

$\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ does not satisfy Normann's condition

Matthias Schröder

Abstract

We prove that the Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ does not satisfy Normann's condition. A topological space X is said to fulfil Normann's condition, if every functionally closed subset of X is an intersection of clopen sets. The investigation of this property is motivated by its strong relationship to a problem in Computable Analysis. D. Normann has proved that in order to establish non-coincidence of the extensional hierarchy and the intensional hierarchy of functionals over the reals it is enough to show that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ fails the above condition.

Keywords: Kleene-Kreisel spaces, Sequential Spaces, QCB-spaces, Computable Analysis, Coincidence Problem

1 Introduction

The Kleene-Kreisel continuous functionals over the natural numbers play an important role in mathematical logic as well as in higher type computability [5, 6, 7]. A simple way of defining this hierarchy is to construct it as a sequence of exponentials in an appropriate cartesian closed category by applying the recursion formula $\mathbb{N}\langle 0 \rangle := \mathbb{N}$ and $\mathbb{N}\langle k + 1 \rangle := \mathbb{N}^{\mathbb{N}\langle k \rangle}$. In this paper we use the cartesian closed category **QCB** as our ambient category [12]. This full subcategory of **Top** has as objects all quotients of countably based topological spaces. Alternatives are the category **Seq** of sequential spaces or the category **kHaus** of Hausdorff Kelley spaces [3].

The main goal of this paper is to prove that the space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}} = \mathbb{N}\langle 2 \rangle$ contains a functionally closed subset¹ which can not be represented as an intersection of clopen sets. Since $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is hereditarily Lindelöf, this is equivalent to saying that the completely regular reflection of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not zero-dimensional. Note that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ means the exponential formed in **QCB** (or equivalently in **Seq** or **kHaus**, see [3]) to the basis \mathbb{N} and the exponent $\mathbb{N}^{\mathbb{N}}$. So $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is topologised by the sequentialisation of the compact-open topology on the set of continuous functions from the Baire space to the discrete space \mathbb{N} , see [3, 9]. It is well-known that this topology is strictly finer than the compact-open topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. From [10] we know that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is neither zero-dimensional nor regular.

As an important consequence of this result, we obtain that the extensional hierarchy and the intensional hierarchy of functionals over the real numbers do not coincide. These two hierarchies have been introduced by Bauer, Escardó and Simpson to model

¹A subset A of a topological space is *functionally closed*, if A is the preimage of 0 under a continuous function into the unit interval $[0; 1]$.

two approaches to higher type computation over the real numbers in functional programming [1]. Normann proved that the two hierarchies agree up to level $k + 1$ if, and only if, the Kleene-Kreisel space $\mathbf{N}\langle k \rangle$ has the property that functionally closed sets are intersections of clopen sets (see [8, Theorems 4.17 & 5.5]). We therefore refer to this property as *Normann's condition*. Bauer, Escardó and Simpson had already observed that the coincidence question for level 3 is related to topological properties of the space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, see [1].

In Section 2 we construct a Polish space \mathbf{M} that arises as the sequential coreflection of some zero-dimensional space, but is not zero-dimensional itself. So \mathbf{M} is a totally disconnected metric space that does not fulfil Normann's condition. In Section 3 we prove that \mathbf{M} is a retract of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. Both results combined entail that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ does not satisfy Normann's condition. In Section 5 we briefly discuss the extensional and the intensional hierarchies of functionals over the reals.

2 Definition of the Polish space \mathbf{M}

We will define \mathbf{M} to be a closed subspace of the real vector space ℓ_1 . The space ℓ_1 consists of those elements $x \in \mathbb{R}^{\mathbb{N}}$ for which the 1-norm $\|x\|_1$ defined by

$$\|x\|_1 := \sum_{i \in \mathbb{N}} |x(i)|$$

is less than ∞ . It is well-known that $(\ell_1, \|\cdot\|_1)$ is a Banach space. In abuse of notation, we henceforth denote by ℓ_1 the countably based topological space that carries the topology induced by the ℓ_1 -metric $(x, y) \mapsto \|x - y\|_1$. We will later need the following characterisation of convergence of sequences in ℓ_1 which is folklore in functional analysis.

