On the Effective Existence of Schauder Bases in Banach Spaces

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Notation

- $\langle \underbrace{\cdot, \ldots, \cdot}_{k \text{ arguments}} \rangle : \mathbb{N}^k \to \mathbb{N}$ standard bijective tupling function
- For a sequence (x_i) in Banach space X, put
  $[x_0, x_1, \ldots] := \mathrm{clo}\big(\mathrm{span}((x_i))\big)$
- Sequence (x_i) complete in Banach space X $:\Longleftrightarrow$
  $[x_0, x_1, \ldots] = X$

Compact operators – Definition

- X, Y Banach spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
- $B_X := \{x \in X \ : \ \|x\| \leq 1\}$
- $F : X \to Y$ linear operator
- F compact $:\iff \overline{F(B_X)}$ compact in Y

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Compact operators – Examples

- All finite rank operators are compact
- $F : C[0,1] \to C[0,1]$ is compact if there is an $f \in C([0,1] \times [0,1])$ such that

$$F(g)(t) := \int_0^1 f(t,s)g(s)\,ds$$

Compact operators – Some classical results

- Space $B(X, Y)$ of bounded linear operators from X to Y equipped with operator norm is a Banach space
- Space $K(X, Y)$ of compact operators is a closed subspace of $B(X, Y)$
- $F \in B(X, Y)$, $G \in B(Y, Z)$, F or G compact $\implies GF$ compact

Compact operators – Theorem of Schauder

- $X^* := B(X, \mathbb{F})$ (topological) dual space of X
- For $F \in B(X, Y)$, define adjoint operator $F^* \in B(Y^*, X^*)$ by $F^*(g)(x) := g(F(x))$
- Theorem (Schauder):
  $F \in K(X, Y) \iff F^* \in K(Y^*, X^*)$

The approximation property

- Banach space X has approximation property (AP) $:\iff$ for every compact $K \subseteq X$ and every $\varepsilon > 0$ there is a finite rank operator $T : X \to X$ such that

$$\|T(x) - x\| \leq \varepsilon \text{ for all } x \in K$$

- Theorem (Grothendieck):
X has AP $\iff$ finite rank operators dense in $K(Y, X)$ for every Banach space Y

Effective versions of the classical results

Vasco Brattka and Ruth Dillhage (2007)

- ... define a natural effective representation of $K(X, Y)$
  (this representation includes information on $F \in K(X, Y)$ as a continuous mapping and on $F(B_X)$ as a compact set)
- ... prove computable versions of the results above,
  but: They assume that X, Y possess computable
  Schauder bases
  (with certain additional properties)

Schauder bases – Definition

- X infinite dimensional Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
- (x_i) sequence in X
- (x_i) (Schauder) basis for X $:\Longleftrightarrow$

  for every $x \in X$ there is a unique sequence (α_i) in $\mathbb{F}$

  such that $x = \lim_{n\to\infty} \sum_{i=0}^{n-1} \alpha_i x_i$

Schauder bases – Definition

- X finite dimensional Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
- $(x_1, \ldots, x_n)$ finite sequence in X
- $(x_1, \ldots, x_n)$ (Schauder) basis for $X :\Longleftrightarrow$

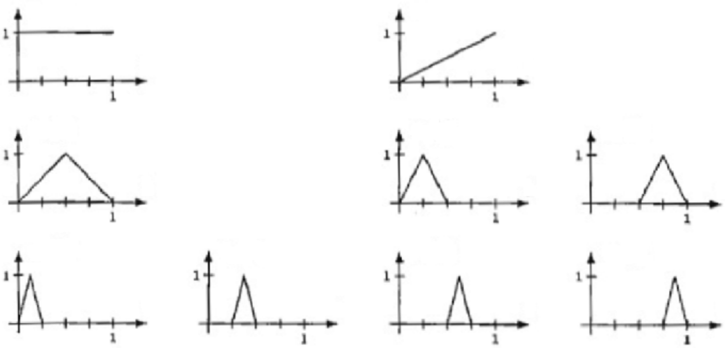
  for every $x \in X$ there are unique $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$
  such that $x = \sum_{i=1}^{n} \alpha_i x_i$

Schauder bases – Examples

- Consider Hilbert space H and
  (e_i) complete orthogonal sequence in H with $e_i \neq 0$ for all i
- Then (e_i) is a Schauder basis
- Consider sequence spaces ℓ_p $(1 \leq p < \infty)$ or c_0
- The unit vectors $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, $(0, 0, 1, \ldots)$, $\ldots$
  form a Schauder basis

