aerial block adjustment.


Fig. 6.7
Nine-frame experimental design with different directions of the camera axesthat makes it possible to eliminate radial image errors.

But there is a difference between aerial and close range photogrammetry as to the exterior orientation of the frames. In aerial block adjustment there are a large number of frames all having different exterior orientation and typic ally 60 \% by $20 \%$ or $60 \%$ by $60 \%$ overlap. In close range we can have groups of frames with the same exterior orientation except for some or all three directions of the camera axis. This can certainly provide advantagescompared to the aerial case. The possibilities have not yet been studied for close range applications. There are some studies on the effect of multiple strips with different flight directions in aerial photography for the determination of additional parameters, e.g. Thomas (1977).

### 6.6 MULTI - STATIONS

The concept of multi-stations is well known in photogrammetry through bundle block adjustment and a series of a nalytic al solutions in close range photogrammetric measurement problems. The ordinary case in aerial block adjustment with vertical photography over flat terrain from the same altitude with one and the same $15 \times 23 \times 23$ camera in parallel strips standardized overlap and control points according to well known thumb-rule pattems has to be generalized in close range photogrammetry with respect to camera orientation (interior and exterior), object geometry, overlaps, control points, object geometry conditions, targeting, etc. It is typic al for high precision close range photogrammetry to try planning the photography in such a way that the ray intersections in the reconstructed model will be as good as possible, preferably under right angles. This leads to convergent photography with nearly $100 \%$ overlap, and to get a homogeneous precision one often useslong focal lengths if the space around the object is large enough, in order to give approximately the same image scale over the entire object. Under these circumstances it is not very convenient to measure the pictures in pairs in a stereo comparator and treat the data in units of stereomodels. The appropriate approach obviously is the bundle adjustment, and thus the pictures preferably are measured one by one, and the image coordinates are the observationsto be adjusted. Most of the theoretic al and practic al investigations into the accuracy of a nalytic al close range photogrammetry have been related to the precision of the object coordinates. Some investigations treat the formulation of the proper mathematic al model for the imaging geometry. Very few reports are given on studies of the relia bility of the experimental design in close range photogrammetry. One recent report is given by Grün (1978b). He applies the "data-snooping" according to Baarda, defines measures for precision and reliability of the experimental design, and he simulates examples with close range bundle block having 2-4 frames. The precision indicator PI defined by Grün is the mean value of the root of the diagonal elements of the inverse of the nomal equation matrix $Q_{x x}$, thus $P I=\operatorname{tr}\left(\sqrt{Q_{x x}}\right) / p$. The reliability indicator $R I$ is defined as the mean of the $q_{v_{i} v_{i}}$ values, thus $R I=t r\left(Q_{v v}\right) / r$. (I think that a more proper definition of the precision indic ator would be $P I=\operatorname{tr}\left(Q_{x x}\right) / p$. Grün's results show that good precision is obtained from good intersection angles of the rays in the object points, i.e. good base-to distance ratios. This holds also for pairs of pictures. Good reliability on the other hand is obtained if four pictures, three of which do not lie in the same straight line, form a block to determine the object points with rays of which not more than two are in the same plane. This holds also for small
base-to-distance ratios. For such a photoquadruple the reliability indicator is $R I=0.62$. In the case of a stereo-pair the $q_{v_{i} v_{i}}$ - values will be zero for the coordinate that is orientated in the epipolarplane direction (zero variance problem). This, of course, is well known; it is just the vertic al coordinate that is controlled in the stereo-pair case (vertical parallax). For three pictures with camera axes in the sample plane the situation is slightly better, but not good enough, as the reliability is much better for $y^{\prime}$ than $x^{\prime}$. Grün suggests that the three camera stations be chosen such that the intersection rays in the object points form three planes which do not coincide. In this way a more uniform reliability is expected in the different coordinate directions. The importance of multiple-ray intersection for the reliability has been further studied by Grün (1979b), but for aerial blocks. It a ppears from that study that, at least four intersecting rays are necessary for a good reliability, and that the rays must form at least four different planes intersecting but not coinciding in the object point. The $q_{v_{i} v_{i}}$ - values are for these points on the average 0.50 . (Grün uses in hispaper $\sqrt{q_{v v}}$,thusobtaining 0.70 as an average). The results presented by Grün (1978b) and (1979b) indicate that the following thumb-rule might be valid: In order to obtain good reliability in photogrammetric block adjustment, it should be such a geometry that the object points are determined from at least four rays that pairwise, form epipolarplanes that do not coincide; and to obta in good precision these epipolarplanes should intersect at right angles. The geometry is demonstrated in Fig. 6.8. The validity of this thumb-rule should be investigated and studied in theory and practice.