Lemma 2.1 *A sequence $(x_n)_n$ converges in ℓ_1 to x_∞ if, and only if, (a) and (b) hold:*

- (a) *For all $i \in \mathbb{N}$, $(x_n(i))_n$ converges to $x_\infty(i)$ in \mathbb{R} .*
- (b) *The sequence $(\|x_n\|_1)_n$ converges to $\|x_\infty\|_1$ in \mathbb{R} .*

For the construction of \mathbf{M} , we define for $i \in \mathbb{N}$ the set M_i by

$$M_i := \{j \cdot 2^{-i} \mid j \in \{0, \dots, 2^i\}\} = \{0, 1 \cdot 2^{-i}, 2 \cdot 2^{-i}, 3 \cdot 2^{-i}, \dots, (2^i - 1) \cdot 2^{-i}, 1\}.$$

Then the subspace \mathbf{M} of ℓ_1 with underlying set

$$|\mathbf{M}| := \left\{x \in \prod_{i \in \mathbb{N}} M_i : \|x\|_1 < \infty\right\} \tag{1}$$

is closed in ℓ_1 . This is due to the fact that the sets M_i are closed in \mathbb{R} . So \mathbf{M} is a Polish space, with the restriction of the ℓ_1 -metric being a complete metric for \mathbf{M} .

We show in a similar way as in [2, Example 6.2.19] that \mathbf{M} is not zero-dimensional.

Lemma 2.2 *The unit ball $B_{\mathbf{M}}(0^\omega; 1) := \{x \in \mathbf{M} : \|x\|_1 < 1\}$ does not contain any clopen neighbourhood of the constant zero-function $0^\omega \in \ell_1$.*

Proof. Let V be any open set with $0^\omega \in V \subseteq B_M(0^\omega; 1)$. By recursion we construct a sequence $(a_k)_k \in \prod_{i \in \mathbb{N}} M_i$ such that

$$\begin{aligned} x_k &:= (a_0, \dots, a_k, 0, 0, \dots) \in V \quad \text{and} \\ y_k &:= (a_0, \dots, a_{k-1}, a_k + 2^{-k}, 0, 0, \dots) \in M \setminus V. \end{aligned}$$

for all $k \in \mathbb{N}$.

“ $k = 0$ ”: Set $a_0 := 0$. Then $x_0 = 0^\omega \in V$ and $y_0 = 10^\omega \notin V$.

“ $k - 1 \rightarrow k$ ”: Assume that a_0, \dots, a_{k-1} are already constructed with $x_{k-1} \in V \not\subseteq y_{k-1}$. Let $z := (a_0, \dots, a_{k-1}, 1, 0, 0, \dots)$. As $1 \leq \|z\|_1 \leq 2$, we have $z \in M \setminus B_M(0^\omega; 1)$. Therefore there is some $b \in M_k \setminus \{1\}$ with $(a_0, \dots, a_{k-1}, b, 0, 0, \dots) \in V$ and $(a_0, \dots, a_{k-1}, b + 2^{-k}, 0, 0, \dots) \notin V$. Since $(\sum_{i=0}^{k-1} a_i) + b + 2^{-k} < \infty$, the number $a_k := b$ satisfies the requirements.

We set $x_\infty := (a_0, a_1, \dots) \in \prod_{i \in \mathbb{N}} M_i$. Clearly, both the sequences $(x_k)_k$ and $(y_k)_k$ converge to x_∞ in $\mathbb{R}^{\mathbb{N}}$. By Lemma 2.1 they converge to x_∞ in the space M as well, because

$$\|x_\infty\|_1 = \sum_{i=0}^{\infty} a_i = \lim_{m \rightarrow \infty} \|y_m\|_1 = \lim_{m \rightarrow \infty} \|x_m\|_1 \leq 1 < \infty.$$

As $M \setminus V$ is closed, we have $x_\infty \notin V$. Thus V is not closed. We conclude that the ball $B_M(0^\omega; 1)$ does not contain any clopen neighbourhood of 0^ω . □

The ball $B_M(0^\omega; 1)$ is a basic open of the metrisable topology of M . Hence the complement $M \setminus B_M(0^\omega; 1)$ is closed and, as M is a metric space, even functionally closed. Lemma 2.2 entails that M is not zero-dimensional and that $M \setminus B_M(0^\omega; 1)$ is not an intersection of clopen sets.