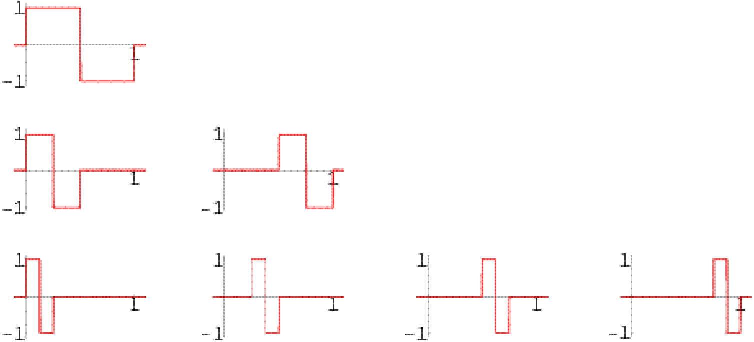
Schauder bases – Examples

- Schauder basis in space $C[0,1]$ of continuous functions



Schauder bases – Examples

- Schauder basis in integration space $L_p([0,1])$, $1 \leq p < \infty$



Today's question and result

- Question:
  Given a computable Banach space that possesses a basis.
  Does it necessarily possess a computable basis?
- Answer for Hilbert spaces (Brattka and Yoshikawa 2006):
  Yes.
- Answer in general (to be demonstrated today):
  No!

Computable Banach spaces

- $X = (X, \|\cdot\|)$ a separable normed space over $\mathbb{F} = \mathbb{R}$ (or $\mathbb{F} = \mathbb{C}$)
- $\alpha_{\mathbb{F}} : \mathbb{N} \to \mathbb{F}$ a standard numbering of $\mathbb{Q}$ (or $\mathbb{Q}[i]$)
- (e_i) a complete sequence in X
- α_e a derived numbering of the rational span of (e_i):

  $$\alpha_e\langle k, \langle n_0, \dots, n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i) e_i$$
- $(X, \|\cdot\|, (e_i))$ computable normed space $:\Longleftrightarrow$

  $m \mapsto \|\alpha_e(m)\|$ computable
- $(X, \|\cdot\|, (e_i))$ computable Banach space $:\Longleftrightarrow$

  $(X, \|\cdot\|, (e_i))$ computable normed space and X complete

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Computable sequences

- Sequence (x_i) in $(X, \|\cdot\|, (e_i))$ computable $:\Longleftrightarrow$
  there is a computable $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that
  $(\forall\, i)\ lim_{j\to\infty} \alpha_e(g(i,j)) = x_i$
  and $(\forall\, i)(\forall\, k > j)\ \|\alpha_e(g(i,j)) - \alpha_e(g(i,k))\| \leq 2^{-j}$

Where to search for an example?

- General intuition:

  Object A has property B, but does not have property B effectively, then A is "close to not having property B"
- First task:
  Find a computable Banach space without a basis!
- Leads to the more general Basis Problem (Banach 1932):

  Is there a separable Banach space without a basis?

The Approximation Problem and its solution

- A Banach space without AP cannot have a basis
- Approximation Problem:
  Is there a (separable) Banach space without AP?
- *Yes*: First example constructed by Enflo in 1973
- Simplified by Davie in same year

Enflo/Davie's space is computable

- Enflo/Davie's space Z lacks AP
- Not hard to verify (when looking at Davie's proof): There are suitable e_i such that $(Z, \|\cdot\|, (e_i))$ is a computable Banach space

A space Y of sequences in Z

- Y Banach space of all sequences (z_i) in Z with $z_i \to 0$ equipped with norm $\|(z_i)\| = \sup_{m \in \mathbb{N}} \|z_m\|$
- $\mathrm{emb}^m : Z \to Y$
  $\mathrm{emb}^m(z) := (0, \ldots, 0, \underbrace{z}_{m\text{-th}}, 0, \ldots)$
- With canonical choice of suitable $g_i \in Y$:
  $(Y, \|\cdot\|, (g_i))$ is a computable Banach space
- Our final example will be a computable subspace X of Y

What will our example space X look like?

- $X_0 \subseteq X_1 \subseteq \ldots \subseteq Z$ sequence of certain fin. dim. subspaces with $\mathrm{clo}\left(\bigcup_i X_i\right) = Z$
- $\tau : \mathbb{N} \to \mathbb{N}$
- $X := \{(z_i) \in Y \ : \ z_m \in X_{\tau(m)}$ for all $m\}$
- Idea:
  - If X had a subspace of form $\mathrm{emb}^m(Z)$, then X would not have AP (and hence would not have a basis)
  - X contains subspaces $\mathrm{emb}^m(X_{\tau(m)})$ with $X_{\tau(m)}$ "close" to Z
  - X is "close" to not having a basis

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It remains to solve two problems

- Problem 1:
  How to (effectively) choose the X_m such that X will have a basis?
- Problem 2:
  Compute τ such that X does not have a computable basis.