Plane 13 P
Plane
24 P
Plane 23 P
Plane 12 P
Plane
14 P
Plane
34 P
Fig. 6.8 Epipolar planes form four pictures 1, 2, 3 and 4 of one point $P$.

### 6.7 MULTI - CONTROL

In most cases control is given as coordinates of full (XYZ), planimetric (XY) or height (Z) control points. In some aerial block adjustment programs there is a possibility to use the information that all points on the shoreline of a lake have the same but unknown height. In some cases auxiliary data from sta tosc ope, APR a nd the like can be used in the adjustment. Similar information originating from both object and photography can efficiently be used in close range photogrammetry.

In a nalogue methods approximate values are used in the orientation process, e.g. in aerial photogrammetry, where vertical photography is assumed, the operatorcan use this information to orient the model even if the control is insufficient, say, the height control is mainly located in a line. Then the operator uses the approximation to the vertic al to orient the model in height so that at least planimetry can be plotted and perhapsalso topographic form-lines if they can serve the purpose for which the map is intended. If there is insufficient control in the numerical orientation, this leads to a singular matrix of observation equation coefficients. A general method to overcome this problem is to add a number of equations which correspond to direct observations on the unknowns in the form of approximate values. These equations are given weights that take into a c count the uncertainty of the approximations. Direct observations on the exterior orientation parameters can also be introduced in the system of observation equations. This is often the case for levelled metric cameras and photo-theodolites. Forstereometric cameras with fixed base the distance between the two projection centres can be introduced as a ficticious observation with weight corresponding to the precision of the base. In the same way fictic ious observations on the interior orientation elements can be introduced, and corrections will be computed to these values in the adjustment. See e.g. Wrobel-Kruck (1978) and Hell (1979).

In this way a singular system of equations will always be avoided. As approximations have to be calculated anyhow, orgiven in input, for the linearization, they all are available, and in the described way the same algorithm can be used for all cases, also such where control is missing. The effect will also show up in the errors of the unknowns and functions of them, in such a way that the elements in the normal equation inverse $Q=\left(A^{T} P A\right)^{-1}$ will be large for the weakly controlled unknowns. For uncontrolled unknowns $x_{i}$ the $Q_{x_{i} x_{i}}$ corresponds to the weight of the approximations $P_{i i}=s_{0}^{2} / s_{i}^{2}$ given a priori.

Utilization of vertical and horizontal planes, straight lines, angles, etc., for the relative and absolute orientation in analogue methods of close range photogrammetry has also to be formulated in a mathematical way and must be included in the adjustment (Kager-Kra us (1976), a nd Hell (1979)). Doing so, there will sometimes be introduced new unknown parameters in the adjustment, e.g. position and attitude of an arbitrary plane in the object space. These condition equations are given weights that corespond to the precision with which the condition can describe the coresponding matter in reality. It happens very often that these conditions are non-linear, and thus they have to be linearized before they are entered into the equation system.

### 6.8 MULTI - PURPOSE PROGRAM

To summarize, we now have a system of equations that has the following structure:

where the
$A B C D E F G$ are coefficient matrices
$Y$ vector of corrections to approximations of exterior orientation elements $\left(X_{O} Y_{O} Z_{O} \omega \varphi \kappa\right)_{i}$ six elements for each photo $i$
$X \quad$ vector of corrections to approximations of object
space coordinates ( $\begin{aligned} & X Y Z)_{k}\end{aligned}$ 3 unknownsforeach point $\kappa$
$Z \quad$ vector of corrections to approximations of interior orientation parameters and additional parameters

a set of selected parameters for each camera $l$ used
$U \quad$ vector of corrections to approximations of unknown parameters in geometric object space conditions
$I \quad$ unit matrix
$L_{1} \ldots L_{7} \quad$ right hand side term in the linearized observation equation (often called discrepancies)
$P_{b} \ldots P_{e} \quad$ weight matrices for the observation equations.

The equations of type 2 and 5 are of the same kind, the only difference being that the weights of the given control points are much higher than those of the unknown points for which we have just approximations. The same holdsfor equations of type 3 and 4.