Lemma 2.3 *The space M is a Polish space that is not zero-dimensional and that does not satisfy Normann’s condition.*

In Section 3 we will prove that M is a retract of the QCB-space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. Hence M is homeomorphic to a closed subspace of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. By being topologised by the sequentialisation of the compact-open topology on the set $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, the QCB-exponential $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is the sequential coreflection of some zero-dimensional space. This property is inherited by closed subspaces. We conclude that M is the sequential coreflection of some zero-dimensional topological space.

3 Embedding M into $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ as a retract

In this section we show that M is a retract of the QCB-space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. We do this via the space $2^{\mathbb{N} \times \mathbb{F}}$, where \mathbb{F} denotes the countable metric *fan* and 2 denotes the two point discrete space with points $0, 1$.

3.1 The countable fan \mathbb{F}

The fan space \mathbb{F} is the “smallest” non-locally-compact metrisable space, in the sense that it embeds into every non-locally-compact metrisable space as a closed subspace. Our version of the countable fan has $\mathbb{N}^2 \cup \{(\infty, \infty)\}$ as underlying set and its topology is induced by the unique metric $d_{\mathbb{F}}$ that satisfies

$$d_{\mathbb{F}}((a, b), (\infty, \infty)) = 2^{-a} \quad \text{and} \quad d_{\mathbb{F}}((a, b), (a', b')) = \max\{2^{-a}, 2^{-a'}\}$$

for $a, b, a', b' \in \mathbb{N}$ with $(a, b) \neq (a', b')$. So every point apart from (∞, ∞) is an isolated point in \mathbb{F} . Furthermore, $(a_n, b_n)_n$ converges to (∞, ∞) in \mathbb{F} if, and only if, $\lim_{n \rightarrow \infty} a_n = \infty$.

By being a zero-dimensional Polish spaces, the product $\mathbb{N} \times \mathbb{F}$ is a retract of the Baire space $\mathbb{N}^{\mathbb{N}}$ in QCB, i.e., there are continuous functions $e: \mathbb{F} \times \mathbb{N} \rightarrow \mathbb{N}^{\mathbb{N}}$ and $r: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N} \times \mathbb{F}$ satisfying $r \circ e = id_{\mathbb{N} \times \mathbb{F}}$ (see [10, Section 3.3] for an explicit construction).

3.2 The zero-dimensional space $M_{\mathbb{P}}$

For any $i \in \mathbb{N}$, we endow the finite set M_i with its discrete topology and denote the corresponding finite discrete space by M_i . The product $M_{\mathbb{P}} := \prod_{i \in \mathbb{N}} M_i$ is a zero-dimensional compact metrisable space. Since the spaces M_i are discrete subspaces of \mathbb{R} , a sequence $(x_n)_n$ of elements of the set $\prod_{i \in \mathbb{N}} M_i$ converges in the space $\mathbb{R}^{\mathbb{N}}$ to some point x_{∞} if, and only if, it does in the space $M_{\mathbb{P}}$. We obtain by Lemma 2.1:

Lemma 3.1 *Let $(x_n)_n$ be a sequence in M and let $x_{\infty} \in M$. Then $(x_n)_n$ converges in M to x_{∞} if, and only if, (a) and (b) hold:*

(a) *For all $i \in \mathbb{N}$ there is some $n_i \in \mathbb{N}$ with $x_n(i) = x_{\infty}(i)$ for all $n \geq n_i$.*

(b) *The sequence $(\|x_n\|_1)_n$ converges to $\|x_{\infty}\|_1$ in \mathbb{R} .*

Hence the injection $id: M \rightarrow M_{\mathbb{P}}$ is sequentially continuous and thus topologically continuous, as M is metrisable. This implies that the topology of M is finer than the subspace topology on the set $|M|$ induced by the topology of $M_{\mathbb{P}}$. In fact, it is strictly finer than the subspace topology, because the sequence $(0^{n+1} \frac{1}{2} 0^{\omega})_n$ converges in $M_{\mathbb{P}}$ to 0^{ω} , but not in M .

