On the Effective Existence of Schauder Bases in Banach Spaces

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Compact operators – Definition

- X, Y Banach spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
- $B_X := \{x \in X \ : \ \|x\| \leq 1\}$
- $F : X \to Y$ linear operator
- F compact $:\Longleftrightarrow \overline{F(B_X)}$ compact in Y

Compact operators – Some facts

- Compact operators form a Banach space $K(X, Y)$ (w.r.t. operator norm)
- Composition of bounded operator with compact operator yields compact operator.
- Operator is compact $\iff$ adjoint operator is compact.
- All finite rank operators

$$x \mapsto \sum_{i=1}^{n} f_i(x) y_i, \quad f_i \in X^*, \, y_i \in Y,$$

are compact.

The approximation property

- Banach space Y has approximation property (AP) $:\Longleftrightarrow$
  for every compact $K \subseteq Y$ and every $\varepsilon > 0$, there is a finite rank operator $T : Y \to Y$ such that
  $$\|T(y) - y\| \leq \varepsilon \quad \text{for all } y \in K.$$
- Theorem (Grothendieck):
  Y has AP $\Longleftrightarrow$ finite rank operators dense in $K(X, Y)$ for every Banach space X

Effective versions of the classical results

Vasco Brattka and Ruth Dillhage (2007)

- ... define a natural effective representation of $K(X, Y)$

  (this representation includes information on $F \in K(X, Y)$ as a continuous mapping and on $\overline{F(B_X)}$ as a compact set)
- ... prove computable versions of the results mentioned above,

  but: They assume that X, Y possess computable Schauder bases

  (with certain additional properties)

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Schauder bases – Definition

- X infinite dimensional Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
- $(x_i)_i \in X^\omega$ is a (Schauder) basis of X $:\Longleftrightarrow$

  for every $x \in X$, there is a unique sequence $(\alpha_i)_i \in \mathbb{F}^\omega$

  such that the series $\sum_n \alpha_n x_n$ converges to x.
- X finite dimensional $\Longrightarrow$ Schauder basis defined just like usual (Hamel) basis

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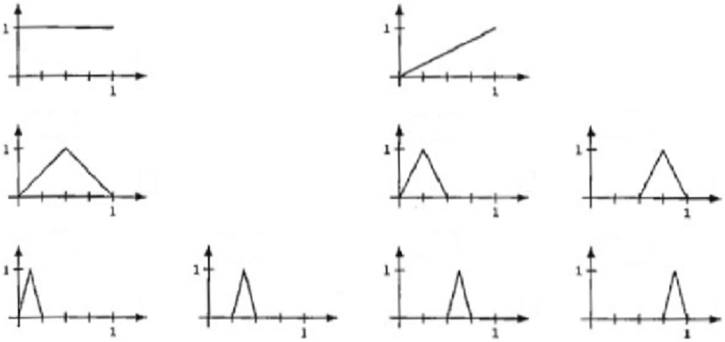
Schauder bases – Examples

- Othonormal bases of separable Hilbert spaces.
- Consider sequence spaces ℓ_p $(1 \leq p < \infty)$ or c_0.

  The unit vectors $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, $(0, 0, 1, \ldots)$, $\ldots$ form a Schauder basis.

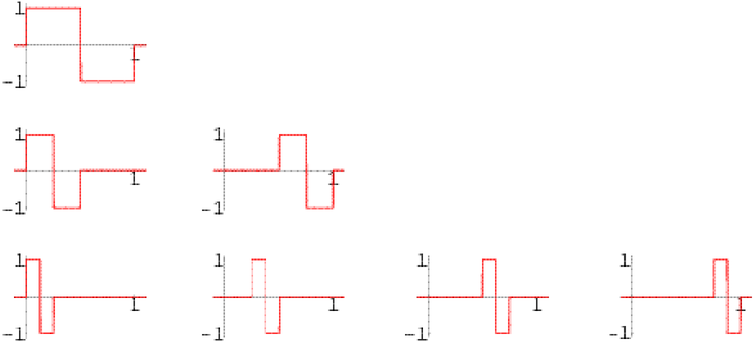
Schauder bases – Examples

- Schauder basis in space $C[0,1]$ of continuous functions



Schauder bases – Examples

- Schauder basis in integration space $L_p([0,1])$, $1 \leq p < \infty$



The question we are going to answer

- Question:
  Given a computable Banach space that possesses a basis.
  Does it necessarily possess a computable basis?
- Answer for Hilbert spaces (Brattka and Yoshikawa 2006):
  Yes.
- Answer in general (to be demonstrated today):
  No!

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Computable Banach spaces

- $X = (X, \|\cdot\|)$ a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.
- $(e_i)_i \in X^\omega$ with $[e_0, e_1, \ldots] := \mathrm{cls}(\mathrm{span}\{e_i\}_i) = X$.
- α_e a derived numbering of the rational span of $\{e_i\}_i$.
- $(X, \|\cdot\|, (e_i)_i)$ computable Banach space $:\Longleftrightarrow$

  $m \mapsto \|\alpha_e(m)\|$ computable
- Cauchy representation $\delta_X :\subseteq \mathbb{N}^\omega \to X$ encodes elements of X as rapidly converging sequences in the rational span of $\{e_i\}_i$

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Where to search for an example?

- General intuition:

  Object *A* has property *B*, but does not have property *B* effectively, then *A* is "close to not having property *B*"
- First task:
  Find a computable Banach space without a basis!
- Leads to the more general Basis Problem (Banach 1932):

  Is there a separable Banach space without a basis?