The model is very general, it can be used forcalibration purposes, for point determination, for resection, etc. Forcamera calibration the interior orientation elements $Z$ are unknowns. If the calibration is based on testfield then $X$ is known with high accuracy.

$$
\begin{aligned}
A Y+B X+G Z & =L_{1}, P_{b} \\
C X & =L_{2}, P_{g} \\
I Y & =L_{4}, P_{n y} \\
+I Z & =L_{6}, P_{n z}
\end{aligned}
$$

If the given points are regarded as free from errors we have $P_{g}=\infty$. Then, $X=C^{-1} L_{2}$. Introducing this in the first set of equations we get

$$
\begin{aligned}
A Y+G Z & =L_{1}-B C^{-1} & L_{2}, P_{b} \\
I Y & =L_{4} & , P_{n y} \\
I Z & =L_{6} & , P_{n z}
\end{aligned}
$$

As here in this case all coordinates of all points are known we have $C=I$.
$L_{1}$ are measured image coordinates and $B C^{-1} L_{2}=B L_{2}$ are given object coordinates $B L_{2}$ using the approximate values on $Y$ and $Z$. Often $P_{n y}$ and $P_{n z}$ are put to zero, so that we only have equations of the first type. The number of photoscan be limited when test-field are used but they must have a point distribution in space such that $(A ; G)$ is non-singular.

If self-calibration is used, all or most of the elements in $X$ are unknowns but more photos of the same object with varying exterior orientation $Y$ are introduced in the adjustment compared to the test-field case. A large number of equations for geometric object space conditionscan be introduced, which is the case fore.g. the plumbline method.

Testfield calibrations and self-calibrations give a set of constants for the interior orientation $Z$ and the intention is to use them in future projects as given values in functions that correct the image coordinates for the systematic errors before adjustment. The type 1 equation then is written in the form

$$
A Y+B X=L_{1}-G Z .
$$

It is important to note here that the set of constants in $Z$ should be the same
as in the calibration. The calibration should also be designed such that the estimate of $Z$ is not correlated so that of $Y$ after the adjustment of the calibration observations. If a subject $Z_{1}$ of the calibration parameters $Z$ will be used in laterprojects, this subset $Z_{1}$ should be chosen such that it is uncorrelated to the remaining set $Z_{2}$. The normal equation in calibration takes the form

$$
\begin{aligned}
& N_{11} Y+N_{12} Z=H_{1} \\
& N_{21} Y+N_{22} Z=H_{2} .
\end{aligned}
$$

Partitioning the calibration variables in two groups, $Z_{1}$ and $Z_{2}$, the inverse $N^{-1}$ can be written in the form

| $Q_{Y Y}$ | $Q_{Y Z}$ |  |
| :---: | :---: | :---: |
| $Q_{Z Y}$ | $Q_{Z_{1} Z_{1}}$ | $Q_{Z_{1} Z_{2}}$ |
|  | $Q_{Z_{2} Z_{1}}$ | $Q_{Z_{2} Z_{2}}$ |

We now require the calibration adjustment to yield

$$
\begin{aligned}
& Q_{Y Z}=0 \\
& Q_{Z_{1} Z_{2}}=0 .
\end{aligned}
$$

The reach this the whole calibration procedure hasto be studied, because $Q$ dependson the design of the experiment.

Wrobel (1978) has drawn attention to the fact that the determined calibration constants should be, if possible, uncorrelated to exterior element and between themselves in order to be easily used lateron.

On-the-job calibration means that all types of unknowns are included in the adjustment, exterior elements, interior elements, point coordinates and geometric condition parameters. The same pictures are, at the same time, used for the calibration and for the project. Grün has disc ussed the problem of over-parameterization (Grün (1978a) and (1978b)) which has the effect that the standard emors of the unknowns and the covariance between them increase although the standard error of unit weight decreases. A possible solution to this is to design the additional parameters to be orthogonal to the other unknowns, and then combine this with statistic al tests on their signific a nce. The general bivariate polynomial approach is thus abandoned. For aerial tria ngulation Grün (1979a) has found that additional parameters i.e. on-the-job-calibration, are somewhat superior to testfield calibration methods. In many close range a pplications it may be diffic ult to get a good design of the project for on-the-job calibration, in other cases it is easier. Our opinion principle is to expect higheraccuracy if more observations are included in the process, which is the case forprojects based on testfield calibration. On the other hand, on-the-job calibration determines the interior orientation of exactly the same photos as for the project, which eliminates the effect of variation of systematic errors between calibration and project.