Basis constants

- (x_i) basic sequence $:\Longleftrightarrow$

  (x_i) is a basis of $[x_0, x_1, \ldots]$
- Characterization (Banach):
  - $x_i \neq 0$ for all i; and
  - there is a $C \geq 1$ such that

$$\|\sum_{i=0}^{m_1} \alpha_i x_i\| \leq C \|\sum_{i=0}^{m_2} \alpha_i x_i\|$$

  for all $0 \leq m_1 \leq m_2$, $\alpha_0, \ldots, \alpha_{m_2} \in \mathbb{Q}[i]$.
- Minimum C as above: basis constant $\mathrm{bc}((x_i))$

Basis constants

- Basis constant $\mathrm{bc}(x_1, \ldots, x_n)$ of lin. indep. $x_1, \ldots, x_n$: minimal C such that

$$\|\sum_{i=1}^{m_1} \alpha_i x_i\| \leq C \|\sum_{i=1}^{m_2} \alpha_i x_i\|$$

  for all $1 \leq m_1 \leq m_2 \leq n$, $\alpha_0, \ldots, \alpha_{m_2}$.
- Basis constant $\mathrm{bc}(X)$ of a Banach space X with basis: infimum over the basis constants of all bases of X

Local basis structure of Z

- Definition (Pujara 1975):
  Banach space X has local basis structure (l.b.s.) $:\Longleftrightarrow$

  there is a constant C such that
  for every fin. dim. subspace $M \subseteq X$
  there is fin. dim. N such that
  $M \subseteq N \subseteq X$ and $\mathrm{bc}(N) < C$.
- A sufficient condition given by Szarek (1987) yields:
  Z has local basis structure

Solution to Problem 1

- L.b.s. can be made effective:

  There is

  a constant C,

  a computable sequence (c_i) in Z, and

  a computable strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$

  such that for $X_m := [c_0, \ldots, c_{\sigma(m)}]$ one has

  $\mathrm{clo}\left(\bigcup_m X_m\right) = Z$, and

  $\mathrm{bc}(X_m) < C$ for all m

- Using "$\mathrm{bc}(X_m) < C$ for all m", one can prove:

  $X = \{(z_i) \in Y \ : \ z_m \in X_{\tau(m)} \text{ for all } m\}$
  has a basis, no matter how τ is chosen

Diagonalization strategy

- $(y_i^{(0)}), (y_i^{(1)}), \ldots$ effective enumeration of all computable sequences in Y

  Acutally, such an enumeration does not exist:

  $(y_i^{(n)})$ necessarily “undefined” for infinitely many n

  This problem can be worked around (no details...)

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Diagonalization strategy

- Idea: Compute τ such that for every n either
  - there is a j such that $\mathrm{emb}^{\langle n,j\rangle}(X_{\tau(\langle n,j\rangle)}) \not\subseteq [y_0^{(n)}, y_1^{(n)}, \ldots]$ (and so $X \not\subseteq [y_0^{(n)}, y_1^{(n)}, \ldots]$); or
  - $(y_i^{(n)})$ is not basic
- Then no computable basic sequence in Y is fundamental in $X \implies$ we're done!

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Details on how to semi-compute $\tau(\langle n, j \rangle)$

- Think of $\tau(\langle n, j \rangle)$ as a variable that may be increased finitely many times
- Initialize $\tau(\langle n, j \rangle) \leftarrow 0$
- Perform the following loop:

Step 1 Semidecide whether "$\mathrm{emb}^{\langle n,j \rangle}(X_{\tau(\langle n,j \rangle)}) \subseteq [y_0^{(n)}, y_1^{(n)}, \ldots]$" is "dangerously close to being fulfilled"

Step 2 If so, put $\tau(\langle n, j \rangle) \leftarrow \tau(\langle n, j \rangle) + 1$ and goto Step 1

Details on how to semi-compute $\tau(\langle n, j\rangle)$

- If $(y_i^{(n)})$ basic, then $\mathrm{emb}^{\langle n,j\rangle}(Z) \not\subseteq [y_0^{(n)}, y_1^{(n)}, \ldots]$

  In this case, $\tau(\langle n, j\rangle)$ eventually so large that Step 1 will not return

  (this is the "final value" for $\tau(\langle n, j\rangle)$)
- Problem: If $(y_i^{(n)})$ not basic, $\tau(\langle n, j\rangle)$ might be increased infinitely often

Assuring finiteness of $\tau(\langle n, j \rangle)$

- Recall: If $(y_i^{(n)})$ not basic, then there are $0 \leq m_1 \leq m_2$ and $\alpha_0, \ldots, \alpha_{m_2} \in \mathbb{Q}[i]$ with

$$\|\sum_{i=0}^{m_1} \alpha_i y_i^{(n)}\| > j \|\sum_{i=0}^{m_2} \alpha_i y_i^{(n)}\|$$

- In parallel to the loop above:
  - Search for such $0 \leq m_1 \leq m_2$ and $\alpha_0, \ldots, \alpha_{m_2}$
  - If found, interrupt the loop
- Finally note:

  If $(y_i^{(n)})$ basic, no such interruption will occur for large enough j

X is a computable subspace of Y

- Requirement: X is computable subspace,
  i.e. $X = [H]$ for a set H which is the range of a computable sequence in Y
- Recall:
  - $X = \{(z_i) \in Y \;:\; z_m \in X_{\tau(m)} \text{ for all } m\}$
  - $X_m := [c_0, \ldots, c_{\sigma(m)}]$ with computable (c_i) and σ
- τ is lower-semicomputable,

  (i.e. $\{(m,k) \;:\; k \leq \tau(m)\}$ r.e.)

  We can choose
  $H := \{\mathrm{emb}^m(c_s) \;:\; m \in \mathbb{N},\, 0 \leq s \leq \tau(\sigma(m))\}$

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