The Approximation Problem and its solution

- Fact: A Banach space without AP cannot have a basis.
- Approximation Problem:
  Is there a (separable) Banach space without AP?
- *Yes*: First example constructed by Enflo in 1973.
- Simplified by Davie in the same year.

Enflo/Davie's space is computable

- Enflo/Davie's space Z lacks AP.
- Not hard to verify (when looking at Davie's proof): There are suitable e_i such that $(Z, \|\cdot\|, (e_i)_i)$ is a computable Banach space.

The form of our example

- $Y := (Z \times Z \times \cdots)_{c_0}$ is the Banach space of all sequences in Z that converge to zero equipped with the sup-norm,
- $Z_0 \subseteq Z_1 \subseteq \ldots$ an ascending sequence of subspaces of Z (to be specified later) with

$$\mathrm{cls}\Big(\bigcup_n Z_n\Big) = Z,$$

- $\tau : \mathbb{N} \to \mathbb{N}$ an arbitrary function.
- Define the subspace Y_τ of Y by $Y_\tau := (Z_{\tau(0)} \times Z_{\tau(1)} \times \cdots)_{c_0}$.

Remaining steps of the proof

- Step 1: Choose the Z_n such that
  - for every τ, the space Y_τ has a basis; and
  - Y_τ is a computable Banach space if τ is lower semi-computable
    (i.e. $\{(n,j) \ : \ j \leq \tau(n)\}$ is r.e.)
- Step 2: Construct a lower semi-computable τ such that Y_τ does not have a computable basis.

Basis constants

- $(x_i)_i$ basic sequence $:\Longleftrightarrow$

  $(x_i)_i$ is a basis of $[x_0, x_1, \ldots]$.
- Characterization (Banach):
  - $x_i \neq 0$ for all i; and
  - there is a constant C such that

$$\|\sum_{i=0}^{m_1} \alpha_i x_i\| \leq C \|\sum_{i=0}^{m_2} \alpha_i x_i\|$$

  for all $0 \leq m_1 \leq m_2$, $\alpha_0, \ldots, \alpha_{m_2} \in \mathbb{F}$.
- Minimum C as above $=:$ basis constant $\mathrm{bc}((x_i)_i)$ of $(x_i)_i$.
- Basis constant $\mathrm{bc}(x_0, \ldots, x_n)$ of linearly independent $x_0, \ldots, x_n$ defined analoguously.

Local basis structure of Z

- A criterion by Szarek (1987) yields that Z has a property called local basis structure, defined by Pujara (1975).
- Consequence: There is a sequence $(c_i)_i \in Z^\omega$ and an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that
  - $[c_0, c_1, \ldots] = Z$; and
  - there is a constant C such that every subspace $[c_0, \ldots, c_{\sigma(n)}]$ has a basis with basis constant less than C.
- Suitable $(c_i)_i$ and σ can be found effectively.

Completion of Step 1

- Choose $Z_n := [c_0, \ldots, c_{\sigma(n)}]$.
- If τ is lower semi-computable, there is a computable sequence in Y that is complete in $Y_\tau = (Z_{\tau(0)} \times Z_{\tau(1)} \times \cdots)$.
  $\Longrightarrow Y_\tau$ is a computable Banach space.
- A basis of Y_τ can be constructed from the bounded-bc bases of the $Z_{\tau(n)}$.

A technical problem

- $(y_i^{(0)})_i, (y_i^{(1)})_i, \ldots$ effective enumeration of all computable sequences in Y

  Actually, such an enumeration does not exist:

  $(y_i^{(n)})_i$ necessarily "undefined" for infinitely many n.

  This problem can be worked around. (no details...)

Making Y_τ "too close to Y" to have a basis

- Any subspace X of Y with

$$(\{0\} \times \cdots \times \{0\} \times \underbrace{Z}_{n\text{-th}} \times \{0\} \times \cdots)_{c_0} \subseteq X$$

for some n lacks AP, hence does not have a basis.

- $\Longrightarrow$ If $(y_i^{(n)})_i$ is basic, we can effectively increase $\tau(n)$ so much that

$$(\{0\} \times \cdots \times \{0\} \times \underbrace{Z_{\tau(n)}}_{n\text{-th}} \times \{0\} \times \cdots)_{c_0} \not\subseteq [y_0^{(n)}, y_1^{(n)}, \ldots].$$

$\Longrightarrow Y_\tau \not\subseteq [y_0^{(n)}, y_1^{(n)}, \ldots]$

$\Longrightarrow (y_i^{(n)})_i$ is not a basis of Y_τ.

Taking care of non-basic $(y_i^{(n)})_i$

- Problem: If $(y_i^{(n)})_i$ is not basic, $\tau(n)$ might be increased infinitely often.
- Taken care of with the following modified algorithm:
  - For every k, try to increase $\tau(\langle n, k \rangle)$ until it is ensured that $Y_\tau \not\subseteq [y_0^{(n)}, y_1^{(n)}, \ldots]$.
  - In parallel, search for $0 \leq m_1 \leq m_2$ and $\alpha_0, \ldots, \alpha_{m_2}$ with

$$\|\sum_{i=0}^{m_1} \alpha_i y_i^{(n)}\| > k \|\sum_{i=0}^{m_2} \alpha_i y_i^{(n)}\|.$$

Once such numbers are found, stop increasing $\tau(\langle n, k \rangle)$.

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