The multi-purpose program concept contains so many different features that great flexibility is needed for the practical use of the program for calculations. The possibility of interactive work to edit input and output seems to be an advantage that is worthwhile to be tried, in order to facilitate the effective use of the computational tools. These tools are partly the software and programs written by the photogrammetrists, partly the hardware such as computer peripherals, output devices, interactive numerical and graphic al terminals, input from image coordinate measuring instruments, and the like. As the design and solutions in a nalytic al close range photogrammetry vary considerably from one project to the other, the interactive approach to the adjustment calculation must be the best way to follow.

## 7 CONCLUSIONS

Classic al theory of errors of measurements divide emors in three groups: random, systematic and gross errors (blunders). The combined effect of these types of errors gives the accuracy. The random errors are supposed to be nomally distributed and usually also independent, the systematic errors follow some known rules and the observations can be corrected for their effect, the gross errors are blunders or mistakes made by observers or malfunctioning equipment.

The random errors and their effect are connected to the concept of precision. The systematic errors are related to the mathematical model for the adjustment of the photogrammetric observations. The better the model fits to the reality, the better is the model fidelity. The gross errors and blunders are related to the concept of reliability of the adjustment. The accuracy thus comprises three parts: precision, model fidelity and reliability.

All three parts of accuracy are improved by redundant observations. Redundancy is needed in each step of the photogrammetric procedure and in the determination of the final result. Some main steps of the procedure where improvement through redundancy can be obtained are:

- photography: the importance of good photographic quality is not always recognized
- comparators: correction of systematic errors determined by calibration
- multi-readings: repeated settingson each image point
- multi-fiducials: more than four fiducial marks are of value
- multi-targets: local irregularities of the imaging and measuring system (atmosphere, lens, emulsion, comparator, operator, etc.) is a veraged out by more fold targets on each object point
- multi-frames: more exposures on each station give an averaged bundle of rays, and the camera can be rotated $\omega, \phi$ and $\kappa$ between the exposures so as to estimate or eliminate systematic errors of the bundle of rays
- multi-stations: more tha $n$ the necessary two photography stations for a stereopair yield very effective means to improve accuracy by having each object point determined by three, four and more rays intersecting from various directions in space
- multi-control: absolute orientation and control of model deformation is obtained not only by given co-ordinates of control points but also by geometric object space conditions such as lines, planes, distances, etc.
- multi-pupose program : a rigorouscomputer program is needed to handle all information, it has to be flexible for various purposes and mathematical models in photogra mmetry, it should preferably be based on interactive checks on intermediate results.

The multi-concept will be very expensive to introduce in all steps. It does not seem to be necessary to do so in order to improve precision, fidelity a nd reliability. Hottier (1976) has given recommendation for combinations of numbers of settings, targets and frames for photogrammetric intersection from two stations to improve precision. Grün (1978) has shown the effect of base-distance-ratio a nd intersection from more than two stations on precision and reliability. Calibration of cameras a nd comparators and additional parameters in adjustments have improved the model fidelity. Provided there is a general computer program available for the adjustment it seems to be reasonable to assume that intersection of object points from four different photogra phy stations yields a good precision and reliability. The improvement of the model fidelity can be achieved by precalibration or multi-fra mes on each station combined with on-the-job calibration. Research, development and practical experience are needed to find the relation between time consumption, cost, and accuracy improvement so as to find the best design of photogrammetric solution of a particularmeasuring task.

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## Appendix

PROGRAM MULTI-PHOTOGRAMMEIRY

## INSTRUCTIONS FOR INPUT OF COMPARATOR MEASUREMENTS

Each record starts with an identifier defining the type of record. The identifier is a two digit number [ 00,99 ]. Measurements from mono- and stereocomparators are accepted. Readings in mm-units. The measuring file starts with a title record (id $=10$ ) and ends with a stoprecord (id =99). The file contains measurements on an a rbitrary number of photos (or pair of photos in case of a stereocomparator). A series of measurements on a photo (pair of photos) begins with id $=12$ and ends with id $=98$. The same photo (pair of photos) can be measured several times with changed positions in the comparator. Such a replication begins id $=12$ and ends id $=98$. With each series the observations on the image points may be repeated once, i.e. double measurement on the same point is allowed between id $=12$ and id $=98$. Such double readings are identified with the point number. There can be single and double readings on the same photo. Foreach photo the readings are of fourkinds: fiducial marks (id $=20-29$ ), control points (id $=30-39,60-69$ ), a uxiliary points (id $=40-49$, $70-79$ ) a nd new points (id $=50-59,80-89$ ). id $=X 0$ : no double readings. id $=$ X1 double readings. If the fiducial marks are crosses, they can be measured on the four bars instead of centrally. id $=24$ means foursingle readings on the bars, id $=28$ means four double readings on the bars. For object points with multi-targets the second digit of the id-number indicates the number of targets measured for that point. If id $\geq 62$ there are double readings on multi-targets. Point numbers are positive and have at most 6 digits ( 000001,999999 ).