3.3 An embedding of M into $M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$

We now start to prove that M is a retract of the QCB-product $M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$. First we define two functions $f: M_{\mathbb{P}} \times \mathbb{N} \times \mathbb{N}^2 \rightarrow \{0, 1\}$ and $g: M \rightarrow 2^{\mathbb{N} \times \mathbb{F}}$ by

$$f(x, k, a, b) := \begin{cases} 0 & \text{if } \sum_{i=a}^{a+b} x(i) \leq 2^{-k} \\ 1 & \text{otherwise} \end{cases} \quad (2)$$

$$g(y)(k, a, b) := f(y, k, a, b) \quad \text{and} \quad g(y)(k, \infty, \infty) := 0 \quad (3)$$

for all $x \in M_{\mathbb{P}}$, $y \in M$ and $k, a, b \in \mathbb{N}$.

Lemma 3.2 *For any $y \in M$, the function $g(y): \mathbb{N} \times \mathbb{F} \rightarrow 2$ is continuous. Moreover, f and g are continuous.*

Proof.

- (1) Let $(x_n, k_n, a_n, b_n)_n$ converge to $(x_\infty, k_\infty, a_\infty, b_\infty)$ in $M_P \times \mathbb{N} \times \mathbb{N}^2$. Then there is some $n_0 \in \mathbb{N}$ such that

$$x_n(i) = x_{n_0}(i), \quad k_n = k_\infty \quad \text{and} \quad (a_n, b_n) = (a_\infty, b_\infty)$$

for all $n \geq n_0$ and all $i \in \{a_\infty, \dots, a_\infty + b_\infty\}$. By the definition of f , we have $f(x_n, k_n, a_n, b_n) = f(x_\infty, k_\infty, a_\infty, b_\infty)$ for all $n \geq n_0$. So f is continuous.

- (2) By the cartesian closedness of QCB it suffices to show the continuity of the function $G: M \times \mathbb{N} \times \mathbb{F} \rightarrow 2$ defined by

$$G(x, k, a, b) = f(x, k, a, b) \quad \text{and} \quad G(x, k, \infty, \infty) := 0.$$

Let $(x_n, k_n, a_n, b_n)_n$ converge to $(x_\infty, k_\infty, a_\infty, b_\infty)$ in $M \times \mathbb{N} \times \mathbb{F}$. If $(a_\infty, b_\infty) \in \mathbb{N}^2$, then the sequence $(G(x_n, k_n, a_n, b_n))_n$ converges to $G(x_\infty, k_\infty, a_\infty, b_\infty)$ by the continuity of f and by the fact that the topology of M is finer than the subspace topology on M induced by the topology of M_P .

Now let $(a_\infty, b_\infty) = (\infty, \infty)$. Since $\sum_{i=0}^{\infty} x_\infty(i) < \infty$, there is some $m \in \mathbb{N}$ with $\sum_{i=m}^{\infty} x_\infty(i) < 2^{-k_\infty-1}$. There exists some $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and all $i < m$,

$$x_n(i) = x_\infty(i), \quad \left| \|x_n\|_1 - \|x_\infty\|_1 \right| < 2^{-k_\infty-1}, \quad k_n = k_\infty \quad \text{and} \quad a_n \geq m.$$

Then all $n \geq n_0$ with $(a_n, b_n) \neq (\infty, \infty)$ satisfy

$$\begin{aligned} \sum_{i=a_n}^{a_n+b_n} x_n(i) &\leq \sum_{i=m}^{\infty} x_n(i) = \|x_n\|_1 - \sum_{i=0}^{m-1} x_n(i) = \|x_n\|_1 - \sum_{i=0}^{m-1} x_\infty(i) \\ &= \|x_n\|_1 - \|x_\infty\|_1 + \sum_{i=m}^{\infty} x_\infty(i) < 2 \cdot 2^{-k_\infty-1} = 2^{-k_n}. \end{aligned}$$

This implies $G(x_n, k_n, a_n, b_n) = 0 = G(x_\infty, k_\infty, a_\infty, b_\infty)$ for all $n \geq n_0$. Hence G is sequentially continuous and thus topologically continuous as a function from the QCB-product $M \times \mathbb{N} \times \mathbb{F}$ to the two-point discrete space 2 . We conclude that g is a continuous function into the space $2^{\mathbb{N} \times \mathbb{F}}$. □

Using the continuous map g , we define the function $e_M: M \rightarrow M_P \times 2^{\mathbb{N} \times \mathbb{F}}$ by

$$e_M(x) := (x, g(x)) \tag{4}$$

for all $x \in M$. From Lemmas 3.1 and 3.2 we obtain:

Lemma 3.3 *The function e_M is injective and continuous.*

3.4 Construction of the retraction map from $M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$ to M

To construct a retraction map pertaining to the section e_M , we define for any $m \in \mathbb{N}$ the set C_m by

$$C_m := \left\{ (x, h) \in M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}} \mid \begin{array}{l} h(k, \infty, \infty) = 0 \text{ and} \\ h(k, a, b) = f(x, k, a, b) \text{ for all } k, a, b \leq m \end{array} \right\}.$$

Using the decreasing sequence $(C_m)_m$, we construct a function r_M from $M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$ to $\prod_{i \in \mathbb{N}} M_i$ by

$$r_M(x, h)(m) := \begin{cases} x(m) & \text{if } (x, h) \in C_m \\ 0 & \text{otherwise} \end{cases}$$

for all $(x, h) \in M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$ and $m \in \mathbb{N}$. It follows immediately from the definitions that the image of e_M lies in $\bigcap_{m \in \mathbb{N}} C_m$ and that $r_M(e_M(x)) = x$ holds for every $x \in M$.

We show that r_M maps into the space M . This fact allows us to consider r_M henceforth as a function of the form $M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}} \rightarrow M$.

Lemma 3.4

- (1) Let $(x, h) \in M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$. Let $(a, k) \in \mathbb{N}^2$ such that $a \geq k$ and $h(k, a, b) = 0$ for all $b \in \mathbb{N}$. Then we have $\sum_{i=a}^{\infty} r_M(x, h)(i) \leq 2^{-k}$.
- (2) The image of e_M is equal to the intersection $\bigcap_{m \in \mathbb{N}} C_m$.
- (3) For every $(x, h) \in M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$, we have $r_M(x, h) \in M$.
- (4) For every $m \in \mathbb{N}$, the set C_m is clopen in $M_{\mathbb{P}} \times 2^{\mathbb{N} \times \mathbb{F}}$.

Proof.

- (1) By induction on m we show $\sum_{i=a}^m r_M(x, h)(i) \leq 2^{-k}$ for all $m \geq a$.

“ $m = a$ ”: If $(x, h) \in C_a$, then $f(x, k, a, 0) = h(k, a, 0) = 0$ and $r_M(x, h)(a) = x(a) = \sum_{i=a}^a x(i) \leq 2^{-k}$. Otherwise we have $r_M(x, h)(a) = 0 \leq 2^{-k}$.

“ $m > a$ ”: If $(x, h) \in C_m$, then we have $r_M(x, h)(i) = x(i)$ for all $i \leq m$ and $f(x, k, a, m-a) = h(k, a, m-a) = 0$, because $k \leq m$. This implies $\sum_{i=a}^m r_M(x, h)(i) = \sum_{i=a}^m x(i) \leq 2^{-k}$.
Otherwise $r_M(x, h)(m)$ is equal to 0 and the induction hypothesis yields $\sum_{i=a}^m r_M(x, h)(i) = \sum_{i=a}^{m-1} r_M(x, h)(i) \leq 2^{-k}$.

We conclude $\sum_{i=a}^{\infty} r_M(x, h)(i) \leq 2^{-k}$.

- (2) We have already noticed $e_M[M] \subseteq \bigcap_{m \in \mathbb{N}} C_m$. To show “ \supseteq ”, let $(x, h) \in \bigcap_{m \in \mathbb{N}} C_m$. The continuity of h implies that there is some $a_0 \in \mathbb{N}$ such that $h(0, a, b) = h(0, \infty, \infty) = 0$ for all $a \geq a_0$ and $b \in \mathbb{N}$. From (1) we obtain

$$\begin{aligned} \|x\|_1 &= \sum_{i=0}^{\infty} x(i) = \sum_{i=0}^{\infty} r_M(x, h)(i) = \sum_{i=a_0}^{\infty} r_M(x, h)(i) + \sum_{i=0}^{a_0-1} r_M(x, h)(i) \\ &\leq 2^0 + \sum_{i=0}^{a_0-1} r_M(x, h)(i) < \infty, \end{aligned}$$

hence $x \in M$. This allows us to apply g and e_M to x . Since $(x, h) \in \bigcap_{m \in \mathbb{N}} C_m$, we have $g(x) = h$ and $e_M(x) = (x, h)$. Therefore (x, h) lies in the image of e_M .