Spec ific ation for the identifiers.
id $=10 \quad$ 'text string with max 72 characters"
id $=11 \quad$ comp. no.
Identification no for the comparator. Hasto be the same as in the comparator calibration file.
$\mathrm{id}=12 \quad \mathrm{p} 1, \mathrm{k} 1(\mathrm{p} 2, \mathrm{k} 2)$
p1 is an identification number for the photo,
k 1 is the camera number (When a stereocomparator is used, p 2 is the photo on the parallaxcarriage taken with camera no k2.).
id $=20-$
$-i d=89 \quad p t, x, y,(p x, p y)$
pt: point no (000 001-999 999)
$x$ : xcoordinate in mm
$y$ : y coordinate in mm
$p x$ : $x$-parallax, in case of stereocomparator
py: y-parallax, in case of stereocomparator
id $=20$
single reading on fiducial mark
id $=21$
id $=24$
id $=28$
id $=30$
id $=31$
id $=32$
id $=62$
id $=33$
id $=63$
id $=34$
id $=64$
id $=35$
id $=65$
id $=36$
id $=66$
id $=37$
id $=67$
id $=4 X$
id $=7 X$ double reading on fiducial mark single reading on a bar of fiducial mark double reading on a bar of fiducial mark single reading on control point, single target double reading on control point, single target single reading on control point, 2-fold ta rget double reading on control point, 2-fold target single reading on control point, 3 -fold ta rget double reading on control point, 3-fold target single reading on control point, 4-fold target double reading on control point, 4-fold target single reading on control point, 5-fold ta rget double reading on control point, 5-fold target single reading on control point, 6-fold target double reading on control point, 6-fold target single reading on control point, 7-fold ta rget double reading on control point, 7-fold target
single readings on auxilia ry points
double readings on a uxilia ry points

The second digit $X$ has the same meaning asfor id $=3 X$ and $i d=6 X$ series above. The a uxiliary points are observed extra to give a strong connection between the photos. A pair of photos has a common object space. Auxiliary points can be taken in such positions as to
a) cover the common image area
b) cover the object space volume.

```
id \(=5 X \quad\) single readings on new points to be determined
id \(=8 \mathrm{X} \quad\) double readings on new points to be determined
The second digit \(X\) has the same meaning as for id \(=3 X\) and id \(=6 X\)
series above.
id \(=98 \quad \mathrm{pt}, \mathrm{x}, \mathrm{y}(, \mathrm{px}, \mathrm{py})\)
End of photos (Dummy recording of \(p t, x, y\) )
id \(=99 \quad p t, x, y(, p x, p y)\)
End of mea surement file (Dummy, recording of \(p t, x, y\) )
```


## Note 1:

id $=24 \quad$ Readings on bars of the fiducial mark is for a single four bar recording done such that the two first readingsare taken on opposite bars and the two last on those bars perpendic ular to the first ones. The point no is that of the fiducial mark for all 4 settings.

## Note 2:

id $=28 \quad$ Fordouble readings on the bars the pattem above is repeated later on, preferably at the end of the photo readings. The point no is that of the fiducial mark for all 8 settings.

## Note 3:

id $=24 \quad$ Forcameras without fiducial marks the side lines (the edges) of the image format can be used. The fourbarreadings on each fiducial mark and here instead four readings on the two sides forming the image comer. The location of readings $1-4$ should be the same as when calibrating the camera to avoid bias. The point no is the same for all 4 settings, i.e. that of the fiducial mark, here = image comer.

## Note 4:

id $=28$ Double readings on side linesare possible. The point no is the same for a ll 8 settings, i.e. that of the fiducial mark, here $=$ image comer.

## Note 5:

Photos are measured in a right handed xy system with the photos in positions a s dia positives. Parallax readings are defined from the comparator file.