(3) Clearly, $r_M(x, h)(i) \in M_i$ holds for every $i \in M$. If $(x, h) \in e_M[M]$, then we have $(x, h) \in \bigcap_{m \in \mathbb{N}} C_m$ and thus $r_M(x, h) = x \in M$. Otherwise, if $(x, h) \notin e_M[M]$, then by (2) there is some $m \in \mathbb{N}$ with $(x, h) \notin C_m$. This implies $r_M(x, h)(i) = 0$ for all $i \geq m$. Hence $\|r_M(x, h)\|_1 < \infty$ and thus $r_M(x, h) \in M$.

(4) Let $k, a, b \in \{0, \dots, m\}$. For every $s \in \{0, 1\}$ the set

$$D_s := \{x \in M_P \mid f(x, k, a, b) = s\}$$

is clopen in M_P by the continuity of f (see Lemma 3.2). Moreover, the set

$$E_s := \{h \in 2^{\mathbb{N} \times \mathbb{F}} \mid h(k, \infty, \infty) = 0 \text{ and } h(k, a, b) = s\}$$

is clopen w.r.t. the compact-open topology and thus w.r.t. the sequential topology on $2^{\mathbb{N} \times \mathbb{F}}$, because the latter is finer than the former. Hence the set

$$\{(x, h) \in M_P \times 2^{\mathbb{N} \times \mathbb{F}} \mid h(k, \infty, \infty) = 0 \text{ and } h(k, a, b) = f(x, k, a, b)\}$$

is clopen in $M_P \times 2^{\mathbb{N} \times \mathbb{F}}$ by being equal to $(D_0 \times E_0) \cup (D_1 \times E_1)$. Therefore C_m is clopen by being a finite intersection of clopen sets.

□

We need the following lemma about converging sequences in the QCB-exponential $2^{\mathbb{N} \times \mathbb{F}}$. It can be easily deduced from the fact that the convergence relation of QCB-exponentials is continuous convergence², see [3, 9].

Lemma 3.5 *Let $(h_n)_n$ converge to h_∞ in $2^{\mathbb{N} \times \mathbb{F}}$. Then for every $k \in \mathbb{N}$ there exists some $m \in \mathbb{N}$ with $h_n(k, a, b) = h_\infty(k, \infty, \infty)$ for all $n \geq m$ (including $n = \infty$), all $a \geq m$ and all $b \in \mathbb{N}$.*

Now we are able to show that r_M is a retraction map.

Proposition 3.6 *The space M is a retract of $M_P \times 2^{\mathbb{N} \times \mathbb{F}}$ in QCB. The functions e_M and r_M form a section-retraction-pair.*

²A sequence $(f_n)_n$ of continuous functions between two sequential spaces X and Y is said to *converge continuously* to a continuous function $f_\infty: X \rightarrow Y$, if $(f_n(x_n))_n$ converges to $f_\infty(x_\infty)$ in Y , whenever $(x_n)_n$ converges to x_∞ in X .

Proof. We have already observed $r_M \circ e_M = id_M$ and the continuity of e_M .

It remains to prove the continuity of r_M . Let $(x_n, h_n)_n$ converge to (x_∞, h_∞) . We set $z_n := r_M(x_n, h_n)$ for all $n \in \mathbb{N} \cup \{\infty\}$. For every $m \in \mathbb{N}$, $(z_n(m))_n$ converges to $z_\infty(m)$ in M_m , because C_m is clopen by Lemma 3.4 and $(x_n(m))_n$ converges to $x_\infty(m)$ in M_m . We consider two cases.

- (a) Let $(x_\infty, h_\infty) \in e_M[M]$. Then $z_\infty = x_\infty$, because $(x_\infty, h_\infty) \in \bigcap_{m \in \mathbb{N}} C_m$. To show $\lim_{n \rightarrow \infty} \|z_n - z_\infty\|_1 = 0$, let $k \in \mathbb{N}$. Then $h_\infty(k, \infty, \infty) = 0$, because $(x_\infty, h_\infty) \in C_k$. By Lemma 3.5 there exists some $m \in \mathbb{N}$ such that

$$h_n(k, a, b) = h_\infty(k, \infty, \infty) = 0$$

for all $n \geq m$ (including $n = \infty$), $a \geq m$ and $b \in \mathbb{N}$. We set $c := \max\{m, k\}$. Since $(z_n)_n$ converges pointwise to z_∞ , there is some $n_1 \in \mathbb{N}$ with $z_n(i) = z_\infty(i)$ for all $n \geq n_1$ and $i < c$. By Lemma 3.4(1), all $n \geq \max\{m, n_1\}$ satisfy

$$\begin{aligned} \|z_n - z_\infty\|_1 &= \left| \sum_{i=0}^{\infty} z_n(i) - \sum_{i=0}^{\infty} z_\infty(i) \right| = \left| \sum_{i=c}^{\infty} z_n(i) - \sum_{i=c}^{\infty} z_\infty(i) \right| \\ &\leq \max \left\{ \sum_{i=c}^{\infty} z_n(i), \sum_{i=c}^{\infty} z_\infty(i) \right\} \leq 2^{-k}. \end{aligned}$$

We conclude that $(z_n)_n$ converges to z_∞ in ℓ_1 and thus in the subspace M .

- (b) Let $(x_\infty, h_\infty) \notin e_M[M]$. By Lemma 3.4(2) there is some $m \in \mathbb{N}$ with $(x_\infty, h_\infty) \notin C_m$. Since $(z_n)_n$ converges pointwise to z_∞ and C_m is closed, there is some $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $i < m$,

$$(x_n, h_n) \notin C_m \quad \text{and} \quad z_n(i) = z_\infty(i).$$

For all $n \geq n_0$ we have $z_n = z_\infty$, because $z_n(i) = 0 = z_\infty(i)$ holds for all $i \geq m$. Therefore $(z_n)_n$ converges to z_∞ in M . □

3.5 Establishing M as a retract of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$

As M_P is a zero-dimensional compact metrisable space without isolated points, M_P is homeomorphic to the Cantor space $2^{\mathbb{N}}$ by Theorem 7.4 in [4]. Moreover, the product $2^{\mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{F}}$ is homeomorphic to $2^{\mathbb{N} \times \mathbb{F}}$, hence $M_P \times 2^{\mathbb{N} \times \mathbb{F}}$ is homeomorphic to $2^{\mathbb{N} \times \mathbb{F}}$. Since $\mathbb{N} \times \mathbb{F}$ is a retract of $\mathbb{N}^{\mathbb{N}}$ (see Section 3.1), $2^{\mathbb{N} \times \mathbb{F}}$ is a retract of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. We obtain by Proposition 3.6:

Proposition 3.7 *The Polish space M is a retract of $2^{\mathbb{N} \times \mathbb{F}}$ and of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$.*

4 The main result

To establish our main result that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ does not satisfy Normann's condition, it remains to verify that forming retracts preserves Normann's condition.

Lemma 4.1 *Let X be a retract of some qcb-space Y . If Y satisfies Normann's condition, then so does X .*

Proof. Let $e: X \rightarrow Y$ and $r: Y \rightarrow X$ be continuous functions with $r \circ e = id_X$. Let A be a functionally closed subset of X . Then $r^{-1}[A]$ is a functionally closed subset of Y . As Y satisfies Normann’s condition, there is a family of clopen subsets $(C_i)_{i \in I}$ of Y with $\bigcap_{i \in I} C_i = r^{-1}[A]$. By the continuity of e , the sets $e^{-1}[C_i]$ are clopen in X . One easily verifies $A = \bigcap_{i \in \mathbb{N}} e^{-1}[C_i]$. Therefore A is an intersection of clopen sets of X . \square

Hence Lemma 2.3 and Proposition 3.7 imply our main result stating that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ contains functionally closed sets that are not intersections of clopens.

Theorem 4.2 *The space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ does not satisfy Normann’s condition.*

Analogously, neither $M_P \times 2^{\mathbb{N} \times \mathbb{F}}$ nor $2^{\mathbb{N} \times \mathbb{F}}$ satisfies Normann’s condition. By induction on k one can show that $N\langle k \rangle$ is a retract of $N\langle k + 1 \rangle$. We conclude by Lemma 4.1:

Corollary 4.3 *For every $k \geq 2$, the sequential space $N\langle k \rangle$ of Kleene-Kreisel functionals of level k does not satisfy Normann’s condition.*

5 An application in Computable Analysis

In [1], Bauer, Escardó and Simpson formalised two approaches to higher type computation over the real numbers in functional programming by defining two “real” objects in the category **Equ** of equilogical spaces [11]. The first object, R_E , models the *external reals* describing the approach of introducing the reals as an own datatype. The object R_I models the concept of representing real numbers via infinite streams. These reals are called *internal reals*.

Using exponentiation in the cartesian closed category **Equ**, the application of the natural recursion formulae

$$R_E\langle 0 \rangle := R_E \quad \text{and} \quad R_E\langle k + 1 \rangle := R_E^{R_E\langle k \rangle} \quad (5)$$

$$R_I\langle 0 \rangle := R_I \quad \text{and} \quad R_I\langle k + 1 \rangle := R_I^{R_I\langle k \rangle} \quad (6)$$

yields two hierarchies of functionals over the real numbers. The hierarchy of the underlying sets of the sequence $(R_E\langle k \rangle)_k$ is called the *extensional hierarchy*. The underlying sets of $(R_I\langle k \rangle)_k$ form the *intensional hierarchy*.

The natural question arises whether the two hierarchies of functionals coincide. This question is referred to as the *Coincidence Problem*. From [1] we know that both hierarchies agree up to level 2. Normann’s equivalence result (Theorem 4.17 and 5.5 in [8]) states that the two hierarchies agree on level $k + 1$ if, and only if, every functionally closed subset of the Kleene-Kreisel space $N\langle k \rangle$ is an intersection of clopen sets. Therefore our main result (Theorem 4.3) along with Corollary 4.3 solves the Coincidence Problem negatively.

Theorem 5.1 *The extensional hierarchy and the intensional hierarchy of functionals over the reals do not coincide from level 3 on.*

Hence both hierarchies disagree from the first previously unknown level on. It is known that the extensional hierarchy coincide with the sequential hierarchy. The latter is formed by the underlying sets of the sequence

$$R_S\langle 0 \rangle := \mathbb{R} \text{ and } R_S\langle k + 1 \rangle := \mathbb{R}^{R_S\langle k \rangle}$$

formed in the category QCB (equivalently in Seq or kHaus). So Theorem 5.1 states there is a continuous functional $F: \mathbb{R}^{\mathbb{R}^{\mathbb{R}}} \rightarrow \mathbb{R}$ that is not an element of the space $R_I\langle 3 \rangle$.

References

- [1] A. Bauer, M. Escardó, A. Simpson: “Comparing Functional Paradigms for Exact Real-number Computation”; Proc. ICALP ’02, Lecture Notes in Computer Science 2380 (2002), 488–500.
- [2] R. Engelking: “General Topology”; Heldermann, Berlin (1989).
- [3] M. Escardó, J. Lawson, A. Simpson: “Comparing Cartesian closed Categories of Core Compactly Generated Spaces”; Topology and its Applications 143 (2004), 105–145.
- [4] A. Kechris: “Classical Descriptive Set Theory”; Springer, Berlin (1995).
- [5] S.E. Kleene: “Countable Functionals”; Constructivity in Mathematics, North-Holland (1959), 81–100.
- [6] G. Kreisel: “Interpretation of Analysis by means of Functionals of Finite Type”; Constructivity in Mathematics, North-Holland (1959), 101–128.
- [7] D. Normann: “Recursion on the Countable Functionals”; Lecture Notes in Mathematics 811 (1980).
- [8] D. Normann: “Comparing Hierarchies of Total Functionals”; Logical Methods in Computer Science 1,2:4 (2005), 1–28.
- [9] M. Schröder: “Admissible Representations for Continuous Computations”; PhD Thesis, Fachbereich Informatik, FernUniversität Hagen (2002).
- [10] M. Schröder: “The sequential topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not regular”; Mathematical Structures in Computer Science 19 (2009), 943–957.
- [11] D. Scott, A. Bauer, L. Birkedal: “Equilogical Spaces”; Theoretical Computer Science 415 (2004), 35–59.
- [12] A. Simpson: “Towards a Convenient Category of Topological Domains”; Proceedings of thirteenth ALGI Workshop, RIMS, Kyoto University (2003).

Fakultät für Informatik, Universität der Bundeswehr, Munich, Germany
Email: matthias.schroeder@unibw.de